Using Special Relativity (SR) as a starting point, then noting a few empirical 4-Vector facts, one can derive the Principles that are normally considered to be Axioms of Quantum Mechanics (QM).

Since many of the QM Axioms are rather obscure, this seems a more logical and understandable paradigm than QM as a separate theory from SR, and sheds light on the origin and meaning of the QM Principles.

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

or: Why General Relativity (GR) is *NOT* wrong
or: Don’t bet against Einstein ;)
or: QM, the easy way...

And yes, I did the Math...
4-Vectors are a fantastic language/tool for describing the physics of both relativistic and quantum phenomena. They easily show many interesting properties and relations of our Universe, and do so in a simple and concise mathematical way. Due to their tensorial nature, these SR 4-Vectors are automatically coordinate-frame invariant, and can be used to generate *ALL* of the physical SR Lorentz Scalar tensors and higher-index-count SR tensors. Let me repeat: You can mathematically build *ALL* the Lorentz Scalars and larger SR tensors from SR 4-Vectors.

4-Vectors are likewise easily shown to be related to the standard 3-vectors that are used in Newtonian classical mechanics, Maxwellian classical electromagnetism, and standard quantum theory.

Why 4-Vectors as opposed to some of the more abstract mathematical approaches to QM? Because the components of 4-Vectors are things that can actually be empirically measured. Experiment is the ultimate arbiter of which theories actually correspond to reality. If your quantum logics and string theories give no testable/measurable predictions, then they are basically useless for real physics.

In this treatise, I will demonstrate how 4-Vectors are used in the context of Special Relativity, and then show that their use in Relativistic Quantum Mechanics is really not fundamentally different. Quantum Principles then emerge in a natural and elegant way.

I also introduce the SRQM Diagramming Method, an instructive graphical charting-method, which visually shows how the SRQM 4-Vectors, Lorentz 4-Scalars, and 4-Tensors are all related to each other. This symbolic representation clarifies a lot of physics and is a great tool for understanding and teaching.
**SRQM**

**Some Physics Abbreviations & Notation**

- **GR** = General Relativity
- **SR** = Special Relativity
- **CM** = Classical Mechanics
- **EM** = ElectroMagnetism
- **QM** = Quantum Mechanics
- **RQM** = Relativistic Quantum Mechanics
- **NRQM** = Non-Relativistic Quantum Mechanics
- **QFT** = Quantum Field Theory
- **QED** = Quantum ElectroDynamics
- **RWE** = Relativistic Wave Equation
- **KG** = Klein-Gordon (Relativistic Quantum) Eqn
- **H** = The Hamiltonian = \( \gamma (P_T \cdot U) \)
- **L** = The Lagrangian = \(-\frac{\partial}{\partial \tau} [S] \)
- **∇** = 3-gradient = \((\partial_x, \partial_y, \partial_z)\) = \((\partial/\partial x, \partial/\partial y, \partial/\partial z)\)
- **∂** = 4-Gradient = \(\partial^\mu = (\partial/c, -\nabla)\); \(\partial_\mu = (\partial/c, \nabla)\)
- **S** = The Action (4-TotalMomentum \( P_T = -\partial[S] \))
- **Φ** = The Phase (4-TotalWaveVector \( K_T = -\partial[\Phi] \))
- **τ** = Proper Time (Invariant Rest Time) = \(t_0\)
- **Σ** = Sum of Range ; **Π** = Product of Range
- **Δ** = Difference ; **d** = Differential ; **∂** = Partial

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td>Relativistic Beta = (v/c = {0..1} \hat{n}); (v) = 3-velocity = ({0..c} \hat{n})</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>Relativistic Gamma = (1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta \cdot \beta}) = ({1..\infty})</td>
</tr>
<tr>
<td>(D)</td>
<td>Relativistic Doppler = (1/\gamma(1-</td>
</tr>
<tr>
<td>(\Lambda^{\mu\nu})</td>
<td>Lorentz (SpaceTime) Transform</td>
</tr>
<tr>
<td>(I_{(3)})</td>
<td>3D Identity Matrix ; (I_{(4)}) = 4D Identity Matrix = Diag([1,1,1,1])</td>
</tr>
<tr>
<td>(\delta^{ij})</td>
<td>3D Kronecker Delta</td>
</tr>
<tr>
<td>(\delta^{\mu\nu})</td>
<td>4D Kronecker Delta</td>
</tr>
<tr>
<td>(\xi^{\mu\nu}_{\gamma\delta})</td>
<td>3D Levi-Civita anti-symmetric permutation symbol</td>
</tr>
<tr>
<td>(\epsilon^{ijk})</td>
<td>3D Levi-Civita anti-symmetric permutation symbol</td>
</tr>
<tr>
<td>(\epsilon^{\mu\nu\rho\sigma})</td>
<td>4D Levi-Civita anti-symmetric permutation symbol</td>
</tr>
</tbody>
</table>

**Tensor-Index & 4-Vector Notation:**

- \(A^i = A = (a^1, a^2, a^3): 3\)-vector [Latin index \(\{1..3\}\)]
- \(A^\mu = A = (a^0, a^1, a^2, a^3): 4\)-Vector [Greek index \(\{0..3\}\)]
- \(A^\mu B_\mu = A \cdot B: 3\)-vector |
- \(A^\mu B^\nu = A \cdot B: 4\)-Vector |
- \(A^\mu B^\nu - A^\nu B^\mu = A^\mu B^\nu: 4\)-Vector |
- \(A^\mu B^\nu - B^\mu A^\nu = [A^\mu, B^\nu]: Commutation |

SRQM = The [SR→QM] Interpretation of Quantum Mechanics, by John B. Wilson
See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document, most current ver at scirealm.org)

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM Study: Physical/Mathematical Tensors

Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

Component Types: Temporal, Spatial, Mixed

Matrix Format

SR 4-Scalar S

SR 4-Vector Vμ

SR 4-Tensor Tμν

SRQM Diagram Format

SR 4-Scalar (0,0)-Tensor S

SR 4-Vector (1,0)-Tensor Vμ

SR 4-Tensor (2,0)-Tensor Tμν

SRQM Diagram Ellipse:
4-Scalars, 0 index
4^0 = 0 corners
4^1 = (1) = 1 component

SRQM Diagram Rectangle:
4-Vectors, 1 index
4^1 = 4 corners
4^3 = (1+3) = 4 components

SRQM Diagram Octagon:
4-Tensors, 2 index
4^2 = 8 corners
4^3 = (1+6+9) = 16 components

arily, all these objects are “SR 4-Tensors”, but we usually reserve the name “4-Tensor” for objects with 2 or more indices, and use the (m,n)-Tensor notation to specify all the objects more precisely.

Mixed 4-Tensor (1,1)-Tensor

Mixed 4-Tensor (1,1)-Tensor

Mixed 4-Tensor (0,2)-Tensor

Lowered 4-Tensor (0,2)-Tensor

Technically, all these objects are “SR 4-Tensors”, but we usually reserve the name “4-Tensor” for objects with 2 or more indices, and use the (m,n)-Tensor notation to specify all the objects more precisely.
The SRQM Diagramming Method shows the properties and relationships of various physical objects in a graphical way. This "flowchart" method aids understanding.

**Representation:** 4-Scalars by ellipses, 4-Vectors by rectangles, 4-Tensors by octagons. Physical/mathematical equations and descriptions inside each shape/object. Sometimes there will be additional clarifying descriptions around a shape/object.

**Relationships:** Lorentz Scalar Products or tensor compositions of different 4-Vectors are on simple lines between the related 4-Vectors. Lorentz Scalar Products of a single 4-Vector, or Invariants of Tensors, are next to that object and highlighted in a different color.

**Flow:** Objects that are some function of a Lorentz Scalar with another 4-Vector or 4-Tensor are on lines with arrows indicating the direction of flow. (ex. multiplication)

**Properties:** Some objects will also have a symbol representing its properties nearby, and sometimes there will be color highlighting within the object to emphasize temporal-spatial properties. I typically use blue=Temporal, red= Spatial, purple=mixed TimeSpace.

Alternate ways of writing 4-Vector expressions in physics:

\[(A \cdot B)\] is a 4-Vector style, which uses vector-notation (ex. inner product "dot= " or exterior product "wedge="), and is typically more compact, always using **bold** UPPERCASE to represent the 4-Vector, ex. \((A \cdot B) = (A^\mu \eta_{\mu \nu} B^\nu)\), and **bold** lowercase to represent 3-vectors, ex. \((a \cdot b) = (a^i \delta_{ij} b^j)\). Most 3-vector rules have analogues in 4-Vector mathematics.

\((A^\mu \eta_{\mu \nu} B^\nu)\) is a Ricci Calculus style, which uses tensor-index-notation and is useful for more complicated expressions, especially to clarify those expressions involving tensors with more than one index, such as the Faraday EM Tensor \(F^{\mu \nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial \wedge A)\)

**SRQM Diagramming Method**

\[\partial[R] = \partial^\mu [R^\nu] = \eta^{\mu \nu} \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\delta_{\mu \nu}] \]

Minkowski Metric

4-Displacement

\[\Delta R = (c \Delta t, \Delta r) \]

4-Position

\[R = (ct, r) \rightarrow \text{<Event>} \]

4-Gradient

\[\partial = (\partial/c, -\nabla) \]

Lorentz

\[\partial_\nu [R^\mu] = \partial R^\mu/\partial R_\nu = \Lambda^\mu_\nu \]

Transform

4-Scalar

\[\Lambda_{\mu \nu} \Lambda^{\mu \nu} = 4 \]

Dimension

\[\Lambda_{\mu \nu} \Lambda^{\mu \nu} = 4 \]

\[\text{Tr}[\eta^{\mu \nu}] = 4 \]

4-Momentum

\[P = (mc, p) = (E/c, p) = m_0 U \]

\[d^3p/E \]

4-Velocity

\[U = \gamma(c, u) \rightarrow dR/d\tau [\ldots] \]

U \cdot U = c^2

\[\gamma d/dt [...] \]

Proper-Time Derivative

**SR 4-Vector**

\[V = (V_0, V) \]

SR 4-Scalar

\[S \]

Lorentz Scalar

**SR 4-Tensor**

\[(2,0)-Tensor T^{\mu \nu}, \text{ or } T^{\mu}_{\nu}, \text{ or } T^{\mu}_{\nu}, \text{ or } T_{\mu \nu}, \]

\[(1,1)-Tensor V^\nu = V = (V_0, V) \]

\[(0,2)-Tensor V_{\nu} = (V_0, V) \]

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson

A Tensor Study of Physical 4-Vectors

The Special Relativity → Quantum Mechanics SRQM Diagramming Method shows the properties and relationships of various physical objects in a graphical way. This "flowchart" method aids understanding.

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\[\partial[R] = \partial^\mu [R^\nu] = \eta^{\mu \nu} \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\delta_{\mu \nu}] \]

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4-Displacement

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Lorentz

\[\partial_\nu [R^\mu] = \partial R^\mu/\partial R_\nu = \Lambda^\mu_\nu \]

Transform

4-Scalar

\[\Lambda_{\mu \nu} \Lambda^{\mu \nu} = 4 \]

Dimension

\[\Lambda_{\mu \nu} \Lambda^{\mu \nu} = 4 \]

\[\text{Tr}[\eta^{\mu \nu}] = 4 \]

4-Momentum

\[P = (mc, p) = (E/c, p) = m_0 U \]

\[d^3p/E \]

4-Velocity

\[U = \gamma(c, u) \rightarrow dR/d\tau [\ldots] \]

U \cdot U = c^2

\[\gamma d/dt [...] \]

Proper-Time Derivative

**SR 4-Vector**

\[V = (V_0, V) \]

SR 4-Scalar

\[S \]

Lorentz Scalar

**SR 4-Tensor**

\[(2,0)-Tensor T^{\mu \nu}, \text{ or } T^{\mu}_{\nu}, \text{ or } T^{\mu}_{\nu}, \text{ or } T_{\mu \nu}, \]

\[(1,1)-Tensor V^\nu = V = (V_0, V) \]

\[(0,2)-Tensor V_{\nu} = (V_0, V) \]
One of the extremely important properties of Tensor Mathematics is the fact that there are numerous ways to generate Tensor Invariants. These Invariants lead to Physical Properties that are fundamental in our Universe. They are totally independent of the coordinate systems used to measure them. Thus, they represent symmetries that are inherent in the fabric of SpaceTime. See the Cayley-Hamilton Theorem.

Trace Tensor Invariant: \( \text{Tr}[T^\mu] = \eta_{\mu\nu}T^\nu = T_\mu = \Sigma \text{[EigenValues } \lambda_n \text{]} \) for \( T^\mu \),

Determinant Tensor Invariant: \( \text{Det}[T^\mu] = \Pi \text{[EigenValues } \lambda_n \text{]} \) for \( T^\mu \),

Inner Product Tensor Invariant: \( IP[T^\mu] = T^\mu T_\mu \),

4-Divergence Tensor Invariant: \( 4\text{-Div}[T^\mu] = \partial_{\mu}T^\nu = \partial \cdot T = \partial T^\nu/\partial x^\nu \) : \( 4\text{-Div}[T^\mu] = S^\nu = \partial_{\nu}T^\mu \),

Lorentz Scalar Product Tensor Invariant: \( \text{LSP}[T^\mu,S^\nu] = T^\mu\eta_{\mu\nu}S^\nu = T^\mu S_\mu = T\cdot S = T\cdot S \),

Phase Space Tensor Invariant: \( \text{PS}[T^\mu] = (d^4t/t^4) \) for \( (T\cdot T) \) = constant

The Ratio of 4-Vector Magnitudes (Ratio of Rest Value 4-Scalars): \( T\cdot T/S\cdot S = (t^0/s^0)^2 \)

Tensor Eigenvalues \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \}

The various AntiSymmetric Tensor Products, etc.

These are not all always independent, some invariants are functions of other invariants.
### SRQM Study: Physical/Mathematical Tensors

#### Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

**Examples – Venn Diagram**

<table>
<thead>
<tr>
<th>0 index-count Tensors</th>
<th>1 index-count Tensors</th>
<th>2 index-count Tensors</th>
<th>Higher index-count Tensors</th>
</tr>
</thead>
<tbody>
<tr>
<td>EM Charge (Q=</td>
<td>dtdx)</td>
<td>SR 4-Vector (0,0)-Tensors</td>
<td>Lorentz Scalar S</td>
</tr>
<tr>
<td>Speed-of-Light (c=√[U·U])</td>
<td>4-Position R=R'=(ct,r)→(ct,x,y,z)</td>
<td>4-Velocity U=U'=c,u=dt/dτ</td>
<td>SR 4-Covector = “Dual” 4-Vector</td>
</tr>
<tr>
<td>#Dimensionless</td>
<td>4-Momentum P=P'=mc=mv</td>
<td>Proper Time δ=R=δR'=4 Dimension</td>
<td>SR 4-CoVector aka. One-Forms</td>
</tr>
<tr>
<td>RestMass (m₀)</td>
<td>δ'[X·X]</td>
<td>δ'=v·dt/dτ</td>
<td>Gradient One-Form δ=δ(c,ν)</td>
</tr>
<tr>
<td>Planck’s Const (h)</td>
<td>d²P/E</td>
<td>V=γd²x</td>
<td>(δ/c,δ,δ)</td>
</tr>
<tr>
<td>2 index-count Tensors: Minkowski η=δ[μν]=v·v+H²</td>
<td>Projected (Mixed) Tensors Pν=</td>
<td>Lorentz Transform</td>
<td>Lowered Minkowski δ₁[R]=η=1</td>
</tr>
<tr>
<td>SR 4-Vector (0,2)-Tensors R⁰⁰, R⁰¹, R⁰²</td>
<td>Temporal Projection Pν=Vν</td>
<td>Lorentz ParityInverse</td>
<td>Projection Tensors Pν=</td>
</tr>
<tr>
<td>T⁰⁰ =</td>
<td>Spatial Projection Pν=Hν</td>
<td>Lorentz Boost Λν=</td>
<td>Temporal Proj. Pν=Vν</td>
</tr>
<tr>
<td>[T⁰⁰, T¹¹, T²²]</td>
<td></td>
<td>Λν=</td>
<td>Spatial Proj. Pν=Hν</td>
</tr>
<tr>
<td>Perfect Fluid 4-Tensor T⁰⁰ = (ρv)⁰⁰</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Faraday EM 4-Tensor F⁰¹ = ∂A¹ - ∂A⁰</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Riemann Curvature Tensor Ricci Decomposition of Riemann Tensor</td>
<td>Weyl (Conformal) Curvature Tensor</td>
<td>Ricci Decomposition of Riemann Tensor</td>
<td></td>
</tr>
<tr>
<td>R⁰⁰ = δ₅₆₅₆ - δ₅₆₅₆ + δ₅₆₅₆ = R⁻¹ = 0⁰⁰ =</td>
<td>Lorentz ParityInverse</td>
<td>Trace[Tν]= [ η¹ν[Tν=Vν = Tν = Tν = Vν = Vν = (Vν)² - (Vν)² = (Vν)²</td>
<td></td>
</tr>
<tr>
<td>SR 4-Scalar (0,0)-Tensors R⁰⁰</td>
<td>Lorentz Scalar</td>
<td>Lorentz Scalar</td>
<td>= Lorentz Scalar</td>
</tr>
</tbody>
</table>
### 4-Tensors = (1,0)-Tensors

**SL Units**

- **[Temporal : Spatial] components**
  - [ Temporal : Spatial ] components
    - [T = kg/C·s]
      - [0 : 3-Electric-Field (e = e)]: 3-Magnetic-Field (b = b)]
    - [F = ∂A/A - ∂A'μ]
    - [J = N·m·s = kg·m²/s]
      - [0 : 3-Mass-Moment (n = n)]: 3-Angular-Momentum (l = l)]
      - [M = X·P = X·P' - X·Pμ]

### 4-Tensors = (2+ index)-Tensors

- **Minkowski Metric ημν**
  - [diagonal]
  - [1 : 0 : -1(3)]
    - [1 : 0 : -δk]
    - [δμν]
    - [Vμν]
    - [Hμν]

- **Perfect-Fluid Stress-Energy Tμν → Diag[ρ, p, p, p, p]**
  - [J/m² = N/m² = kg·m²/s²]
  - [ρφ : 0 : pI(3)]
  - [ρφ : 0 : pδk]

### 4-Vectors = Type (1,0)-Tensor

**SI Units**

- **[Temporal : Spatial] components**
  - [T = kg/C·s]
    - [0 : 3-Electric-Field (e = e)]: 3-Magnetic-Field (b = b)]
    - [F = ∂A/A - ∂A'μ]

**4-Tensor**

- **Position R** = Rμν = (ct, r)
- **Velocity U** = Uμ = γ(c, u)
- **Unit Temporal T** = Tμν = γ(1, β)
- **Momentum P** = Pμ = (E/c, c, pν)
- **Total Momentum** P = Pμ = (E/c = H/c, pν) = n[Pμ]
- **Force F** = Fμν = γ(E/c, f)
- **Wave Vector K** = Kμ = (ω/c, k)
- **Total Wave Vector** K = Kμ = (ω/c, k) = n[Kμ]
- **Angular Momentum** = n[Γμν] = n[(nc, u) = n(1, β)]
- **Spins S** = Sμ = (s, s) = (β, s)

**4-Vectors**

- **Energy (E)** = 3-Momentum **p**
- **Total Energy** (E) = Hamiltonian **H** = 3-TotalMomentum (pμ)
- **Angular Frequency** (ω) = 3-AngularWaveNumber (k)
- **Total Angular Frequency** (ω) = 3-TotalAngularWaveNumber (kμ)
- **ChargeDensity (ρ)** = 3-CurrentDensity = 3-ChargeFlux (j)
- **ScalarPotential (φ)** = 3-VectorPotential (a), typically the EM versions (φEM) : (aEM)
- **Time Differential (∂t) : Spatial 3-Gradient (∇ = ∂t) : Spatial 3-Angular-Momentum (l = l)
- **WaveNumber (k)** : 3-TotalWaveNumber (kμ)
- **Spatial 3-Velocity Factor (γ)**
- **Spatial 3-Velocity (u)**
- **Spatial 3-Spin (s)**
- **Spatial 3-Spin (s)**

**SR 4-Vector**

- **(2,0)-Tensor Tμν** or Tμν
- **(1,0)-Tensor Vμ = (vμ, ν)**

**SR 4-Scalar**

- **(0,0)-Tensor S** Lorentz Scalar

4-Tensors can be constructed from the Tensor Products of 4-Vectors. Technically, 4-Tensors refer to all SR objects (4-Scalars, 4-Vectors, etc), but we typically reserve the name 4-Tensor for SR Tensors of 2 or more indices.
SRQM 4-Scalars = (0,0)-Tensors = Lorentz Scalars → Physical Constants

4-Scalar = Type (0,0)-Tensor

<table>
<thead>
<tr>
<th>SL Units:</th>
<th>4-Scalar = Type (0,0)-Tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t) = [R·U]/[U·U] = [R·R]/[R·U] <strong>Time as measured in the at-rest frame</strong></td>
<td></td>
</tr>
<tr>
<td>(dt) = [dR·U]/[U·U] <strong>Differential Time as measured in the at-rest frame</strong></td>
<td></td>
</tr>
<tr>
<td>(c) = sqrt(U·U) = [T·U] with 4-UnitTemporal T = γ(1,β)</td>
<td></td>
</tr>
<tr>
<td>(m_e) = [P·U]/[U·U] = [P·R]/[R·U] (m_e→m_e) as Electron RestMass</td>
<td></td>
</tr>
<tr>
<td>(E_e) = [P·U]</td>
<td></td>
</tr>
<tr>
<td>(ω_e) = [K·U]</td>
<td></td>
</tr>
<tr>
<td>(p_e) = [J·U]/[U·U] = (q)/[N·U]/[U·U] = (q)/[n_e]</td>
<td></td>
</tr>
<tr>
<td>(Φ_{phase}) = [K·R] = (k·r - ωt) : (Φ_{phase}) = -[K·R] = (k·r - ωt)</td>
<td></td>
</tr>
<tr>
<td>(S_{action}) = -[P·R] = (p·r - Et) : (S_{action}) = -[P·R] = (p·r - Et) <strong>Units [Action] = [Momentum] [Length] = [Energy] [Time]</strong></td>
<td></td>
</tr>
<tr>
<td>(h) = [P·U]/[K·U] = [P·R]/[K·R]</td>
<td></td>
</tr>
<tr>
<td>(4) = [b·r] = Tr[e^{αβ}] <strong>4-Divergence[4-Position] = Trace[MinkowskiMetric] = SR Dimension</strong></td>
<td></td>
</tr>
<tr>
<td>δF_{αβ} = (μ_e)J = (1/ε_o)c^2J Maxwell EM Eqn. μ_e,ε_o = 1/c^2</td>
<td></td>
</tr>
<tr>
<td>δF_{αβ} = (μ_e)J = (1/ε_o)c^2J Maxwell EM Eqn. μ_e,ε_o = 1/c^2</td>
<td></td>
</tr>
<tr>
<td>U·F_{αβ} = (q)/c Lorentz Force Eqn. (q→-e) as Electron Charge</td>
<td></td>
</tr>
<tr>
<td>(Q) = [p·d·x] = [p·γ d·x] Integration of volume charge density</td>
<td></td>
</tr>
<tr>
<td>(N) = [j·d·x] = [j·γ d·x] Integration of volume number density</td>
<td></td>
</tr>
<tr>
<td>(V_o) = [V·d·x] Integration of volume elements (Riemannian Volume Form)</td>
<td></td>
</tr>
<tr>
<td>(p_o) = [V_o]T_{αβ} Spatial &quot;Vertical&quot; Projection of Perfect Fluid Stress-Energy Tensor</td>
<td></td>
</tr>
<tr>
<td>(p_o) = (-1/3)H_{αβ}T_{αβ} Spatial &quot;Horizontal&quot; Projection of Perfect Fluid Stress-Energy Tensor</td>
<td></td>
</tr>
</tbody>
</table>

Lorentz Scalars can be constructed from the Lorentz Scalar Product of 4-Vectors
### SRQM Study: Physical 4-Vectors

#### Some SR 4-Vectors and Symbols

| 4-Gradient | \( \partial = \partial^\mu = (\partial / c, \nabla) \) |
| 4-Displacement | \( \Delta R = R^\mu = (c \Delta t, \Delta r) = R_2 - R_1 \) |
| 4-Position | \( R = R^\mu = (ct, r) = \langle \text{Event} \rangle \) |
| 4-Velocity | \( U = U^\mu = (\gamma(c, u)) \) |
| 4-Acceleration | \( A = A^\mu = (\gamma(c', u + \gamma a)) \) |
| 4-Tensor | \( \delta[R] = \delta[R^\mu] = \delta^\mu \) |

#### Symbols

- \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \)
- \( T = T^\mu = (1, \beta) \)
- \( U = U^\mu = (c, 0) \)
- \( A = A^\mu = (\phi/c, \mathbf{a}) = (\phi, c^2)U \)
- \( Q = Q^\mu = (\mathbf{q}/c, \mathbf{q}) = (\mathbf{q}, c^2)U \)
- \( J = J^\mu = (\mathbf{p}/c, \mathbf{p}) = (\mathbf{p}, c^2)U \)
- \( N = N^\mu = (\mathbf{n}/c, \mathbf{n}) = (\mathbf{n}, c^2) \)
- \( S_{\text{ent pure}} = S_{\text{ent pure}}^\mu \)
- \( S_{\text{ent heat}} = S_{\text{ent heat}}^\mu \)
- \( S_{\text{ent}} = S_{\text{ent}}^\mu \)

#### Equations

- 4-Momentum: \( P = P^\mu = (mc, \mathbf{p}) = mcU \)
- 4-Force: \( F = F^\mu = (E/c, \mathbf{f}) \)
- 4-Wavevector: \( K = K^\mu = (\omega/c, k) = (\omega, c^2)U \)
- 4-Mass Flux: \( G = G^\mu = (\rho, c) = \rho c (u) \)
- 4-Energy Flux: \( Q = Q^\mu = (\rho E/c, q) = \rho E (c, u) \)
- 4-Heat Entropy Flux: \( S_{\text{ent}} = S_{\text{ent}}(\mathbf{n}/c_0, \mathbf{n}) = S_{\text{ent}}(\mathbf{n}, c_0) \)
- 4-Entropy Flux: \( S_{\text{ent}} = S_{\text{ent}}(\mathbf{n}/c_0, \mathbf{n}) = S_{\text{ent}}(\mathbf{n}, c_0) \)

---

**SR 4-Tensor**

- (2,0)-Tensor \( T_{\mu} = (1,1)-\text{Tensor} \ T^\mu, \text{or } T^\mu \)
- (0,2)-Tensor \( T_{\mu} \)

**SR 4-Vector**

- (1,0)-Tensor \( V^\mu = (v^x, v) \)
- (0,1)-Tensor \( V_{\mu} = (v, v^x) \)

**SR 4-Scalar**

- (0,0)-Tensor \( S \)

**4-Vector**

- \( V = V^\mu = (v^x, v^y, v^z) = (v^\mu) \)

**Trace**

- \( \text{Trace}[T^\mu] = \eta_{\mu\nu} T^\nu = T^\mu = T \)
- \( V \cdot V = (V^\mu)(V^\nu) = (V^\mu V^\nu) = (\mathbf{v} \cdot \mathbf{v}) = (v^\mu v^\nu) = (v^\mu v^\nu) = (1/v^2) = \text{Lorentz Scalar} \)
SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

Lorentz Identity Transform \( \Lambda^\mu_\nu = \eta^\mu_\nu = I_{(4)} \)

Lorentz Time-Reverse Transform \( \Lambda^\mu_\nu \rightarrow T^\mu_\nu = -I_{(4)} \)

Lorentz Space-Parity Transform \( \Lambda^\mu_\nu \rightarrow P^\mu_\nu = \sigma^\mu_\nu \)

Lorentz ComboPT Transform \( \Lambda^\mu_\nu \rightarrow (PT)^\mu_\nu = -I_{(4)} \)

SR: Minkowski Metric
\( \delta^\mu_\nu = \delta^\mu_\nu = V^\mu_\nu + H^\mu_\nu \)
\( \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\delta^\mu_\nu] \)

Perfect Fluid
\( T^\mu_\nu = (\rho_0) V^\mu_\nu + (-p_0) H^\mu_\nu \)

Faraday EM
\( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \partial^\nu A^\mu \)

4-AngularMomentum
\( M^\mu = X^\mu P^\nu - X^\nu P^\mu = X^\mu P^\nu \)

4-Tensor
- Symmetric
- Anti-symmetric

Trace\([T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T^\mu_\mu = T \)
\( V\cdot V = V^\mu_\nu V^\mu_\nu = (V^\mu_\nu)^2 = (v^\mu)^2 = \text{Lorentz Scalar} \)
SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

4-ForceDensity
\[ T^{\mu\nu} \rightarrow (\rho_{\infty} V^{\mu\nu} + (-p_{\infty}) H^{\mu\nu}) \]

Perfect Fluid Stress-Energy

Null-Dust=Photon Gas

EM (Maxwell) Stress-Energy Tensor
\[ T^{\mu\nu} \rightarrow (1/\mu_0[\rho F^{\mu\nu} - (1/4)\eta F^{\rho\sigma} F_{\rho\sigma}]) \]

(Cold) Matter-Dust
\[ T^{\mu\nu} \rightarrow (\rho_{\infty} \eta^{\mu\nu} = (\lambda) \eta^{\mu\nu}) \]

Lambda Vacuum
\[ T^{\mu\nu} \rightarrow 0 \]

Zero:Nothing Vacuum
\[ T^{\mu\nu} \rightarrow 0 \]

Faraday EM Tensor
\[ E^{\mu\nu} = \partial A^{\mu} - \partial A^{\nu} = \partial \mathbf{A} \]

SR: Minkowski Metric
\[ \delta[R] = \delta R^{\mu\nu} = \eta^{\mu\nu} + H^{\mu\nu} \]

4-Vector
\[ \mathbf{V} = (V^t, V^x, V^y, V^z) \]

4-Scalar
\[ S = \rho + p \]

Tr\[\mathbf{T}^{\mu\nu}\] = 1

4-Tensor
\[ \mathbf{F}_{\mu\nu} = \mathbf{E}^{\mu\nu} + \mathbf{B}^{\mu\nu} \]

Tr\[\mathbf{E}^{\mu\nu}\] = 3

Tr\[\mathbf{B}^{\mu\nu}\] = 3

Addition
\[ \mathbf{A} + \mathbf{B} = \mathbf{C} \]

Subtraction
\[ \mathbf{A} - \mathbf{B} = \mathbf{C} \]

Multiplication
\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{C} \]

Division
\[ \mathbf{A} / \mathbf{B} = \mathbf{C} \]

Note that the Projection Tensors and the Minkowski Metric are dimensionless.

Equation of State
\[ (\text{EoS}) \ w = (p_{\infty}/\rho_{\infty}) \]

Trace[\[\mathbf{T}^{\mu\nu}\] = 0

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson
**SRQM Study: Physical 4-Tensors**

### Projection 4-Tensors

#### Temporal "(V)ertical" Projection (2,0)-Tensor

\[ P_{\mu\nu} \rightarrow V_{\mu} = T^\mu T_{\nu} \rightarrow \text{Diag}[1,0] \]

#### Temporal "(V)ertical" Projection (1,1)-Tensor

\[ P_{\mu\nu} \rightarrow V_{\mu} = T^\mu T_{\nu} \rightarrow \text{Diag}[1,0] \]

#### Temporal "(V)ertical" Projection (0,2)-Tensor

\[ P_{\mu\nu} \rightarrow V_{\mu} = T^\mu T_{\nu} \rightarrow \text{Diag}[1,0] \]

#### Spatial "(H)orizontal" Projection (2,0)-Tensor

\[ P_{\mu\nu} \rightarrow H_{\mu} = n_{\mu} - T^\mu T_{\nu} \rightarrow \text{Diag}[0,1,0,0] \]

#### Spatial "(H)orizontal" Projection (1,1)-Tensor

\[ P_{\mu\nu} \rightarrow H_{\mu} = n_{\mu} - T^\mu T_{\nu} \rightarrow \text{Diag}[0,1,0,0] \]

#### Spatial "(H)orizontal" Projection (0,2)-Tensor

\[ P_{\mu\nu} \rightarrow H_{\mu} = n_{\mu} - T^\mu T_{\nu} \rightarrow \text{Diag}[0,1,0,0] \]

#### Perfect Fluid

The projection tensors can work on 4-Vectors to give a new 4-Vector, or on 4-Tensors to give either a 4-Scalar component or a new 4-Tensor.

#### 4-Unit Temporal Tensor

\[ T^\mu = \gamma(1,\beta) \]

#### 4-Generic A

\[ A^\mu = (a^2,a) = (a^2,a,a^2,a^2) \]

#### Vector V

\[ V^\mu = (0,a^2+0+a^2+0-a^2,0-a^2,0-a^2+0+a^2,0-a^2,0-a^2+0+a^2,0-a^2) = (a^2,0,0,0) = (a^2,0): \text{Temporal Projection} \]

\[ H^\mu = (0-a^2,0-a^2,0-a^2,0-a^2,0-a^2,0-a^2,0-a^2+0+a^2+1+a^2) = (0,a^2,a^2,a^2) = (0,a): \text{Spatial Projection} \]

#### Lorentz Scalar

\[ \eta_{\mu\nu} V^\mu V^\nu = (0,0,0,0) \]

#### Lorentz Vector

\[ V^\mu = (0,0,0,0) \]

#### Lorentz Pseudoscalar

\[ \eta_{\mu\nu} V^\mu V^\nu = (0,0,0,0) \]

#### Traceless Tensor

\[ T_{\mu

\[ V^\mu = (0,0,0,0) \]

Note that the Projection Tensors are dimensionless.
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

{c,τ,m₀,ℏ,i}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors: Related by these SR Lorentz Invariants

| 4-Position | R = (ct,r) | = <Event> | (R·R) = (ct)² |
| 4-Velocity | U = γ(c,u) | = (U·∂)R = (d/dτ)R = dR/dτ | (U·U) = (c)² |
| 4-Momentum | P = (E/c,p) | = m₀U | (P·P) = (m₀c)² |
| 4-WaveVector | K = (ω/c,k) | = P/ℏ | (K·K) = (m₀c/ℏ)² |
| 4-Gradient | ∂ = (∂/c, -∇) | = -iK | (∂·∂) = -(m₀c/ℏ)² = KG Eqn→RQM→QM |

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit { |v| << c }, giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM 4-Vector Topics Covered

SR & QM via 4-Vector Diagrams

Mostly SR Stuff

- 4-Vector Basics
- Paradigm Assumptions, Where is Quantum Gravity?
- Minkowski SpaceTime, <Events>, WorldLines, Minkowski Metric
- 4-Scalars, 4-Vectors, 4-Tensors & Tensor Invariants, Cayley-Hamilton
- SR 4-Vector Connections
- SR Lorentz Transforms, CPT Symmetry, Trace Identification, Antimatter
- Fundamental Physical Constants = Lorentz Scalar Invariants
- Projection Tensors: Temporal (V) & Spatial (H)
- Stress-Energy Tensors, Perfect Fluids, Special Cases (Dust, Radiation, etc)
- Invariant Intervals, Measurement, Causality, Relativity
- SpaceTime Kinematics & Dynamics, ProperTime Derivative
- Einstein’s E = mc^2 = γm_o c^2, Rest Mass, Rest Energy, Invariants
- SpaceTime Orthogonality: Time-like velocity, Space-like acceleration
- Relativity of Simultaneity, Time Dilation, Length Contraction
- SR Motion * Lorentz Scalar = Interesting Physical 4-Vector
- SR Conservation Laws & Local Continuity Equations, Symmetries
- Relativistic Doppler Effect, Relativistic Aberration Effect
- SR Wave-Particle Relation, Invariant d’Alembertian Wave Eqn, SR Waves
- SpaceTime is 4D: \( \partial \cdot R = \partial \mu R^\mu = 4, \Lambda^\mu \Lambda_{\mu} = 4, \text{Tr}[\eta^{\mu \nu}] = 4, A = A^\mu = (a^0, a^1, a^2, a^3) \)
- Minimal Coupling = Interaction with a (Vector)Potential
- Conservation of 4-TotalMomentum
- SR Hamiltonian: Lagrangian Connection
- Lagrangian, Lagrangian Density
- Hamilton-Jacobi Equation (differential), Relativistic Action (integral)
- Euler-Lagrange Equations
- Relativistic Equations of Motion, Lorentz Force Equation
- \( c^2 \) Invariant Relations, The Speed-of-Light (c)
- Thermodynamic 4-Vectors, Unruh-Hawking Radiation, Particle Distributions

Mostly QM & SRQM Stuff

- Relativistic Quantum Wave Equations
- Klein-Gordon Equation/Relation
- RoadMap from SR to QM: SR→QM, SRQM 4-Vector Connections
- QM Schrödinger Relation
- QM Axioms? - No, (QM Principles derived from SR) = SRQM
- Relativistic Wave Equations: based on mass & spin & velocity
- Klein-Gordon, Dirac, Proca, Maxwell, Weyl, Pauli, Schrödinger, etc.
- Classical Limits |v|<<c
- Photon Polarization
- Linear PDE’s →
  - (Principle of Superposition, Hilbert Space, <Bra|Ket> Notation)
  - Canonical QM Commutation Relations – derived from SR
  - Heisenberg Uncertainty Principle (due to non-zero commutation)
  - Pauli Exclusion Principle (Fermion), Bose Aggregation Principle (Boson)
- Complex 4-Vectors
- CPT Theorem, Lorentz Invariance, Poincaré Invariance, Isometry
- Hermetian Generators, Unitarity, Anti-Unitarity
- Quantum → Classical Correspondence Principle, similar to SR → Classical
- Quantum Probability
- The Compton Effect = Photon:Electron Interaction (neglecting Spin Effects)
- Photon Diffraction, Crystal-Electron Diffraction, The Kapitza-Dirac Effect
- The h Relation, Einstein-de Broglie, Planck:Dirac
- The Aharonov-Bohm Effect, The Josephson Junction Effect
- Noether’s Theorem, Continuous Symmetries, Conservation Laws
- Dimensionless Quantities
- Quantum Relativity: GR is \( \textbf{NOT} \) wrong, *Never bet against Einstein* :)
- Quantum Mechanics is Derivable from Special Relativity, SR→QM, SRQM

SRQM = The [SR→QM] Interpretation of Quantum Mechanics

= Special Relativity → Quantum Mechanics

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
### There are some paradigm assumptions that need to be cleared up:

Relativistic Physics **IS NOT** the generalization of Classical Physics.

Classical Physics **IS** the low-velocity limiting-case approximation of Relativistic Physics $\{ |v| \ll c \}$.

This includes (Newtonian) Classical Mechanics and Classical QM, (meaning the non-relativistic Schrödinger QM Equation).

However, Classical EM doesn't include intrinsic spin, even though spin is a result of SR Poincaré Invariance, not QM.

So far, in all of my research, if there was a way to get a result classically, then there was usually a much simpler way to get the result using 4-Vectors and SRQM relativistic thinking.

Likewise, a lot of QM results make much more sense when approached from SRQM.

<table>
<thead>
<tr>
<th>Equations</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SR Wave Eqn</strong>: $K_T = -\partial[\Phi, \text{phase}] = P_T/h \rightarrow { \omega_T = -\partial[\Phi] : k_T = \nabla[\Phi] }$</td>
<td>$P = m_0U \rightarrow { E = mc^2 = \gamma m_0 c^2 = \gamma E_0 : p = mu = \gamma m_0 u }$</td>
</tr>
<tr>
<td><strong>Hamilton-Jacobi Eqn</strong>: $P_T = -\partial[S, \text{action}] = hK_T \rightarrow { E_T = -\partial[S] : p_T = \nabla[S] }$</td>
<td>$H = \gamma(P_TU)$ { Relativistic } $\rightarrow (T + V) = (E\text{kinetic} + E\text{potential}) { \text{Classical-limit only, }</td>
</tr>
<tr>
<td><strong>Action Equation</strong>: $\Delta S_{\text{action}} = -\int_{\text{path}} P_T \cdot dX = \int_{\text{path}} (P_T \cdot U)dt = \Delta S_{\text{action}}/h$</td>
<td>$L = -\gamma(P_TU)/\gamma { \text{Relativistic} } = (T - V) = (E\text{kinetic} - E\text{potential}) { \text{Classical-limit only, }</td>
</tr>
<tr>
<td><strong>Euler-Lagrange Equation</strong>: $(U = (d/d\tau)R) \rightarrow (\partial R = (d/d\tau)\partial_0)$</td>
<td>$SR \rightarrow QM$</td>
</tr>
<tr>
<td><strong>Hamilton’s Equations</strong>: $(d/d\tau)[X] = (\partial/\partial P_T)[H_o] &amp; (d/d\tau)[P_T] = (\partial/\partial X)[H_o]$</td>
<td><strong>4-Vector SRQM Interpretation of QM</strong></td>
</tr>
</tbody>
</table>
| **d’Alembertian Wave Equation**: $\partial^2 \Phi = (\partial^2/c^2) - \nabla \cdot \nabla, \text{ with solutions } \sim \sum_n e^{i(Kn \cdot X)}$ | **SciRealm.org**

4-Vector formulations are all extremely easy to derive in SRQM and are all relativistically covariant.

**SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)**
There are some paradigm assumptions that need to be cleared up:

SR 4D Physical 4-Vectors *ARE NOT* generalizations of Classical/Quantum 3D Physical 3-vectors. While a “mathematical” Euclidean \((n+1)D\)-vector is the generalization of a Euclidean \((n)D\)-vector, the “physical” analogy ends there.

Minkowskian SR 4-Vectors *ARE* the primitive elements of 4D Minkowski SR SpaceTime. Classical/Quantum Physical 3-vectors are just the spatial components of SR Physical 4-Vectors. There is also a fundamentally-related Classical/Quantum Physical scalar related to each 3-vector, which is just the temporal component scalar of a given SR Physical 4-Vector.

ex. 4-Position \( R = (r^\mu) = (r^0, r) = (ct, r) \rightarrow (ct, x, y, z) \) : 4-Momentum \( P = (p^\mu) = (p^0, p) = (E/c, p) \rightarrow (E/c, p^x, p^y, p^z) \)

These Classical/Quantum \( \{\text{scalar}\} + \{\text{3-vector}\} \) are the dual \( \{\text{temporal}\} + \{\text{spatial}\} \) components of a single SR 4-Vector = \((\text{temporal scalar } \times c^{\pm1}, \text{ spatial 3-vector})\) with SR lightspeed factor \((c^{\pm1})\) to give correct overall dimensional units.

While different observers may see different "values" of the Classical/Quantum components \((v^0, v^1, v^2, v^3)\) from their point-of-view in SpaceTime, each will see the same actual SR 4-Vector \( V \) and its magnitude \(|V|\) at a given \(<\text{Event}>\) in SpaceTime.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
There are some paradigm assumptions that need to be cleared up:

We will **NOT** be employing the commonly-(mis)used Newtonian classical limits \( \{ c \to \infty \} \) and \( \{ \hbar \to 0 \} \). Neither of these is a valid physical assumption, for the following reasons:

1. Both \( c \) and \( \hbar \) are unchanging Physical Constants and Lorentz Invariants. Taking a limit where these change is non-physical. They are CONSTANT.

Many, many experiments verify that these constants have not changed over the lifetime of the universe. This is one reason for the 2019 Redefinition of SI Base Units on Fundamental Constants \( (c, \hbar, e, k_B, N_A, K_{CD}, \Delta \nu_{Cs}) \).

2. Let \( E = pc \). If \( c \to \infty \), then \( E \to \infty \). Then Classical EM light rays/waves have infinite energy.
   Let \( E = \hbar \omega \). If \( \hbar \to 0 \), then \( E \to 0 \). Then Classical EM light rays/waves have zero energy.

   Obviously neither of these is true in the Newtonian limit.

   In Classical EM and Classical Mechanics, \( (c) \) remains a large but finite constant.
   Likewise, \( (\hbar) \) remains very small but never becomes zero.

   The correct way to take the limits is via:
   The low-velocity non-relativistic limit \( \{ |v| << c \} \), which is a physically-occurring situation.
   The Hamilton-Jacobi non-quantum limit \( \{ \hbar | \nabla \cdot p | << (p \cdot p) \} \), which is a physically-occurring situation.
There are some paradigm assumptions that need to be cleared up:

We will *NOT* be implementing the common {→ lazy and extremely misguided} convention of setting physical constants to the value of (dimensionless) unity, often called “Natural Units”, to hide them from equations; nor using mass (m) instead of (m₀) as the RestMass. Likewise for other components vs Lorentz Scalars with naughts, like energy (E) vs (E₀) as the RestEnergy.

One sees this very often in the literature. The usual excuse cited is “For the sake of brevity”.

Well, the “sake of brevity” forsakes “clarity”

The *ONLY* situation in which setting constants to unity is practical or advisable is in numerical simulation.

When teaching physics, or trying to understand physics, it helps when equations are dimensionally correct. In other words, the technique of dimensional analysis is a powerful tool that should not be disdained.

i.e. Brevity aids speed of computation, Clarity aids understanding.

The situation of using “naught = o” for rest-values, such as (m₀) for RestMass and (E₀) for RestEnergy:

Is intrinsic to SR, is a very good idea, absolutely adds clarity, identifies Lorentz Scalar Invariants, and will be explained in more detail later.

Essentially, the relativistic gamma (γ) pairs with a (Lorentz scalar:rest value o) to make a relativistic component: m = γm₀, E = γE₀

Note the multiple equivalent ways that one can write 4-Vectors using these rules:

4-Momentum \( P = P^\mu = (p^0, p^i) = (m_0, \gamma(c,u)) = m_0\gamma(c,u) = \gamma m_0 (c,u) = m(c,u) = (mc, mu) = (mc, p) \)

\( = (E/c, p) = (Ec/c^2)U = (Ec/c^2)\gamma(c,u) = \gamma(Ec/c^2)(c,u) = (Ec^2)(c,u) = (Ec, EcLu/c^2) = (E/c, p) \)

It is damn hard enough just to get the minus-signs right in GR/SR, as there are different metric-conventions available.

This notation makes clear what is relativistic vs. invariant, temporal vs. spatial

BTW, I prefer the “Particle Physics” Metric-Convention (+,-,-,-). {Makes rest values positive, fewer minus signs to deal with}

Show the physical constants and naughts in the work. They deserve the respect and you will benefit.

You can always set constants to unity later, when you are doing your numerical simulations.
There are some paradigm assumptions that need to be cleared up:

Many physics books say that the Electric field $E$ and the Magnetic field $B$ are the “real” physical objects, and that the EM scalar-potential $\phi$ and the EM 3-vector-potential $A$ are just “calculational/mathematical” artifacts.

Neither of these statements is relativistically correct.

All of these physical EM properties: \{E, B, $\phi$, A\} are actually just the components of SR tensors, and as such, their magnitudes will vary in different observers’ reference-frames.

The truly SR invariant physical objects are:
The 4-Gradient $\partial$, the 4-VectorPotential $A$, and their combination via exterior (wedge=$\wedge$) product into the Faraday EM Tensor $F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha} = \partial \wedge A$

Given this SR knowledge, we demote the physical property symbols, (the tensor components) to their lower-case equivalents \{e, b, $\phi$, a\}.

Temporal-spatial components of 4-Tensor $F^{\alpha\beta}$: electric 3-vector field $e$.

Spatial-spatial components of 4-Tensor $F^{\alpha\beta}$: magnetic 3-vector field $b$.

Temporal component of 4-Vector $A$: EM scalar-potential $\phi$.

Spatial components of 4-Vector $A$: EM 3-vector-potential $a$.

Note that the speed-of-light ($c$) plays a prominent role in the component definitions. Also, QM requires the 4-VectorPotential $A$ as explanation of the Aharonov-Bohm Effect. Again, all the higher-index-count SR tensors can be built from fundamental 4-Vectors.
There are some paradigm assumptions that need to be cleared up:

A number of QM philosophies make the assertion that particle "properties" do not "exist" until measured. The assertion is based on the Heisenberg Uncertainty Principle, and more specifically on quantum non-zero commutation, in which a measurement on one property of a particle alters a non-commuting property of the same particle.

That is an incorrect analysis. Properties define particles: what they do, how they interact with other particles. Particles and their properties "exist" independently of human intervention or observation. The correct way to analyze this is to understand what a measurement is: the arrangement of some number of fundamental particles in a particular manner as to allow an observer to get information about one or more of the subject particle's properties. Typically this involves "counting" spacetime events and using SR invariant intervals as a basis of measurement.

Some properties are indeed non-commuting. This simply means that it is not possible to arrange a set of particles in such a way as to measure (ie. obtain "complete" information about) both of the subject particle's non-commuting properties at the same spacetime event. The measurement arrangement events can be done at best sequentially, and the temporal order of these events makes a difference in observed results. EPR-Bell, however, allows one to "infer" properties on a subject particle by making a measurement on a different (space-like separated but entangled) particle. This does *not* imply FTL signalling. It just updates local partial-information about particles that interacted/entangled then separated.

So, a better way to think about it is this: The "measurement" of a property does not "exist" until a physical setup event is arranged. Non-commuting properties require different physical arrangements in order for the properties to be measured, and the temporally-first measurement alters the particle's properties in a minimum sort of way, which affects the latter measurement. All observers agree on the order of temporally-separated spacetime events. However, individual observers may have different sets of partial information about the same particle(s).

This makes way more sense than the subjective belief that a particle's property doesn't exist until it is observed, which is about as unscientific and laughable a statement as I can imagine.

*Relativity is the system of measurement that QM has been looking for*
There are some paradigm assumptions that need to be cleared up:

Correct Notation is critical for understanding physics

Unfortunately, there are a number of “sloppy” notations in relativistic and quantum physics.

Incorrect: Using \( T^i \) as a Trace of tensor \( T^{ij} \), or \( T^{\mu\mu} \) as a Trace of tensor \( T^{\mu\nu} \)

- \( T^i \) is just the diagonal part of 3-tensor \( T^{ij} \), the components: \( T^{ij} = \text{Diag}[T^{11}, T^{22}, T^{33}] \)
- \( T_i \) is the Trace of 3-tensor \( T^{ij} \): \( T_i = T_1^{1} + T_2^{2} + T_3^{3} = 3\text{-trace}[T^{ij}] = T^{11} + T^{22} + T^{33} \) in the Euclidean Metric \( E^{ij} = \delta^{ij} \)

Incorrect: Hiding factors of \( c \) in relativistic equations, ex. \( E = m \)

The use of “natural units” leads to a lot of ambiguity, and one loses the ability to do dimensional analysis.

Wrong: \( E = m \): Energy is *not* identical to mass.

Correct: \( E = mc^2 \): Energy is related to mass via the speed-of-light, ie. mass is a type of concentrated energy.

Incorrect: Using \( m \) instead of \( m_0 \) for rest mass, Using \( E \) instead of \( E_0 \) for rest energy

Correct: \( E = mc^2 = \gamma m_0 c^2 = \gamma E_0 \)

\( E \) & \( m \) are relativistic internal components of 4-Momentum \( P = (mc, p) = (E/c, p) \) which vary in different reference-frames.

\( E_0 \) & \( m_0 \) are Lorentz Scalar Invariants, the rest values, which are the same, even in different reference-frames: \( P = m_0 U = (E_0/c^2)U \)
There are some paradigm assumptions that need to be cleared up:

Incorrect: Using the same symbol for a tensor-index and a component
The biggest offender in many books for this one is quantum commutation. Unclear because (i) means two different things in one equation.
Better: (i = √[-1]) is the imaginary unit; {j,k} are tensor-indicies
In general, any equation which uses complex-number math should reserve (i) for the imaginary, not as a tensor-index.

Incorrect: Using the 4-Gradient notation incorrectly
The 4-Gradient is a 4-Vector, a (1,0)-Tensor, which uses an upper index, and has a negative spatial component in SR.
The Gradient One-Form, its natural tensor form, a (0,1)-Tensor, uses a lower index in SR.
4-Gradient: ∂=∂μ=(∂t/c,-∇)
Gradient One-Form: ∂μ=(∂t/c,∇)

Incorrect: Mixing styles in 4-Vector naming conventions
There is pretty much universal agreement on the 4-Momentum $P=Pμ=(E/c,p)=(mc,p)$
Do not in the same document use 4-Potential $A=(φ,A)$: This is wrong on many levels.
The correct form is 4-VectorPotential $A=Aμ=(φ/c,a)$, with (φ) as the scalar-potential & (a) as the 3-vector-potential
For all 4-Vectors, one should use a consistent notation:
The Upper-Case SpaceTime 4-Vector Names match the lower-case spatial 3-vector names
There is a (c) factor in the temporal component to give overall matching dimensional units for the entire 4-Vector
4-Vector components are typically lower-case with a few historical exceptions, mainly energy E, energy-density e or $ρ$
Old Paradigm: QM (as I was taught)

SR and QM as separate theories

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
- $c, G$ = physical constants

GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle
- Hermitian Generators
- Correspondence Principle to CM
- Born Probability Interpretation
- $\h, \hbar$ = physical constants

SR limiting-case: $|v| \ll c$

QM limiting-case: # particles $N \gg 1$

This was the QM paradigm that I was taught while in Grad School; everyone trying for Quantum Gravity
Old Paradigm: QM (years later)

SR and QM still as separate theories

QM limiting-case better defined, still no QG

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
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- Pauli Exclusion Principle
- Correspondence Principle to CM
- Born Probability Interpretation
- $\hbar, \hbar =$ physical constants

GR

SR

QM

RQM

QFT

SR limiting-case:
$|v| < c$

GR limiting-case:
$g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

QM-limiting case:
$\hbar |\nabla \cdot p| < (p \cdot p)$
or $\psi \rightarrow \text{Re}(|\psi|)$

It is known that QM + SR “join nicely” together to form RQM, but problems with RQM + GR...

Yet another “would be” fortuitous merging??

50+ years searching for QG with no success...

A fortuitous merging?

A Tensor Study of Physical 4-Vectors

A fortuitous merging??

SRQM Interpretation of QM

SciRealm.org
John B. Wilson
Physical Theories as Venn Diagram
Which regions are real?

**GR:** General Relativity

**SR:** Special Relativity

**QM:** Quantum Mechanics

**CM:** Classical Mechanics

**RQBM:** Relativistic Quantum Mechanics

---

**Reality**

- **Reality**
- **Quantum Gravity?**

---

**GR limiting-case:** $g^{\mu\nu} \to \eta^{\mu\nu}$ Minkowski "Flat" SpaceTime = (Curvature $\sim 0$)

**SR limiting-case:** $|v| \ll c$

**QM limiting-case:** $\hbar |\nabla \cdot p| \ll (p \cdot p)$

---

**Many QM physicists believe that the regions outside of QM don't exist...**

---

**SRQM Interpretation would say that the regions outside of GR probably don't exist...**

---

**Hence the attempt to Quantize Gravity:**
- Unsuccessful for 50+ years...
- A new approach is needed: Try SRQM.

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**SRQM Interpretation of QM**

- Many-Worlds Interpretations
- Non-local interactions
- Instantaneous QM entangled connections
- Instantaneous Physical Wavefunction Collapse
- Spacetime Dimensions $>4$
- Hidden:Alternate Dimensions
- Super-Symmetry
- String Theory
- Alternate Gravity Theories
- etc.

---

**Quantum Mysticism...**
- Basically lots of stuff for which there is little to no empirical evidence...
Instead of taking the Physical Theories as set, examine Physical Reality and then apply various limiting-conditions.

What do we then call the various regions?

As we move inwards from any region on the diagram, we are adding more stringent conditions which give physical limiting-cases of “larger” theories.

If one is in Classical GR, one can get Classical SR by moving toward the Minkowski SpaceTime limit.

If one is in RQM, one can get Classical SR by moving toward the Hamilton-Jacobi non-QM limit, or to standard QM by moving toward the SR low-velocity limit.

Looking at it this way, I can define SRQM to be equivalent to Minkowski SpaceTime, which contains ROM, and leads to Classical SR, or QM, or CM by taking additional limits.

My assertion:
There is no “Quantized Gravity”
Actual GR contains SRQM and Classical GR.
Perhaps “Gravitizing QM”...

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties. Likewise, Relativistic Quantum Mechanics (RQM) and Quantum Mechanics (QM) have passed all tests within their realms of validity: 
{generally micro-scale systems, but a few special macro-scale systems ex. Bose-Einstein condensates, superfluids, etc.}

To date, however, there is no experimental indication that quantum effects “alter” the fundamentals of either SR or GR. Likewise, there are no known violations, QM or otherwise, of Local Lorentz Invariance (LLI) nor of Local Position/Poincaré Invariance (LPI). In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR, whereas SR/GR modify the results of QM. All tested quantum-level particles, atoms, isotopes, super-positions, spin-states, etc. obey GR's Universality of FreeFall & Equivalence Principle and SR's \(E = mc^2\) and speed-of-light (c) communication limit. Quantum-level atomic clocks are used to measure gravitational red:blue-shift effects, i.e. GR gravitational frequency-shift (time-dilation) alters atomic=quantum-level timing.

Some might argue that QM modifies the results of SR, such as via non-commuting measurements. However, that is an alteration of CM expectations, not SR expectations. In fact, there is a basic non-zero commutation relation fully within SR: \([\varpi^\mu, X^\nu] = \eta^{\mu\nu}\) which will be derived from purely SR Principles in this treatise.

On the other hand, GR Gravity *does* induce changes in quantum interference patterns and hence modifies QM: See the COW gravity-induced neutron QM interference experiments and the LIGO gravitational-wave detections via QM interferometry. Likewise, SR induces fine-structure splitting of spectral lines of atoms, “quantum” spin, spin magnetic moments, spin-statistics (fermions & bosons), antimatter, QED, Lamb shift, etc. - essentially requiring QM to be RQM to be valid.

Some QM scientists say that quantum entanglement is "non-local", but you still can't send any real messages/signals/information/particles faster than SR's speed-of-light (c). The only "non-local" aspect is the alteration of probabilities based on knowledge gained via measurement. A local measurement can alter the “partial information” known about a distant (entangled) system. There is no FTL communication to or alteration of the distant particle. Getting a Stern-Gerlach “up” here doesn’t cause the distant entangled particle to suddenly start moving “down”. One only knows “now” that it “will” go down if the distant experimenter actually performs the measurement.

QM respects the principles of SR/GR, whereas SR/GR modify the results of QM.
Special Relativity → Quantum Mechanics

**Background: GR Principles**

### Principles/Axioms and Mathematical Consequences of GR:

**Equivalence Principle:** Inertial Motion = Geodesic Motion, Universality/Equivalency of Free-Fall, Mass\(_{\text{inertial}}\) = Mass\(_{\text{gravitational}}\)

**Relativity Principle:** SpaceTime M has a Lorentzian/pseudo-Riemannian Metric \(g^{\mu\nu}\), SR:Minkowski Space rules apply locally (\(\eta^{\mu\nu}\))

**General Covariance Principle:** Tensors describe Physics, Laws of Physics are independent of Coordinate System

**Invariance Principle:** Invariant Interval Measure comes from Tensor Invariance Properties, 4D SpaceTime from Invariant Trace[\(g^{\mu\nu}\)] = 4

**Causality Principle:** Minkowski Diagram/Light-Cone give {Time-Like, Light-Light(null), Space-Like} Measures and Causality Conditions

Einstein: Riemann’s Ideas about Matter & Curvature:
\(\text{Riemann}(g)\) has 20 independent components → too many
\(\text{Ricci}(g)\) has 10 independent components = enough to describe gravitational field

\(\{c,G\}\) are Fundamental Physical Constants

---

**To date, there are no violations of any of these GR Principles.**

It is vitally important to keep the mathematics fixed to known physics. There are too many instances of trying to apply theoretical math to the physics. It doesn’t work that way. Nature is the arbiter of what math works with physics. Tensor mathematics applies well to SR and GR, which have been extremely well-tested now in a variety of physical situations.

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All known experiments to date comply with all of these Principles, including QM
Old Paradigm: QM (for comparison)
SR and QM still as separate theories
QM limiting-case better defined, still no QG

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric \( g^{\mu\nu} \)
- \( c, G = \text{physical constants} \)

SR limiting-case:
- \( |v| << c \)

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle
- Correspondence Principle to CM
- Born Probability Interpretation
- \( h, \hbar = \text{physical constants} \)

GR limiting-case:
- \( g^{\mu\nu} \rightarrow \eta^{\mu\nu} \)
- Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

It is known that QM + SR “join nicely” together to form RQM, but problems with RQM + GR...

Yet another “would be” fortuitous merging??
50+ years searching for QG with no success...
A fortuitous merging?

SR limiting-case:
- \( |v| << c \)

QM limiting-case:
- \( \hbar |\nabla \cdot p| << (p \cdot p) \) or \( \psi \rightarrow \text{Re} [\psi] \)

A Tensor Study of Physical 4-Vectors

4-Vector SRQM Interpretation of QM
SciRealm.org
John B. Wilson
New Paradigm: SRQM or [SR→QM]  
QM derived from SR + a few empirical facts

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR
New Paradigm: SRQM w/ EM

QM, EM, CM derived from SR + a few empirical facts

Simple GR Axioms:
- Principle of Equivalence:
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
  \[ c,G = \text{physical constants} \]

Derived RQM **Principles**:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle
- Hermitian Generators
  \[ h,h = \text{physical constants} \]

Derived QM **Principles**:
- Correspondence Principle to CM
- Born Probability Interpretation

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR
Classical SR w/ EM Paradigm (for comparison)

CM & EM derived from SR + a few empirical facts

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g_{\mu \nu}$
- $c, G =$ physical constants

GR limiting-case: $g_{\mu \nu} \rightarrow \eta_{\mu \nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

Background Inherent Assumption
- QM limiting-case: $\frac{\hbar}{|\nabla \cdot p|} \ll (p \cdot p)$
- Hamilton-Jacobi non-quantum Change by a few quanta has negligible effect on overall state

SR limiting-case: $|v| \ll c$

SR → EM, CM paradigm has been working successfully for decades
New Paradigm: SRQM View as Venn Diagram

GR
General Relativity

SRQM
Special Relativity $\rightarrow$ Relativistic QM
GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$ Minkowski “Flat” Space-Time = (Curvature $\sim 0$)

QM
Non-relativistic Quantum Mechanics
SR limiting-case: $|v| << c$

CM
Classical Mechanics
QM limiting-case: $\hbar|\nabla \cdot p| << (p \cdot p)$ or $\psi \rightarrow \text{Re}[\psi]$
Change by a few quanta has negligible effect on overall state

The SRQM view: Each level (range of validity) is a subset of the larger level.
New Paradigm:  
SRQM View w/ EM as Venn Diagram

The SRQM view: Each level (range of validity) is a subset of the larger level
Newton's laws of classical physics are greatly simplified by the use of physical 3-vector notation, which converts 3 separate space components, which may be different in various coordinate systems, into a single invariant object, a vector: The basis values of these components can differ, yet still refer to the same overall 3-vector object.

\[
\begin{align*}
&\text{Cartesian/Rectangular 3D basis} \\
&\quad (a_x, a_y, a_z) \\
&\text{Polar/Cylindrical 3D basis} \\
&\quad (a'_r, a'_\theta, a_z) \\
&\text{Spherical 3D basis} \\
&\quad (a'_r, a'_\theta, a'_\phi)
\end{align*}
\]

The scalar products of either type are basis-independent

\[
\begin{align*}
&\text{Cartesian/Rectangular 4D basis} \\
&\quad (a_t, a_x, a_y, a_z) \\
&\text{Polar/Cylindrical 4D basis} \\
&\quad (a_t, a'_r, a'_\theta, a_z) \\
&\text{Spherical 4D basis} \\
&\quad (a_t, a'_r, a'_\theta, a'_\phi)
\end{align*}
\]

SR is able to expand the concept of mathematical vectors into the Physical 4-Vector, which combines both (time) and (space) components into a single TimeSpace object: These 4-Vectors are elements of Minkowski 4D SR SpaceTime. Typically there is a speed-of-light factor \(c\) in the temporal component to make the dimensional units match. eg. \(R = (ct, r)\): overall dimensional units of [length] = SI Unit [m] This also allows the 4-Vector name to match up with the 3-vector name.

In this presentation: I use the \((+, -, +, +)\) metric signature, giving \(A \cdot A = \sum_{\mu \nu} A^\mu \eta_{\mu \nu} A^\nu = (a_0^2 - \mathbf{a} \cdot \mathbf{a}) = (\mathbf{a}_0)^2\) 4-Vectors will use Upper-Case Letters, ex. \(A\); 3-vectors will use lower-case letters, ex. \(a\) Vectors of both types will be in bold font; components and scalars in normal font and usually lower-case. 4-Vector name will match 3-vector name. Tensor form will usually be normal font with a tensor index, ex. \(A^\mu\) or \(A_{\mu}^\nu\), with Greek TimeSpace index \((0,1,3)\); Latin SpaceOnly index \((1,3)\)
SR 4-Vectors & Lorentz Scalars

Frame-Invariant Equations

SRQM Diagramming Method

4-Vectors are type (1,0)-Tensors, Lorentz 4-Scalars are type (0,0)-Tensors, 4-CoVectors are type (0,1)-Tensors, (m,n)-Tensors have (m) upper-indices and (n) lower-indices. \( V^\mu, S, C^\mu, T^{\alpha\beta\gamma...\mu}...\mu\nu...\{m\ \text{indices}\}..\{n\ \text{indices}\} \)

Any equation which employs only Tensors, such as those with only 4-Vectors and Lorentz 4-Scalars, (ex. \( P = m_0 U \)) is automatically Frame-Invariant, or coordinate-frame-independent. One’s frame-of-reference plays no role in the form of the overall equations. This is also known as being “Manifestly-Invariant”. This is exactly what Einstein meant by his postulate: “The laws of physics should have the same form for all inertial observers”. Use of the RestFrame-naught (\(_o\)) helps show this.

Unlike the 3D case, the 4D magnitude can be (+/-) The components \( (a^0,a^1,a^2,a^3) \) of the 4-Vector \( A \) can vary depending on the observer and their choice of coordinate system, but the 4-Vector \( A \) itself is invariant. Equations using only 4-Tensors, 4-Vectors, and Lorentz 4-Scalars are true for all inertial observers. The SRQM Diagramming Method makes this easy to see in a visual format, and will be used throughout this treatise.

The following examples are frame-invariant equations:

**Equation Form**

\[
\begin{align*}
U \cdot U &= (c)^2 \\
U &= \gamma(c,u) \\
P &= (mc,p) = (E/c,p) = m_0 U = (E_0/c^2)U \\
K &= (\omega/c,k) = (\omega_0/c^2)U \\
P \cdot U &= E_0
\end{align*}
\]

**SRQM Diagram Form**

The SRQM Diagram Form has all of the info of the Equation Form, but shows overall relationships and symmetries among the 4-Vectors much more clearly.

Blue: Temporal components

Red: Spatial components

Purple: Mixed TimeSpace components

\[
\begin{align*}
P \cdot U &= E_0 \\
4-Velocity: U &= \gamma(c,u) \\
U \cdot U &= c^2 \\
4-Momentum: P &= (mc,p) = (E/c,p) \\
K &= (\omega/c,k) = (\omega_0/c^2)U \\
4-WaveVector: K &= (\omega/c,k) = (\omega_0/c^2)U
\end{align*}
\]
SR 4-Vectors are primitive elements of Minkowski SpaceTime (4D)

We want to be clear, however, that SR 4-Vectors are NOT generalizations of Classical or Quantum 3-vectors.

SR 4-Vectors are the primitive elements of Minkowski SpaceTime (4D) which incorporate both:

- a \{temporal scalar element\}
- a \{spatial 3-vector element\}

4-Vector \( \mathbf{A} = (a^0, a^1, a^2, a^3) \) = \((a^0, a) \rightarrow (a^t, a_x, a_y, a_z)\) with scalar \((a^t)\) & 3-vector \(a \rightarrow (a^x, a^y, a^z)\)

It is the Classical or Quantum 3-vector \(a\) which is a limiting-case approximation of the spatial part of SR 4-Vector \(\mathbf{A}\) for \(|v| << c\).

i.e. The Energy (E) and 3-momentum \((p)\) as “separate” entities occurs only in the low-velocity limit \(|v| << c\) of the Lorentz Boost Transform. They are actually part of a single 4D entity: the 4-Momentum \(\mathbf{P} = (E/c, p)\); with the components: temporal \((E)\), spatial \((p)\), dependent on a frame-of-reference, while the overall 4-Vector \(\mathbf{P}\) is invariant. Likewise with \((t)\) and \((r)\) in the 4-Position \(\mathbf{R}\).

SR is Minkowskian; obeys Lorentz Invariance.

CM is Euclidean; obeys Galilean Invariance.
Relations among 4-Vectors and Lorentz 4-Scalars are Manifestly Invariant, meaning that they are true in all inertial reference frames.

Consider a particle at a SpaceTime \(<\text{Event}>\) that has properties described by 4-Vectors \(A\) and \(B\):

One possible relationship is that the two 4-Vectors are related by a Lorentz 4-Scalar (\(S\)): \(B = (S)A\).

How can one determine this? Answer: Make an experiment that empirically measures the tensor invariant \(\frac{B\cdot A}{A\cdot A}\).

If \(B = (S)A\)

\[B\cdot A = (S)\frac{A\cdot A}{A\cdot A}\]

Note that this basically a vector projection.

Run the experiment many times. If you always get the same result for \((S)\), then it is likely that the relationship is true, and invariant.

Example: Measure \((S_P) = \frac{P\cdot U}{U\cdot U}\) for a given particle type.

Repeated measurement always give \((S_P) = (\frac{\omega_o}{c^2})\)

This makes sense because we know \(P\cdot U = \gamma(E - p\cdot u) = E_o\) and \(U\cdot U = c^2\)

Thus, 4-Momentum \(P = (\frac{E_o}{c^2})U = (m_o)U = (m_o)c^2\) 4-Velocity \(U\)

Example: Measure \((S_K) = \frac{K\cdot U}{U\cdot U}\) for a given particle type.

Repeated measurement always give \((S_K) = (\frac{\omega_o}{c^2})\)

This makes sense because we know \(K\cdot U = \gamma(\omega - k\cdot u) = \omega_o\) and \(U\cdot U = c^2\)

Thus, 4-WaveVector \(K = (\frac{\omega_o}{c^2})U = (\frac{\omega_o}{c^2})^2\) 4-Velocity \(U\)

Since \(P\) and \(K\) are both related to \(U\), this would also mean that the 4-Momentum \(P\) is related to the 4-WaveVector \(K\) in a particular manner for each given particle type... a hint for later...
Some SR Mathematical Tools

Definitions and Approximations

\(\beta = v/c\): dimensionless Velocity Beta Factor

\(\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta\cdot\beta}\): dimensionless Lorentz Relativistic Gamma Factor

\((1+x)^\gamma \sim (1 + nx + O[x^2])\) for \(|x| << 1\) Approximation used for SR \(\rightarrow\) Classical limiting-cases

Lorentz Transformation \(\Lambda^\nu_\mu = \frac{\partial x^\nu}{\partial \xi^\mu} = \delta_{\mu}^\nu, \Lambda^\nu_\mu\): a relativistic frame-shift, such as a rotation or velocity boost

It transforms a 4-Vector in the following way: \(X^\mu = \Lambda^\nu_\mu X^\nu\): with Einstein summation over the paired indicies

A typical Lorentz Boost Transformation \(\Lambda^\nu_\nu \rightarrow B^\nu_\nu\) for a linear-velocity frame-shift \((x,t)\)-Boost in the \(x\)-direction:

\(\beta = v/c\): dimensionless Velocity Beta Factor

\(\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta\cdot\beta}\): dimensionless Lorentz Relativistic Gamma Factor

\((1+x)^\gamma \sim (1 + nx + O[x^2])\) for \(|x| << 1\) Approximation used for SR \(\rightarrow\) Classical limiting-cases

Lorentz Transformation \(\Lambda^\nu_\mu = \frac{\partial x^\nu}{\partial \xi^\mu} = \delta_{\mu}^\nu, \Lambda^\nu_\mu\): a relativistic frame-shift, such as a rotation or velocity boost

It transforms a 4-Vector in the following way: \(X^\mu = \Lambda^\nu_\mu X^\nu\): with Einstein summation over the paired indicies

A typical Lorentz Boost Transformation \(\Lambda^\nu_\nu \rightarrow B^\nu_\nu\) for a linear-velocity frame-shift \((x,t)\)-Boost in the \(x\)-direction:

\(\Lambda = \Lambda^\nu_\mu \rightarrow \text{Boosted Lorentz Transform}\)

\(\Lambda^\nu_\nu = (a^\nu, a^\nu, a^\nu, a^\nu)\)

Original \(A^\nu = (a\nu, a\nu, a\nu, a\nu)\)

Boosted \(B^\nu = (a\nu, a\nu, a\nu, a\nu)\)

\(\Lambda^\nu_\mu = (a\nu, a\nu, a\nu, a\nu)\)

\(\text{Using the Einstein summation convention where upper:lower paired-indices are summed over}\)

\(\delta(X) = \delta^{\nu\mu}(X) = (\delta_{\nu'}c,-\nabla)(ct.x) = \text{Diag}(\delta/c/\text{ct}.,-\nabla[X]) = \text{Diag}[1,-I_{ab}] = \text{Diag}[1,-1,-1,-1] = \eta^{\nu\mu}\) Minkowski "Flat" SpaceTime Metric

\(\eta = \text{Diag}[1,-1,-1,-1] = \gamma\text{Minkowski Metric}\)

\(\text{SR: Lorentz Transform}\)

\(\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} = V^{\nu\mu} + H^{\nu\mu}\)

\(\text{SR: Minkowski Metric}\)

\(\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} = V^{\nu\mu} + H^{\nu\mu}\)

\(\text{SR: Minkowski Metric}\)

\(\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} = V^{\nu\mu} + H^{\nu\mu}\)

\(\text{SR: Minkowski Metric}\)

\(\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} = V^{\nu\mu} + H^{\nu\mu}\)

\(\text{SR: Minkowski Metric}\)

\(\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} = V^{\nu\mu} + H^{\nu\mu}\)

\(\text{SR: Minkowski Metric}\)

\(\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} = V^{\nu\mu} + H^{\nu\mu}\)
SRQM Diagram:
The Basis of Classical SR Physics
Special Relativity via 4-Vectors

Focus on three of the main SR Physical 4-Vectors.

**4-Position**
\[ R = R^\mu = (ct, r) \Rightarrow \text{<Event>} \]

**4-Velocity**
\[ U = U^\mu = \gamma (c, u) \rightarrow \text{<Event>} \]

**4-Gradient**
\[ \partial = \partial^\mu = \left( \frac{\partial}{c \partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) \Rightarrow \text{<Event>} \]

These three give some of the main classical results of Special Relativity, including SR concepts like:
- The Minkowski Metric, SpaceTime Dimension = 4, Lorentz Transformations
- <Events>, Invariant Interval Measure, Causality (=Temporal Ordering)
- The Invariant Speed-of-Light (c)
- Invariant ProperTime (clock at rest), Invariant ProperLength (ruler at rest)
- Time Dilation (clock moving), Length Contraction (ruler moving)
- Relativity of Simultaneity, Minkowski Diagrams, Light Cone
- Use of the Lorentz Scalar Product to make Lorentz Invariants
- Invariant SR Wave Equations, via the d'Alembertian
- Continuity Equations
- etc.

Trace\[T^\mu_\nu]\ = \eta_{\mu\nu} T^\mu_\nu = T^\mu_\mu = T

\[ V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = (v^0)^2 - (v^\mu)^2 = (v^0)^2 \]

= Lorentz Scalar
SRQM Diagram:
The Basis of Classical SR Physics
Special Relativity via 4-Vectors

The Basis of most all Classical SR Physics is in the SR Minkowski Metric of “Flat” SpaceTime, which can be generated from the 4-Position and 4-Gradient, and determines the measurement between <Event>’s.

This Metric provides the relations between the 4-Vectors of SR: 4-Position R, 4-Gradient \( \partial \), 4-Velocity U.

The Tensor Invariants of these 4-Vectors give the: Invariant Interval Measures & Causality, from R-R Invariant d’Alembertian Wave Equation, from \( \partial \partial \) Invariant Magnitude LightSpeed (c), from U-U The relation between 4-Gradient \( \partial \) and 4-Position R gives the Dimension of SpaceTime (4), the Minkowski Metric \( \eta^{\mu\nu} \), and the Lorentz Transformations \( \Lambda^{\mu\nu} \).

The relation between 4-Gradient \( \partial \) and 4-Velocity U gives the ProperTime Derivative d\( \partial \)/d\( \tau \): Rearranging gives the ProperTime Differential d\( \tau \), which leads to Time Dilation & Length Contraction.

The ProperTime Derivative d\( \partial \)/d\( \tau \): acting on 4-Position R gives 4-Velocity U acting on the SpaceTime Dimension Lorentz Scalar gives the Continuity of 4-Velocity Flow.

The relation between 4-Displacement \( \Delta R \) and 4-Velocity U gives Relativity of Simultaneity.

One of the most important properties is the Tensor Invariant Lorentz Scalar Product ( dot = \( \eta_{\mu\nu} \)), provided by the lowered- index form of the Minkowski Metric \( \eta_{\mu\nu} \).

SR is a theory about the relations between 4D SpaceTime <Event>’s, i.e. how they are measured.
The 4-Position is essentially one of the fundamental 4-Vectors of SR. It is the SpaceTime location of an Event, the basic element of Minkowski SpaceTime: a time (t) & a place (r) → (when,where) = (ct,r).

The 4-Position relates time to space via the fundamental physical constant (c): the speed-of-light = "(c)elerity ; (c)eleritas", which is used to give consistent dimensional units across all SR 4-Vectors.

The 4-Position is a specific type of 4-Displacement, for which one of the endpoints is the origin, or 4-Zero. As such, the 4-Position and 4-Zero are Lorentz Invariant (point rotations and boosts), but not Poincaré Invariant (Lorentz + time & space translations), which can move the Origin.

The general 4-Displacement and 4-Differential(Displacement) are invariant under both Lorentz and Poincaré transformations, since neither of their endpoints are pinned this way.

The 4-Differential(Displacement) is just the infinitesimal version of the finite 4-Displacement, and is used in the calculus of SR. \( \mathbf{U} = \frac{d\mathbf{R}}{dt} \), where \( d\mathbf{R} = (cdt, dr) \).

\[
\Delta \mathbf{R} = (c\Delta t, \Delta r) = \mathbf{U} \Delta t = (c\Delta t, c\Delta r)
\]

\[
d\Delta \mathbf{R} = (cdt, dr) = \frac{d\mathbf{R}}{dt} d\Delta t
\]

\[
\mathbf{R} = (ct,r) \text{<Event>}
\]

The 4-Position is a specific type of 4-Displacement, for which one of the endpoints is the origin, or 4-Zero.

\[\Delta \mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 \rightarrow \mathbf{R} - \mathbf{Z} = \mathbf{R}
\]

As such, the 4-Position and 4-Zero are Lorentz Invariant (point rotations and boosts), but not Poincaré Invariant (Lorentz + time & space translations), which can move the Origin.

The general 4-Displacement and 4-Differential(Displacement) are invariant under both Lorentz and Poincaré transformations, since neither of their endpoints are pinned this way.

The 4-Differential(Displacement) is just the infinitesimal version of the finite 4-Displacement, and is used in the calculus of SR. \( \mathbf{U} = \frac{d\mathbf{R}}{dt} : \mathbf{dR} = (cdt, dr) \).
The Basis of Classical SR Physics

Invariant Intervals, Causality, TimeSpace

The Invariant Interval is the Lorentz Scalar Product of the {4-Position, 4-Displacement, 4-Differential} with itself, giving a magnitude-squared.

\[ \Delta R \cdot \Delta R = \left( c \Delta t \right)^2 - \left( \Delta r \right)^2 \]

4-Displacement \( \Delta R = (c \Delta t, \Delta r) = U \Delta t = R_2 - R_1 = (ct_2 - ct_1, r_2 - r_1) \): {finite}

4-Differential \( d\Delta R = (cdt, dr) = U dt \): {infinitesimal}

4-Position \( R = (ct, r) = \langle \text{Event} \rangle \)

\( d\Delta R = (cdt)^2, d\Delta r = (cdt)^2 \)

\( -ct \quad \gamma \)

\( \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{d^2}{c^2}}} \)

\( \Delta t \quad \Delta r \)

\( c \)

LightCone

SR 4-Vector

\( (0,1) \)-Tensor \( V^\mu = V = (\hat{v}^\mu, v^\mu) \)

\( \text{SR 4-Scalar} \)

\( (0,0) \)-Tensor \( S \)

\( \text{Lorentz Scalar} \)
These are of course the ones that appear in the 3 spatial (1 temporal)

All empirical evidence to-date indicates that there are only the 4 known dimensions:

This Tensor Invariant Lorentz Scalar relation gives the dimension of SpaceTime.

The only way there can more dimensions is if there is another SpaceTime direction available. The 4-Divergence is also used in SR Conservation Laws, ex. (\(\partial \cdot J\))

SRQM Diagram

SR : Minkowski
SpaceTime is 4D

The Tesseract, a 4D cube, symbolizes 4D SpaceTime

\[ \text{Trace}[T]^\mu = \eta_{\mu \nu} T^\nu = T \]

\[ V \cdot V = V^\nu \eta_{\nu \nu} V^\nu = (v^\nu)^2 - (v^\nu)^2 = (v^\nu)^2 = \text{Lorentz Scalar} \]
The Minkowski Metric (\(\eta^{\mu\nu}\)), Measurement

\[
\begin{align*}
\text{SR: Minkowski Metric} & \quad \delta[R] = \delta^R = \eta^{\mu\nu} = V^{\mu\nu} + H^{\mu\nu} \\
\text{SR: Spatial Projection} & \quad "\text{Horizontal}" \quad H^{\mu\nu} = \eta^{\mu\nu} T^{\mu} T^{\nu} \\
& \quad \text{Diag}[0,-1,-1,-1] = \text{Diag}[0,-\delta^0] \\
\text{SR: Temporal Projection} & \quad "\text{Vertical}" \quad V^{\mu\nu} = T^{\mu} T^{\nu} \\
& \quad \text{Diag}[1,0,0,0] = \text{Diag}[1,0^0] \\
\text{SR: 4-Vector} & \quad (1,1) \text{-Tensor} \quad V^{\mu} = \mathbf{V} = (\hat{v}^{\mu}) \\
\text{SR 4-Scalar} & \quad (0,0) \text{-Tensor} \quad S = \text{Lorentz Scalar} \\
\end{align*}
\]

The SR Minkowski Metric is the fundamental SR (2,0)-Tensor, which shows how intervals are measured in SR SpaceTime. It is itself the low-mass = (Curvature = 0) limiting-case of the more general GR Metric \(g^{\mu\nu}\). It can be divided into temporal and spatial parts. The Minkowski Metric is used to raise/lower indices on other tensors and 4-Vectors. Alt. Derivation: \(\delta^X = \eta^{\mu\nu} \delta_\mu \delta_\nu = \eta^{\mu\nu} \delta_\mu \delta_\nu = \eta^{\mu\nu} \delta_\mu \delta_\nu = \eta^{\mu\nu}\)
SR Diagram: The Basis of Classical SR Physics
The Lorentz Transform \( \partial_v[R^\mu]=\Lambda^\mu_v \)

A Tensor Study of Physical 4-Vectors

SRQM Diagram

4-Vector SRQM Interpretation of QM
SciRealm.org
John B. Wilson

4-Gradient \( \partial=(\partial \cdot \nabla) \)

4-Position \( R=(ct,\vec{r})=\text{Event} \)

Tensorial Lorentz Transform \( \Lambda^\mu_v \) acting on 4-Vector \( [R^\mu]=\Lambda^\mu_v \cdot R^\mu \)
\[ \partial[R^\mu]=\Lambda^\mu_v \cdot R^\mu \]
\[ \Lambda^\mu_v \cdot \Lambda^\nu_\nu=4 \]
\[ \text{Det}[\Lambda^\nu_\nu]=\pm1 \]

General Lorentz Boost Transform (symmetric continuous): for a linear-velocity time-space-mixing frame-shift (Boost) in the \( \vec{v}/c=\beta'(\gamma^{1},\beta^{2},\beta^{3}) \)-direction: \( \Lambda^\mu_v \rightarrow B^\nu_v = \gamma \begin{pmatrix} 1 & -\gamma \beta^{i}/(\beta \cdot \beta) \end{pmatrix} \delta^i_j \)

General Lorentz Rotation Transform (non-symmetric continuous): for an angular-displacement spatial-only frame-shift (Rotation) angle \( \theta \) about the \( \vec{n}=(n^i,n^j,n^k) \)-direction: \( \Lambda^\mu_v \rightarrow R^\mu_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

Lorentz Transform Properties:
\[ \Lambda^\mu_v = (\Lambda^{-1})^\nu_\mu \]
\[ \Lambda^\mu_v \Lambda^\nu_\nu = \eta^\nu_\nu \]
\[ \Lambda_v^\mu \Lambda_{\nu}'^\alpha = 4 \times \text{SpaceTime Dimension} \]
\[ \partial[R^\mu]=\Lambda_{\nu}'^\mu \cdot R^\mu \]

SR: Lorentz Transform
\[ \partial[R^\mu]=\partial R^\mu/\partial R^\nu = \Lambda^\mu_v \]
\[ \Lambda^\mu_v = (\Lambda^{-1})^\nu_\mu \cdot \Lambda^\nu_\alpha \Lambda^\alpha_v = \eta^\nu_\nu \delta^\nu_v \]
\[ \eta_{\mu
u} \Lambda_{\alpha}^\alpha \Lambda^\nu_\nu = \eta_{\mu
u} \delta^\nu_v \]
\[ \text{Det}[\Lambda^\nu_\nu]=\pm1 \]
\[ \Lambda^\mu_v \Lambda^\nu_\nu=4 \]

4-Displacement \( \Delta=R-c\Delta t,dr \)
4-Position \( R=(ct,\vec{r}) \)

Invariant Interval \( R=R^i=(ct,\vec{r})=\text{Diag}[1,-1,-1,-1] \)
\[ =\text{Diag}[1,-5\beta^2] \]

Minkowski Metric
\[ \eta_{\mu
u}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

ProperTime Derivative
\[ \partial=\gamma(c,u) \partial_{c',c} \gamma(c,u) \]
\[ =\gamma(dv/dt) \partial_{c',c} \gamma(c,u) \]
\[ =\gamma(dv/dt) \partial_{c',c} \gamma(c,u) \]
\[ =\gamma(dv/dt) \partial_{c',c} \gamma(c,u) \]

Continuity of 4-Velocity Flow
\[ \partial=0 \]

4-Velocity Flow \( U_\mu=U_\mu/c^2 \)

Invariant Magnitude LightSpeed
\[ U_\mu=U_\mu/c^2 \]

SRQM Diagram

4-Gradient \( \partial=(\partial \cdot \nabla) \)

4-Vector (2,0)-Tensor \( T_{iv} \)
(1,1)-Tensor \( T^iv \)
(0,2)-Tensor \( T_{iv} \)

SR 4-Vector (1,0)-Tensor \( V^\nu=\text{V}=(\vec{v}\cdot\vec{V}) \)

SR 4-Scalar (0,0)-Tensor \( S \)

SR 4-Scalar

Trace identifies CPT Symmetry
SRQM Diagram: The Basis of Classical SR Physics
SpaceTime Dimension = 4D, again!

\[ \partial R = Tr[\eta^{\mu\nu}] = \Lambda^{\mu}_{\nu} \Lambda^{\nu}_{\mu} = 4: \text{The SpaceTime Dimension Relations} \]

Tensor Invariants include: Trace, InnerProduct, Determinant, etc. The Trace of the Minkowski Metric & the InnerProduct of any of the Lorentz Transforms give the Dimension of SR SpaceTime = 4D.

**Minkowski Metric**
- 4-Divergence Trace Invariant
- Lorentz Transform Inner Prod Invariant

**Trace[\eta^{\mu\nu}]**
\[ = \partial R \]
\[ = \eta^{\mu\nu} \partial R^{\mu\nu} \]
\[ = \eta^{\mu\nu} \partial R^{\mu\nu} \]
\[ = \delta^{\mu\nu} \partial R^{\mu\nu} \]
\[ = (1+1+1+1) \]
\[ = 4 \]

**General Tensor Trace Invariant**
\[ Tr[T^{\mu\nu}] = \eta^{\nu\nu}(1) - (-1) - (-1) - (-1) = 4 \]

**Minkowski Metric \( \eta^{\mu\nu} \)**
\[ \Rightarrow \]

\[ \partial \eta^{\mu\nu} = \partial R^{\mu\nu} \]
\[ = \eta^{\mu\nu} \partial R^{\mu\nu} \]
\[ = \eta^{\mu\nu} \partial R^{\mu\nu} \]
\[ = \delta^{\mu\nu} \partial R^{\mu\nu} \]
\[ = 0' \]

**Conservation of Minkowski Metric**
- \[ \eta^{\mu\nu} = 0' \]
- \[ \partial R^{\mu\nu} = 0' \]
- \[ \partial R^{\mu\nu} = 0' \]

**SRQM Diagram**
- **4-Vector SRQM Interpretation of QM**
- **John B. Wilson**
- **SciRealm.org**

**Interpretation**
- **SR 4-Scalar**
  - Lorentz Scalar
- **SR 4-Vector**
  - **(2,0)-Tensor** \( T^{\mu\nu} \)
  - **(1,1)-Tensor** \( T^{\mu} = \eta^{\mu\nu} V_{\nu} \)
  - **(0,1)-Tensor** \( V_{\mu} = \partial R^{\mu\nu} \)
- **SR 4-Tensor**
  - **(2,0)-Tensor** \( T^{\mu\nu} = T_{\mu\nu} = T^{\mu\nu} \)
  - **(1,1)-Tensor** \( T^{\mu} = \eta^{\mu\nu} V_{\nu} \)
The Tensor Invariant Lorentz Scalar Product is the SR 4D (Dot) Product. It is used to make Invariant Lorentz Scalars from two 4-Vectors. 

\( \eta_{\mu\nu} \) = is itself just the lowered-index form of the SR Minkowski Metric (\( \eta_{\mu\nu} \)), with individual components \( \{ \eta_{\mu\nu} \} = \{ \eta \} \), else 0. In Cartesian basis, this gives \( \{ \eta_{\mu\nu} = \eta \} \).

It is used in just about every relation between any two interesting 4-Vectors. It also gives the Invariant Magnitude of a single 4-Vector. If the 4-Vector is temporal, then it is used to make Invariant Lorentz Scalars from two 4-Vectors. 

\[
\eta_{\mu\nu} \cdot \eta_{\nu\lambda} = \eta_{\mu\lambda}
\]

(\( \eta_{\mu\nu} \) is itself just the lowered-index form of the SR Minkowski Metric (\( \eta_{\mu\nu} \)), with individual components \( \{ \eta_{\mu\nu} \} = \{ \eta \} \), else 0. In Cartesian basis, this gives \( \{ \eta_{\mu\nu} = \eta \} \).

It is used in just about every relation between any two interesting 4-Vectors. It also gives the Invariant Magnitude of a single 4-Vector. If the 4-Vector is temporal, then the spatial component can be set to zero, giving the rest-frame invariant value, or the (o)observer rest value (‘naught’ = 0).

\[
\eta_{\mu\nu} = \eta_{\nu\mu} = \delta_{\mu\nu} \quad \text{(Cartesian)}
\]

\[
\eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \text{ in } \text{(Cartesian)} \\ 0 & \text{else} \end{cases}
\]

\[
\eta_{\mu\nu} \cdot \eta_{\nu\lambda} = \eta_{\mu\lambda}
\]

The Basis of Classical SR Physics

Lorentz Scalar (Dot) Product \( (\eta_{\mu\nu} \cdot \cdot) \)

SRQM Diagram: 4-Vector SRQM Interpretation of QM

SciRealm.org
John B. Wilson
4-Velocity $U = \gamma(c,u) = (\gamma c, \gamma u) = (U \partial)R = \frac{d}{dt}R$ is the ProperTime Derivative $\frac{d}{dt}$ of the 4-Position $R$ or of the 4-Displacement $\Delta R$.

It is the SR 4-Vector that describes the motion of <Event>’s through SpaceTime. For an un-accelerated observer, the 4-Velocity is along the WorldLine at all points. For an accelerated observer, the 4-Velocity is still tangent to the WorldLine at each point, but changes direction as the WorldLine bends.

The 4-Velocity is unlike most of the other SR 4-Vectors in that it only has 3 independent components, whereas the others usually have 4. This is due to the constraint placed by the Tensor Invariant of the 4-Velocity. $U \cdot U$ has a constant magnitude, the speed-of-light $(c)$ in SpaceTime.

Components:
- 3 independent components
- + 1 independent
- = 4 independent

4-Velocity $U = \gamma(c,u) = (\gamma c, \gamma u)$
4-Momentum $P = (mc, p) = (E/c, p)$

The 4-Velocity also usually has the Relativistic Gamma factor ($\gamma$) exposed in component form, whereas most of the other temporal 4-Vectors have it absorbed into the Lorentz 4-Scalar factor that goes into their components.

4-Velocity $U = \gamma(c,u) = (\gamma c, \gamma u)$
4-Momentum $P = (mc, p) = mU = \gamma m(c, u) = m(c, u) = (mc, mu) = (E/c, p)$

SR 4-Tensor
- (2,0)-Tensor $T^\mu_\nu$
- (1,1)-Tensor $T^\nu_\nu$, or $T^2$
- (0,2)-Tensor $T^\mu_\mu$

SR 4-Vector
- (1,0)-Tensor $V^\nu = V = (v^\nu, v)$

SR 4-CoVector
- (0,1)-Tensor $V_\nu = (v_\nu, v)$

SR 4-Scalar
- (0,0)-Tensor $S = Lorentz$ Scalar

4-Gradient $\delta = (\delta/c, \delta V)$

SRQM Diagram:
The Basis of Classical SR Physics
4-Velocity $U$, <Event> Motion
SRQM Diagram: The Basis of Classical SR Physics

4-Velocity Magnitude = Invariant Speed-of-Light (c)

SRQM Diagram:

An interesting thing to note is that all <event>s move at the speed-of-light (c) in 4D magnitude. Massive at-rest particles simply travel at (c) temporally while photons can move at (c) spatially as well.

If (c) was not a constant, but varied somehow, then all 4-Vectors made from the 4-Velocity would have more than 4 independent components, which is not observed. It seems a compelling argument against variable light-speed theories.
SRQM Diagram:
The Basis of Classical SR Physics
Relativity of Simultaneity

Relativity of Simultaneity:
\[ \Delta X = \gamma(c,u) \Delta t \Delta x = \gamma(c^2 \Delta t - u \Delta x) = c^2 \Delta x \]
If Lorentz Scalar \((U \cdot \Delta X = 0 = c^2 \Delta t)\),
then the Proper Time displacement \((\Delta t) \) is zero,
and the \( \bar{\text{Event}} \)’s separation \((\Delta x = X^2 - X_i)\) is orthogonal
to the worldline at \( U \).

\( \bar{\text{Event}} \)'s \( X_1 \) and \( X_2 \) are therefore simultaneous \((\Delta t = 0)\)
for the observer on this worldline at \( U \).

Examining the equation we get \( \gamma(c^2 \Delta t - u \Delta x) = 0 \).
The coordinate time difference between the events is \((\Delta t = u \cdot \Delta x/c^2)\).
The condition for simultaneity in an alternate frame
(moving at \( v \) velocity wrt. the worldline \( U \)) \((\Delta t = 0)\),
which implies \((u \cdot \Delta X) = 0 \).

This condition can be met by:
\[ \langle u \rangle = 0 \]
the alternate observer is not moving wrt. the events,
i.e. on worldline \( U \) or on a worldline parallel to \( U \).
\((\Delta x) = 0 \), the events are at the same spatial location \( \text{co-local} \).
\((u \cdot \Delta X = 0) \Rightarrow \Delta x = 0 \text{cos}(\theta) \) to the spatial separation \( \Delta x \)
of the events in that frame.

If none of these conditions is met,
then the events will not be simultaneous
in the alternate reference-frame.
This can be shown on a Minkowski Diagram.

\[ \Delta t = 0 \]
Simultaneous in \((t^0,x^0)\)
\[ \Delta t \neq 0 \]
Not Simultaneous in \((t,x)\)

Invariant Interval
\[ R \cdot R' = (ct)^2 - r \cdot r = (ct)^2 \]
\[ \Delta R \cdot \Delta R' = \gamma(c^2 \Delta t - u \cdot \Delta r) \]
\[ dR \cdot dR' = (c dt, dr) = (c dx, dy, dz) \]

Rest-Frame Proper Time
\( t' = \tau \)
Lorentz Boost-Frame
\( t \)

4-Displacement
\[ \Delta R = (\Delta t, \vec{\Delta r}) \]
\[ dR = (c dt, dr) \]
4-Position
\[ R = (ct, \vec{r}) \]

\[ U \cdot \Delta x = \gamma(c^2 \Delta t - u \cdot \Delta r) = c^2 \Delta x \]

\[ \text{Relativity of Simultaneity:} \]
\[ U \cdot \Delta R = \bar{\gamma}(c, u) \cdot (\Delta t, \Delta r) = \gamma(c^2 \Delta t - u \cdot \Delta r) = c^2 \Delta r \]

4-Gradient
\[ d = (\partial/c, \partial V) \]
\[ = (\partial/c, \partial x, \partial y, \partial z) \]

SRQM Diagram

Realizing that simultaneity is not an invariant concept
was a breakthrough that lead Einstein to Special Relativity
The Basis of Classical SR Physics

The ProperTime Derivative (d/dτ)

The derivation shows that the ProperTime Derivative d/dτ is an Invariant Lorentz Scalar. Therefore, all observers must agree on its magnitude, regardless of their frame-of-reference.

It can be used to make new 4-Vectors from existing 4-Vectors, as it is taking the derivative of an existing 4-Vector by a Lorentz Scalar: the ProperTime τ.

SRQM Diagram
The ProperTime Derivative acting on SR 4-Vectors:

\[ U \cdot \partial \gamma(c, u) \cdot (\partial/c, -\nabla) = \gamma(d \cdot \nabla) \gamma = \gamma(d/dt) = d/d\tau \]

4-Vectors:
- 4-Position \( R \rightarrow <\text{Event}> \)
- 4-Velocity \( U = d \cdot \nabla \; \text{d}/\text{d}t \)
- 4-Acceleration \( A = dU/d\tau \)
- 4-Momentum \( P = m \cdot U \)
- 4-Force \( F = dP/d\tau \)

As one can see from the list, the ProperTime Derivative gives the 4-Vectors that are the change in status of the 4-Vector that ProperTime Derivative acts on. It can also act on Scalar Values to give deep SR results.

\[ \partial \cdot R = 4: \text{SpaceTime Dimension} \]

\[ \partial \cdot U = 0: \text{Conservation of the SR 4-Velocity Flow} \]

\[ U \cdot U = c^2: \text{Tensor Invariant of 4-Velocity} \]

\[ d/d\tau[U \cdot U] = d/dt[c^2] = 0 \]

\[ d/d\tau[U \cdot U] = d/dt[U \cdot U] + U \cdot d/d\tau[U] = 2(U \cdot A) = 0 \]

\[ U \cdot A = U \cdot U' = 0: \text{The 4-Velocity is SpaceTime orthogonal to it's own 4-Acceleration} \]

SRQM Diagram: The Basis of Classical SR Physics

ProperTime Derivative on SR 4-Vectors and Scalars

SRQM Diagram

John B. Wilson
SciRealm.org
There are several ways to derive Time Dilation.

ProperTime Derivative (Lorentz 4-Scalar):
\[ U \cdot \partial = \gamma(c,u) \cdot (\partial / \gamma + u \cdot V) = \gamma d\tau / d\tau = d\tau / d\tau \]

ProperTime Differential (Lorentz 4-Scalar):
\[ dr = (ct, dr) \]

4-Differential
\[ dr = (cdt, dr) \]

4-Position
\[ R = (ct, r) \]

ProperTime Differential
\[ dr = (cdt, dr) \]

ProperTime Derivative
\[ U \cdot \partial = \gamma(c,u) \cdot (\partial / \gamma + u \cdot V) \]

ProperTime Derivative
\[ \partial R = 4 \text{ SpaceTime Dimension} \]

Relativity of Simultaneity
\[ U \cdot \partial = \gamma(c,u) \cdot (\partial / \gamma + u \cdot V) = \gamma(c^2 \Delta t - u \cdot \Delta r) = c^2 \Delta t - c \Delta t \]

Continuity of 4-Velocity Flow
\[ \partial U = 0 \]

Red and Blue lengths equal in the moving frame, blue is contracted
\[ L = (1/c)L_o : \text{Length Contraction} \]

Now multiply both sides by the moving-frame speed \[ v \Delta t = \gamma v \Delta t \]
\[ v \Delta t = \text{distance } L \text{, the moving clock travels wrt. frame, which is a proper (fixed-to-frame) displacement length} \]
\[ L_o = \gamma L \]
\[ L = (1/c)L_o : \text{Length Contraction} \]
The 4-Gradient in SR is a major role in advanced physics, showing how SR waves are formed, the Hamilton-Jacobi equations, the Schrödinger equations, Maxwell’s Equations, the Lorentz Transformations, and the Minkowski Metric. It provides the Dimension of SpaceTime, the Invariant Interval, the ProperTime Derivative, and the ProperTime Differential. It is instrumental in creating the ProperTime Derivative $d\tau = \gamma d\tau$. The 4-Gradient is a 4D version of the partial derivative function of calculus.

It is a 4-Vector function that can act on other 4-Vectors and 4-Scalars. The 4-Gradient tells how things change wrt. time and space. It is instrumental in creating the ProperTime Derivative $d\tau = \gamma d\tau$. The 4-Gradient plays a major role in advanced physics, showing how SR waves are formed, the Hamilton-Jacobi equations, the Euler-Lagrange equations, Conservation equations, Maxwell’s Equations, the Lorentz Transformations, and the Minkowski Metric. It gives the Dimension of SpaceTime, the Minkowski Metric, and the Lorentz Transformations. In QM, it provides the Schrödinger relations.

It is fundamental in connecting SR to QM.

SR 4-Tensor
(0,2)-Tensor $T^{\mu\nu}$ (1,1)-Tensor $T^\mu$, or $T_\mu$, (0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V$ (0,1)-Tensor $V_\mu = (v_\mu, v)$

SR 4-Scalar
(0,0)-Tensor $S$ Lorentz Scalar

SR 4-Gradient
$\partial = \partial^\mu (\partial^\mu, -V^\mu)$

SR 4-Momenta
$P^\mu = (E/c, p_\mu)$

SR 4-WaveVector
$K^\mu = (\omega/c, k_\mu)$

SR 4-Position
$R = (ct, r)$

SR 4-Momentum Action
$\delta R = (\partial t, \Delta r)$

SR 4-Acceleration
$\Delta R = (\partial^2 t, \Delta^2 r)$

SR 4-Displacement
$\Delta R = (d^2 t/c^2, d^2 r/c^2)$

SR 4-Gradient
$\partial^\mu = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$

SR 4-Scalar
$\psi = \psi (x, y, z, t)$

SR 4-Vector Function
$V \cdot \partial V = \partial (V \cdot V) = \partial V^\mu \partial V_\mu$

SR 4-Position
$R = (ct, r)$

SR 4-Momenta
$P^\mu = (E/c, p_\mu)$

SR 4-WaveVector
$K^\mu = (\omega/c, k_\mu)$

SR 4-Position
$R = (ct, r)$

SR 4-Momentum Action
$\delta R = (\partial t, \Delta r)$

SR 4-Acceleration
$\Delta R = (\partial^2 t, \Delta^2 r)$

SR 4-Displacement
$\Delta R = (d^2 t/c^2, d^2 r/c^2)$

SR 4-Gradient
$\partial^\mu = (\partial/\partial t, \partial/\partial x, \partial/\partial y, \partial/\partial z)$

SR 4-Scalar
$\psi = \psi (x, y, z, t)$

SR 4-Vector Function
$V \cdot \partial V = \partial (V \cdot V) = \partial V^\mu \partial V_\mu$

SR 4-Position
$R = (ct, r)$

SR 4-Momenta
$P^\mu = (E/c, p_\mu)$

SR 4-WaveVector
$K^\mu = (\omega/c, k_\mu)$

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$R = (ct, r)$

SR 4-Momentum Action
$\delta R = (\partial t, \Delta r)$

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$V \cdot \partial V = \partial (V \cdot V) = \partial V^\mu \partial V_\mu$
The Lorentz Scalar Invariant of the 4-Gradient gives the Invariant d’Alembertian Wave Equation, describing SR wave motion. It is seen in the SR Maxwell Equation for EM light waves. The Lorentz Scalar Invariant of the 4-Gradient gives the Invariant d’Alembertian Wave Equation, describing SR wave motion. It is seen in the SR Maxwell Equation for EM light waves.

Importantly, the d’Alembertian is fully from basic SR rules, with no quantum axioms required. However, it will be seen again in the Klein-Gordon QM wave equation.

It provides for the introduction of an SR Wave 4-Vector, which can also be given by the negative Gradient of a Lorentz Scalar Phase. The usual mathematical (complex) plane-wave solutions apply in SR:

\[ f = (a)e^{4-WaveVector} \]

\[ \text{Lorentz Gauge Conservation of EM Potential: } \partial A = 0 \]

\[ \partial R = \partial (\partial/c)^2 - V \cdot V \]

\[ \text{SR 4-Vector} \]

\[ (1,0)-Tensor V^0 = V = (v^0, v) \]

\[ (0,1)-Tensor V_0 = (v_0, -v) \]

\[ \text{SR 4-Scalar} \]

\[ (0,0)-Tensor S \text{ Lorentz Scalar} \]

\[ \text{SR 4-CoVector} \]

\[ (1,1)-Tensor T^{\nu \mu} = T^{\nu \mu} = T \]

\[ V^0 V^\nu = V^0 T^{\nu \mu} V^\mu = (v^0)^2 \]

\[ \text{SR 4-Tensor} \]

\[ (2,0)-Tensor T^{\mu \nu} \]

\[ \text{SR 4-Position} \]

\[ R = (ct, r) \]

\[ \text{SR 4-Velocity} \]

\[ K = (w, c, k) \]

\[ \text{SR 4-Displacement} \]

\[ \Delta R = (c \Delta t, \Delta r) \]

\[ \text{SR 4-Gradient} \]

\[ \partial \Delta R = \partial (\partial/c)^2 - V \cdot V \]
**SRQM Diagram:**

**The Basis of Classical SR Physics**

**Continuity of 4-Velocity Flow \( \partial \cdot \mathbf{U} = 0 \)**

This leads to all the SR Conservation Laws.

\[
\frac{dR}{d\tau} = 0
\]

\[
\frac{dR}{d\tau} = \frac{d}{d\tau}(c\Delta t, \Delta r) = 0
\]

\[
\frac{dR}{d\tau} = \frac{d}{d\tau}(dx/dt)\gamma + (dy/dt)\gamma + (dz/dt)\gamma = \frac{d}{d\tau} \gamma
\]

\[
\frac{d}{d\tau} \gamma
\]

Conservation of the 4-Velocity Flow

(4-Velocity Flow-Field)

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change of a quantity is balanced by the flow of that quantity into or out of a local region.

Conservation of Charge:

\[
\rho, \partial \cdot \mathbf{U} = \partial \rho, \mathbf{U} = \partial \mathbf{J} = \partial \rho + \nabla \cdot \mathbf{J} = 0
\]

\[
\partial \cdot \mathbf{U} = 0
\]

\[
\gamma d/dt [\gamma] = \partial \mathbf{U} = 0
\]

**Relativity of Simultaneity**

\[
\mathbf{U} \cdot \Delta \mathbf{R} = \gamma(c\mathbf{U}, \mathbf{v}) \cdot (c\Delta t, \Delta \mathbf{r}) = \gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{r})
\]

\[
\mathbf{U} = (c, 0, 0, 0)
\]

\[
\mathbf{U} = \gamma(c\mathbf{u}, \mathbf{v}) \cdot (c\Delta t, \Delta \mathbf{r}) = \gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{r})
\]

\[
\mathbf{U} \cdot \Delta \mathbf{R} = \gamma(c\mathbf{u}, \mathbf{v}) \cdot (c\Delta t, \Delta \mathbf{r}) = \gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{r})
\]

\[
\mathbf{U} = \gamma(c\mathbf{u}, \mathbf{v}) \cdot (c\Delta t, \Delta \mathbf{r}) = \gamma(c^2\Delta t - \mathbf{u} \cdot \Delta \mathbf{r})
\]

**SRQM Diagram**

**4-Gradient**

\[
\partial = (\partial/c, -\mathbf{V}) = (\partial/c, -\mathbf{V}/c^2)
\]

**4-Vector SRQM Interpretation of SR**

**SRQM Diagram**

**SR 4-Tensor**

\( (2,0) \)-Tensor \( T_{\mu\nu} \), or \( T, \) \( (1,1) \)-Tensor \( T^\mu_\nu \), or \( T^\mu_\nu \)

**SR 4-Vector**

\( (1,0) \)-Tensor \( \mathbf{V}^\mu = (\mathbf{\hat{v}}, \mathbf{\hat{v}}) \)

**SR 4-CoVector**

\( (0,1) \)-Tensor \( V_\mu = (\mathbf{\hat{v}}, \mathbf{\hat{v}}) \)

**SR 4-Scalar**

\( (0,0) \)-Tensor \( S \)

\( Lorentz \) \( Scalar \)

\[
\mathbf{V} \cdot \mathbf{V} = \mathbf{V}^\mu \mathbf{V}_\mu = (\mathbf{\hat{v}}^2) = \mathbf{\hat{v}}^2
\]

\[
\mathbf{V} \cdot \mathbf{V} = \mathbf{V}^\mu \mathbf{V}_\mu = (\mathbf{\hat{v}}^2) = \mathbf{\hat{v}}^2
\]

**Trace**

\[
\text{Trace}(\mathbf{T}^\mu_\nu) = \eta_{\mu\nu} T^\mu_\nu + T^\mu_\nu = T^\mu_\nu = T
\]

\[
\mathbf{V} \cdot \mathbf{V} = \mathbf{V}^\mu \mathbf{V}_\mu = (\mathbf{\hat{v}}^2) = \mathbf{\hat{v}}^2
\]
SRQM Diagram: The Basis of Classical SR Physics

<Event> Substantiation

Now focus on a few more of the main SR 4-Vectors.

4-Position $R^\mu = (ct, r) = <Event>

4-Velocity $U^\mu = \gamma (c, u)$

4-Gradient $\partial^\mu = (\partial t/c, -\nabla)$

4-Momentum $P^\mu = (E/c, p) = (mc, \text{mass})$

4-WaveVector $K^\mu = (\omega/c, k) = (\omega/c, \omega n/\text{phase})$

4-CurrentDensity: ChargeFlux $J^\mu = (\rho c, j) = (\rho c, \rho u)$

4-(Dust)NumberFlux $N^\mu = (n c, n) = (n c, n u) = (n u) U$

These 4-Vectors give more of the main classical results of Special Relativity, including SR concepts like:

SR Particles and Waves, Matter-Wave Dispersion
Einstein's $E = mc^2 = \gamma m_0 c^2 = E_0$, Rest Mass, Rest Energy
Conservation of Charge (Q), Conservation of Particle Number (N), Continuity Equations

Trace$[T^\mu] = \eta_{\mu\nu} T^{\mu\nu} = T^\nu_\nu = T$

$V \cdot V = V^\nu \eta_{\nu\nu} V^\nu = [(v^\nu)^2 - V^\nu V^\nu] = (v^\nu)^2$

$Lorentz$ Scalar
SRQM Diagram: The Basis of Classical SR Physics
4-Momentum, Einstein’s $E = mc^2$

4-Position $R = (ct, r)$
4-Gradient $\partial = (\partial / \partial x, \partial / \partial y, \partial / \partial z)$
4-Velocity $U = \gamma(c, u)$

4-Momentum $P = (E/c, p) = m_0 U = \gamma m_0 (c, u) = m(c, u)$

Temporal part: $E = \gamma E_0 = \gamma m_0 c^2 = mc^2$
{energy}
$E = m_0 c^2 + (\gamma - 1)m_0 c^2$
(rest) + (kinetic)

Spatial part: $p = \gamma m_0 u = mu$
{3-momentum}

4-Momentum $P = (E/c, p) = -\partial [S_{\text{action,free}}] = -\partial [\gamma (c, u)]$

Temporal part: $E = -\partial [S_{\text{action,free}}]$
{energy}

Spatial part: $p = +\nabla [S_{\text{action,free}}]$
{3-momentum}

4-Displacement $\Delta R = (c \Delta t, \Delta r)$
$\frac{d R}{d t} = (c d t, d r)$

SRQM Diagram

Hamilton-Jacobi Equation
$P_T = \partial [S_{\text{action,free}}]$ [mR] - [P-dR]

Einstein’s
$E = \gamma E_0 = \gamma m_0 c^2$
Energy/Mass
Equivalence
Rest Mass

$P = (m c, p)$

4-Momentum

$E/c, p = (mc, p)$$P_P = (m c)^2$

$E^2 = (|\mathbf{p}| c)^2 + (m_0 c^2)^2$
$E^2 = (|\mathbf{p}| c)^2 + (E_0)^2$
Einstein Mass:Energy
Relativistic Energy($E$):Mass($m_0$) vs Invariant Rest Energy($E_0$):Mass($m_0$)

ProperTime Derivative: $U \cdot \partial = \gamma \frac{d t}{d t}$

which matches:

$S_{\text{act}} = -m_0 c^2 d t$
$S_{\text{act}} = -E_0 d t$
for a free particle

$S_{\text{act}} = -(m_0 c^2 + V) d t$
$S_{\text{act}} = -(E_0 + V) d t$
in a potential

Trace[$T^{\mu \nu}$] = $\eta_{\mu \nu} T^{\mu \nu} = T^\mu_\mu = T$
$V \cdot V = V^\mu V_\mu = (V^0)^2 - (V^1)^2$
$= Lorentz Scalar$

SR 4-Tensor
{(2,0)}-Tensor $T^{\mu \nu}$
{(1,1)}-Tensor $T^{\mu \nu}$, or $T^\nu_\mu$
{(0,2)}-Tensor $T_{\mu \nu}$

SR 4-Vector
{(1,0)}-Tensor $V^\mu = (\vec{V}, V)$

SR 4-Scalar
{(0,0)}-Tensor $S$
Lorentz Scalar

SciRealm.org
John B. Wilson
SRQM Diagram: The Basis of Classical SR Physics

4-WaveVector, \( u \ast v_{\text{phase}} = c^2 \)

4-Position \( R = (ct, r) \)
4-Gradient \( \partial = (\partial / c, -\vec{V}) \)
4-Velocity \( U = \gamma (c, u) \)

4-WaveVector \( K = (\omega / c, k) = (\omega_o / c^2)U = \gamma (\omega_o / c^2)(c, u) \)

Temporal part: \( \omega = \gamma \omega_o \)
{angular frequency}

Spatial part: \( k = \gamma (\omega_o / c^2)u = (\omega / c^2)u = \omega \hat{n} / v_{\text{phase}} \)
{3-wavevector}
\( |u \ast v_{\text{phase}}| = c^2 \)

4-WaveVector \( K = (\omega / c, k) = -\partial [\Phi_{\text{phase,free}}] = - (\partial / c, -\vec{V})[\Phi_{\text{phase,free}}] \)

Temporal part: \( \omega = -\partial [\Phi_{\text{phase,free}}] \)
{angular frequency}

Spatial part: \( k = +\nabla [\Phi_{\text{phase,free}}] \)
{3-wavevector}

\( (K \cdot K) = (\omega / c)^2 - (k \cdot k) = (\omega_o / c)^2 \)
\( \omega^2 = (|k|c)^2 + (\omega_o)^2 : \text{Matter-Wave Dispersion Relation} \)

Relativistic AngFreq(\( \omega ) \) vs Invariant Rest AngFreq(\( \omega_o ) \)

4-Displacement \( \Delta R = (c \Delta t, \Delta r) \)
\( dR = (c dt, dr) \)

Wave Phase Equation \( K = -\partial [\Phi_{\text{phase}}] \)
\( \omega = c \frac{\partial \Phi_{\text{phase}}}{\partial x} \)

4-Gradient \( \partial = (\partial / c, -\vec{V}) \)
\( = (\partial / c \partial t, -\partial \partial x, -\partial \partial y, -\partial \partial z) \)

which matches:
\( \Phi_{\text{phase}} = -i \omega_o d\tau \) for a free particle
\( \Phi_{\text{phase}} = -i (\omega_o + V/\hat{n}) d\tau \) in a potential
SRQM Diagram: The Basis of Classical SR Physics

4-CurrentDensity, Charge Conservation

4-Position \( R = (ct, r) \)

4-Gradient \( \partial = (\partial/c, -\vec{\nabla}) \)

4-Velocity \( U = \gamma (c, u) \)

4-CurrentDensity \( J = (pc, j) = \rho_o U = \gamma \rho_o (c, u) = \rho (c, u) \)

Temporal part: \( \rho = \gamma \rho_o \) {charge-density}

Spatial part: \( j = \gamma \rho_o u = \rho u \) {3-current-density}

Conservation of Charge (Q)

\[ Q = \int \rho d^3x = \int \gamma \rho_o d^3x \]

\[ \rho_o V_o \]

\[ \rightarrow \]

\[ \rho_o V_o \]

\[ \int dT \cdot J = -cQ/V_o \]

\( J \cdot J = (pc)^2 - (j \cdot j) = (\rho_o c)^2 \)

\( \rho^2 = (jj/c)^2 + (\rho_o)^2 \)

Relativistic ChargeDensity(\( \rho \)) vs Invariant Rest ChargeDensity(\( \rho_o \))
4-Position \( R = (ct, r) \)
4-Gradient \( \partial = (\partial/c, -\nabla) \)
4-Velocity \( U = \gamma (c, u) \)

4-NumberFlux \( N = (nc, n) = n_0 U = \gamma n_0 (c, u) = n(c, u) \)

Temporal part: \( n = \gamma n_0 \) 
{number-density}

Spatial part: \( n = \gamma n_0 u = nu \)  
{3-number-flux}

Conservation of Particle # (\( N \))
\[
\partial \cdot N = (\partial/c, -\nabla) \cdot (nc, n) = (\partial \cdot n + \nabla \cdot n) = 0
\]

Continuity Equation: Noether’s Theorem 
The temporal change in number density is balanced by the spatial change in number-flux. 
Particle # is neither created nor destroyed. It just moves around as number currents...

Relativistic NumberDensity (\( n \)) vs Invariant Rest NumberDensity (\( n_0 \))
\[
|dT \cdot N| = -cn/V_0
\]

Trace \( [T^\mu] = n_0 T_\mu = T_\mu = T \)
\( V \cdot V = V_\mu V^\mu = (V_0)^2 - V^2 = (V_0)^2 \) 
Lorentz Scalar
Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_\nu [X^{\mu'}]$

(Continuous) vs (Discrete)

(Proper Det=+1) vs (Improper Det=-1)

The main idea that makes a generic 4-Vector into an SR 4-Vector is that it must transform correctly according to an SR Lorentz Transformation $\{ \Lambda^{\mu'}_{\nu} = \partial_\nu [X^{\mu'}] \}$, which is basically any linear, unitary or antiunitary, transform (Determinant[$\Lambda^{\mu'}_{\nu}$] = ±1) which leaves the Invariant Interval unchanged.

The SR continuous transforms (variable with some parameter) have {Det = +1, Proper} and include:

- "Rotation" (a mixing of space-space coordinates)
- "Boost" (a mixing of time-space coordinates)

The SR discrete transforms can be {Det = +1, Proper} or {Det = -1, Improper} and include:

- "Space Parity-Inversion" (reversal of the space coordinates)
- "Time-Reversal" (reversal of the temporal coordinate)
- The "Identity" (no change)

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Various single dimension Flips and their combinations.

**Continuous: ex. Boost depends on variable parameter $\beta$, with $\gamma=1/\sqrt{1-\beta^2}$**

**Discrete: ex. Parity has no variable parameters**

**Proper: preserves orientation of basis**

**Improper: reverses orientation of basis**
Lorentz Transforms \( \Lambda^\mu_\nu = \partial_\nu [X^\mu] \)

**Proper Lorentz Transforms (Det=+1):**
Continuous: (Boost) vs (Rotation)

\( \beta = v/c: \) dimensionless Velocity Beta Factor \{ \beta=(0..1), \) with speed-of-light \( (c) \) at \( (\beta=1) \}

\( \gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta\cdot\dot{\beta}}: \) dimensionless Lorentz Relativistic Gamma Factor \{ \gamma=(1..\infty) \}

Typical Lorentz Boost Transform (symmetric):
for a linear-velocity frame-shift \((x,t)-Boost\) in the \( \hat{x} \)-direction:

\[
\Lambda^\mu_\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\beta \gamma & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( \gamma = \text{cosh}[\zeta] \) \quad \( \beta = \text{tanh}[\zeta] \)

\( \Lambda^\mu_\nu A^\nu = (a^0, a^\nu) \)

Typical Lorentz Rotation Transform (non-symmetric):
for an angular-displacement frame-shift \((x,y)-Rotation\) about the \( \hat{z} \)-direction:

\[
\Lambda^\mu_\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos[\theta] & -\sin[\theta] & 0 \\
0 & \sin[\theta] & \cos[\theta] & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( \gamma = \text{cosh}[\zeta] \) \quad \( \beta = \text{tanh}[\zeta] \)

\( \Lambda^\mu_\nu A^\nu = (a^0, a^\nu) \)

SR: Lorentz Transform

\[
\delta_\nu[R^\mu] = \partial_\nu [R^\mu] = \Lambda^\mu_\nu
\]

\( \Lambda^\mu_\nu = \begin{pmatrix}
\Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 & \Lambda^1_4 \\
\Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 & \Lambda^2_4 \\
\Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 & \Lambda^3_4 \\
\Lambda^4_1 & \Lambda^4_2 & \Lambda^4_3 & \Lambda^4_4
\end{pmatrix}
\]

\( \Lambda^\mu_\nu (\Lambda^{-1})_\mu = \delta^\nu_\nu = \eta^\nu_\nu = \delta^\nu_\nu \)

\( \eta^\nu_\nu \Lambda^\mu_\nu A^\nu = (a^0, a^\nu) \)

\( \text{Det}[\Lambda^\mu_\nu] = \pm 1 \) \quad \( \Lambda^\mu_\nu A^\nu = (a^0, a^\nu) \)
Lorentz Transforms $\Lambda^\mu'_{\nu} = \partial_{\nu} [X^\mu']$

Proper Lorentz Transforms (Det=+1): (Boost) vs (Rotation) vs (Identity)

### General Lorentz Boost Transform (symmetric, continuous):
For a linear-velocity frame-shift (Boost) in the $v/c = \beta$ (in $[0,1]$) direction:
- $\Lambda^\mu'_{\nu} = B^\mu_{\nu}$
- $\Lambda^\mu'_{\nu} = \gamma (1-\gamma v \cdot \hat{\nu}) + \gamma v \cdot \hat{\nu}$
- $\Lambda^\mu'_{\nu} = (\gamma^{-1})^\mu_{\nu}$
- $\Lambda^\mu'_{\nu} = \eta^\mu_{\nu}$
- $\Lambda^\mu'_{\nu} = \delta^\mu_{\nu}$

### Lorentz Identity Transform (symmetric, discrete-continuous):
Identical 4-Vector
$A^\mu = A'^\mu = (a^0, a)$

### Lorentz Rotation Transform (non-symmetric, continuous):
- Rotated 4-Vector
$A'^\mu = R^\mu_{\nu} A^\nu = (a^0', a')$

### Lorentz Boost Transform (symmetric, continuous):
- Boosted 4-Vector
$A'^\mu = B^\mu_{\nu} A^\nu = (a^0', a')$

## Lorentz Identity Transform
The Lorentz Identity Transform is the limit of both the Rotation and Boost Transforms when the "rotation angle" is 0.

### Lorentz Identity Transform (symmetric, discrete-continuous):
- for a non-frame-shift (Identity) in any direction
- $\Lambda^\mu'_{\nu} = \eta^\mu_{\nu} = \delta^\mu_{\nu} = \text{Diag}[1, \delta] = I_{4(4)}$

### General Lorentz Rotation Transform (non-symmetric, continuous):
For an angular-displacement frame-shift (Rotation) about $\hat{n}$ in the $\hat{n} = (n^1, n^2, n^3)$-direction:
- $\Lambda^\mu'_{\nu} = R^\mu_{\nu}$
- $\Lambda^\mu'_{\nu} = (\Lambda^{-1})^\mu_{\nu} : \Lambda^\mu_{\nu} \Lambda^\nu_{\mu} = n^\mu_{\nu} = \delta^\mu_{\nu}

### SR-Lorentz Transform
$\delta^\mu_{\nu} [R^\mu_{\nu}] = \partial^\mu [R^\mu_{\nu}] / \partial R^\nu = \Lambda^\mu'_{\nu}$

### Trace
$\text{Trace}[T^\mu_{\nu}] = \eta^\mu_{\nu} T^\mu_{\nu} = T^\mu_{\mu} = T$

### Lorentz Scalars
$V^\mu = V_\nu \eta^\mu_{\nu} = \eta^\mu_{\nu} (v^\nu) = (v^\mu)^2$

$= \text{Lorentz Scalar}$

---

### Tensor Notation
- $T^\mu_{\nu}$
- $V^\mu$
- $S$
- $\eta^\mu_{\nu}$

### Lorentz Scalars
- $\gamma = 1 / \sqrt{1 - \beta^2}$
- $\gamma^{-1} = \sqrt{1 + \beta^2}$
- $\delta_{\mu \nu}$
- $\delta^\mu_{\nu}$
- $\Lambda^\mu'_{\nu}$
- $\Lambda^\mu_{\nu}$
- $\eta^\mu_{\nu}$
- $\delta^\mu_{\nu}$
Lorentz Transforms $\Lambda_{\mu'\nu} = \partial_{\nu}[X_{\mu'}^\nu]$ 

**Discrete (non-continuous) (Parity-Inversion) vs (Time-Reversal) vs (Identity)**

General Lorentz Parity-Inversion Transform: $\Lambda_{\mu'\nu} = P_{\mu'\nu}$ (Improper, symmetric, discrete) $= \begin{bmatrix} 1 & 0 \\ 0 & -\delta^{i}_{j} \end{bmatrix}$

General Lorentz Time-Reversal Transform: $\Lambda_{\mu'\nu} = T_{\mu'\nu}$ (Improper, symmetric, discrete) $= \begin{bmatrix} -1 & 0 \\ 0 & \delta^{i}_{j} \end{bmatrix}$

General Lorentz Identity Transform: $\Lambda_{\mu'\nu} = \delta_{\mu'\nu}$ (Proper, symmetric) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

SR: Lorentz Transform $\Lambda_{\mu'\nu} = (\Lambda_{\mu'\nu}) = \Lambda_{\mu'\nu}$

Both the Parity-Inversion (P) and Time-Reversal (T) have a Determinant of -1, which is an improper transform. However, combinations (PP), (TT), (PT) have overall Determinant of +1, which is proper. Classical SR Time Reversal neglects spin and charge. When included, there is also a Charge-Conjugation(C) transform. Then one gets (CC),(PP),(TT),(PT)(PT) & (CPT) transforms all leading back to the Identity (I).

**4-Vector SRQM Interpretation of QM**

Note that the Trace of Discrete Lorentz Transforms goes in steps from $-4,-2,2,4$. As we will see in a bit, this is a major hint for SR antimatter.
Lorentz Transforms $\Lambda^\mu_\nu = \partial_\nu [X^\mu]$

Discrete & Fixed Rotation $\rightarrow$ Particle Exchange

Lorentz Coordinate-Flip Transforms

\begin{align*}
\text{SR: Lorentz Transform} & \quad \partial_\nu [R^\mu] = \partial R^\mu/\partial R^\nu = \Lambda^\mu_\nu \\
\Lambda^\mu_\nu & = (\Lambda^{-1})^\nu_\mu, \quad \Lambda^\mu_\nu \Lambda^\nu_\alpha = \eta^\mu_\nu = \delta^\mu_\nu, \\
\text{Det}[\Lambda^\mu_\nu] & = \pm 1, \quad \Lambda^\mu_\nu \Lambda^\mu_\nu = 4
\end{align*}

Any single Lorentz Flip Transform is Improper, with a Determinant of -1. However, pairwise combinations are Proper, with a Determinant of +1.

The combination of any two Spatial Flips is the equivalent of a Spatial Rotation by (π) about the associated rotational axis. Since this is a Proper transform, it is also the equivalent of a particle location exchange.

The combination of all three Spatial Flips, Flip-xyz, gives the Lorentz Parity Transform, which is again Improper.

The Flip-t is the standard Lorentz Time-Reversal, Improper.

\begin{align*}
\text{SR 4-Vector} & \quad V^\nu = (\nu^0, \nu^1, \nu^2, \nu^3) \\
\text{SR 4-Scalar} & \quad \text{Lorentz Scalar} \\
\text{SR 4-Tensor} & \quad (0,1)-Tensor T^\mu_\nu = (\nu^0, \nu^1, \nu^2, \nu^3) \\
\text{SR 4-Vector} & \quad (1,0)-Tensor V^\nu = (\nu^0, \nu^1, \nu^2, \nu^3)
\end{align*}
Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

Lorentz Transform Connection Map

**SR: Lorentz Transform**

$\partial_{\nu}[R^{\mu'}] = \partial R^{\mu'}/\partial R^{\mu'} = \Lambda^{\mu'}_{\nu}$

$\Lambda^{\mu'}_{\nu} = (\Lambda^{-1})^{\mu}_{\nu}$ : $\Lambda^{\mu'}_{\nu} = \eta^{\mu'}_{\nu} = \delta^{\mu'}_{\nu}$

$\det[\Lambda^{\mu'}_{\nu}] = \pm 1$

$\det[\Lambda^{\mu'}_{\nu}] = 4$

By CPT Symmetry, this should be equivalent to the Feynman-Stueckelberg

Separate Set of Boosts & Rotations

Charge-Conjugation

$\Lambda^{\mu'}_{\nu} \rightarrow C^{\mu'}_{\nu}$

$-R^* \rightarrow R^*$, $-q \rightarrow q$

Charge Parity anti-unitary

By CPT Symmetry, this should be equivalent to the regular Positive Identity $I^* = C^*$

**CPT Symmetric Equivalent**

**Lorentz Boosts**

Various Continuous $t:x \mid t:y \mid t:z$

Various Discrete $-t$

**Lorentz Transforms**

Identity $I_{(4)}$

$\Lambda^{\mu'}_{\nu} \rightarrow \eta^{\mu'}_{\nu} = \delta^{\mu'}_{\nu}$

No mixing unitary

**Time-reversal**

$\Lambda^{\mu'}_{\nu} \rightarrow T^{\mu'}_{\nu}$

$t \rightarrow -t^*$

Time-parity anti-unitary

**Anti-unitary**

$\Lambda^{\mu'}_{\nu}$

**Unitary**

$\Lambda^{\mu'}_{\nu} = B^{\mu'}_{\nu}$

$\det[B^{\mu'}_{\nu}] = 1$

$\det[B^{\mu'}_{\nu}] = -1$

**Lorentz Rotations**

Continuous Various Rotations

Continuous Rotate-z

Continuous Various Flips

Discrete Various Flips

Discrete Flip-t

Discrete Load Parity=Flip-xyz
Lorentz Transforms
\[ \Lambda_{\mu' \nu} = \partial_{\nu}[X^{\mu'}] \]

Lorentz Transform Connection Map – Discrete Transforms

CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

Examine all possible combinations of Discrete Lorentz Transformations which are Linear (Determinant of \( \pm 1 \)).

A lot of the standard SR texts only mention (P)arity-Inverse and (T)ime-Reversal. However, there are many others, including (F)lips and (R)otations of a fixed amount. However, the (T)imeReversal and Combo(P)arity(T)ime take one into a separate section of the chart. Taking into account all possible discrete Lorentz Transformations fills in the rest of the chart. The resulting interpretation is that there is CPT Symmetry (Charge:Parity:Time) and Dual TimeSpace (with reversed timeflow). In other words, one can go from the Identity Transform (all +1) to the Negative Identity Transform (all -1) by doing a Combo PT Lorentz Transform or by Negating the Charge (Matter→Antimatter). The Feynman-Stueckelberg Interpretation aligns with this as the AntiMatter Side.

This is similar to Dirac’s prediction of AntiMatter, but without the formal need of Quantum Mechanics, or Spin. In fact, it is more general than Dirac’s work, which was about the electron. This is from general Lorentz Transforms for any kind of particle.

<table>
<thead>
<tr>
<th>T</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Discrete NormalMatter (NM) Lorentz Transform Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>Minkowski-Identity : AM-Flip-txyz=AM-ComboPT</td>
</tr>
<tr>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>-1</td>
<td>Flip-z</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
<td>+1</td>
<td>+1</td>
<td>Flip-( \pi )</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>Flip-xyz=\text{Rotate-xyz}(\pi)</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
<td>Flip-xz=\text{Rotate-xz}(\pi)</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>+1</td>
<td>Flip-xy=\text{Rotate-xy}(\pi)</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
<td>+1</td>
<td>-1</td>
<td>Flip-xyz=ParityInverse : AM-Flip-t=AM-TimeReversal</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Trace</th>
<th>Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( X )</td>
</tr>
<tr>
<td>+4</td>
<td>+1</td>
</tr>
<tr>
<td>+2</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>+1</td>
</tr>
<tr>
<td>+2</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>+1</td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Trace : Determinant

Note that the (T)imeReversal and Combo (P)arity(T)ime take NormalMatter

Tao – I Ching – YinYang
fantastic metaphors for SR SpaceTime...
Tao: "Flow of the Universe"
"way, path, route, road"
I Ching: "Book of Changes"
"Transformations"
YinYang: "Positive/Negative"
"complementary opposites"

Discrete Lorentz Transform (1,1)-Tensor
Pair production (+ -) in little circles (••)
Octagon representation

SciRealm.org
John B. Wilson
Lorentz Transforms \( \Lambda_{\mu'}^\nu = \partial_\nu[X^{\mu'}] \)

Lorentz Transform Connection Map – Trace Identification
CPT, Big-Bang, (Matter-AntiMatter), Arrow-of-Time

All Lorentz Transforms have Tensor Invariants: Determinant of \( \pm 1 \) and Inner Product of 4.
However, one can use the Tensor Invariant Trace to identify CPT Symmetry

\[
\begin{align*}
\text{Tr[ NM-Rotate ]} &= \{0\ldots+4\} & \text{Tr[NM-Identity]} &= +4 & \text{Tr[NM-Boost]} &= \{+4\ldots+\infty\} \\
\text{Tr[ AM-Rotate ]} &= \{0\ldots-4\} & \text{Tr[AM-Identity]} &= -4 & \text{Tr[AM-Boost]} &= \{-4\ldots-\infty\}
\end{align*}
\]

Trace : Determinant
\[
\begin{align*}
\text{Tr} &= +4 : \text{Det} = +1 \text{ Proper} \\
\text{Tr} &= +2 : \text{Det} = -1 \text{ Improper} \\
\text{Tr} &= 0 : \text{Det} = +1 \text{ Proper} \\
\text{Tr} &= -2 : \text{Det} = -1 \text{ Improper} \\
\text{Tr} &= -4 : \text{Det} = +1 \text{ Proper}
\end{align*}
\]

Trace Invariant values

Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:

\[
\text{Trace} = \text{Sum} \left( \Sigma \text{ of EigenValues} \right) : \text{Determinant} = \text{Product} (\Pi) \text{ of EigenValues}
\]

As Rank 4 Tensors, each Lorentz Transform has 4 EigenValues (EV's).
Create an Anti-Transform which has all EigenValue Tensor Invariants negated.
\[
\Sigma (-\text{EV's}) = -\Sigma (\text{EV's}) : \text{The Anti-Transform has negative Trace of the Transform.}
\]
\[
\Pi (-\text{EV's}) = (-1)^4 \Pi (\text{EV's}) = \Pi (\text{EV's}) : \text{The Anti-Transform has equal Determinant.}
\]

The Trace Invariant identifies a “Dual” Negative-Side for all Lorentz Transforms.
Based on the Lorentz Transform properties of the last few pages, here is an interesting observation about Lorentz Transforms:

They all have Determinant of ±1, and Inner Product of 4, but the Trace varies depending on the particular Transform.

The Trace of the Identity is at 4. Assume this applies to normal matter particles.

The Trace of the normal matter particle Rotations varies from (0..4)

The Trace of the normal matter particle Boosts varies from (4..Infinity)

So, one can think of Trace = 4 being the connection point between normal matter Rotations and Boosts.

Now, various Flip Transforms (inc. the Time Reversal and Parity Transforms, and their combination as PT transform), take the Trace in steps from (-4,-2,0,+2,+4). Applying a bit of symmetry:

The Trace of the Negative Identity is at -4. Assume this applies to anti-matter particles.

The Trace of the anti-matter particle Rotations varies from (0..-4)

The Trace of the anti-matter particle Boosts varies from (-4..-Infinity)

So, one can think of Trace = -4 being the connection point between anti-matter Rotations and Boosts.

This observation would be in agreement with the CPT Theorem (Feynman-Stueckelberg) idea that normal matter particles moving backward in time are CPT symmetrically equivalent to anti-matter particles moving forward in time.

Now, scale this up to Universe size: The Baryon Asymmetry problem (aka. The Matter-AntiMatter Asymmetry Problem).

If the Universe was created as a huge chunk of energy, and matter-creating energy is always transformed into matter-antimatter mirrored pairs, then where is all the antimatter???

Turns out this is directly related to the Arrow-of-Time Problem as well.

**Answer:** It is temporally on the “Other/Dual side” of the Big-Bang! The antimatter created at the Big-Bang is travelling in the negative time (-t) direction from the Big-Bang creation point, and the normal matter is travelling in the positive time direction (+t).

**Universal CPT Symmetry**. So, what happened “before” the Big-Bang? It “is” the AntiMatter Dual to our normal matter universe!

Pair-production is creation of AM-NM mirrored pairs within SpaceTime. The Big-Bang is the creation of SpaceTime itself.

This also resolves the Arrow-of-Time Problem. If all known physical microscopic processes are time-symmetric, why is the flow of time experienced as uni-directional??? (see Wikipedia “CPT Symmetry”, “CP Violation”, “Andrei Sakharov”)

**Answer:** Time flow on this side of the Universe is in the (+t) direction, while time flow on the dual side of the Universe is in the (-t) direction. The math all works out. Time flow is bi-directional, but on opposite sides of the Big-Bang! Universal CPT Symmetry.

This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=Antimatter, NM=Normal Matter):

- Various (AM_Flips) : Various (NM_Flips)
- -Infinity... (AM_Boosts)... (AM_Identity)=-4)... (AM_Rotations)...0...(NM_Rotations)...(+4=NM_Identity)...(NM_Boosts)...+Infinity

This solves the: Baryon (Matter-AntiMatter) Asymmetry Problem & Arrow(s)-of-Time Problem (+ / -)

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John B. Wilson
Lorentz Transforms $\Lambda_{\mu}^{\nu} = \partial_{\nu}[X^\mu]$

### Lorentz Transform Connection Map – Interpretations 2

**CPT, Big-Bang, (Matter-AntiMatter), Arrows-of-Time**

This idea of Universal CPT Symmetry also gives a Universal Dimensional Symmetry as well.

Consider the well known “balloon” analogy of the universe expansion. The “spatial” coordinates are on the surface of the balloon, and the expansion is in the +t direction. There is symmetry in the +/- directions of the spatial coordinates, but the time flow is always uni-directional, +t, as the balloon gets bigger.

By allowing a “dual side”, it provides a universal dimensional symmetry. One now has +/- symmetry for the temporal directions.

The “center” of the Universe is literally, the Big Bang Singularity. It is the “center=zero” point of both time and space directions.

The expansion gives time flow away from Big Bang singularity in both the Normal Side (+) and the Dual “Side (-). The spatial coordinates expand in both the (+/-) directions on both sides.

Note that this gives an unusual interpretation of what came “before” the Big Bang. The “past” on either side extends only to the BB singularity, not beyond. Time flow is always away from this creation singularity.

This is also in accord with known black hole physics, in that all matter entering a BH ends at the BH singularity. Time and space coordinates both come to a stop at either type of singularity, from the point of view of an observer that is in the spacetime but not at the singularity.

So, the Big Bang is a “starting” singularity, and black holes are “ending” singularities. Also provides for idea of “white holes” actually just being black holes on the alternate side. White hole=time-reversed black hole. This way, the mass is still attractive. Time flow is simply reversed on the alternate side so stuff still goes into the hole...

So, Universal CPT Symmetry = Universal Dimensional Symmetry.

And, going even further, I suspect this is the reason there is a duality in Metric conventions. In other words, physicists have wondered why one can use {+,−,−,−} or {−,+,+}. I submit that one of these metrics applies to the Normal Matter side, while the other complementarily applies to the Dual side. This would allow correct causality conditions to apply on either side. Again, this is similar to the Dirac prediction of antimatter based on a duality of possible solutions.

### Lorentz Transforms

$Lorentz\ Transform\ \Lambda_{\mu}^{\nu} = \partial_{\nu}[X^\mu]$

$\Lambda_{\mu}^{\nu} = (\Lambda^{-1})_{\alpha}^{\mu} : \Lambda_{\mu}^{\alpha}\Lambda_{\alpha}^{\nu} = \eta_{\mu}^{\nu} = \delta_{\mu}^{\nu}$

$\det[\Lambda_{\mu}^{\nu}] = \pm 1$

### SR: Lorentz Transform

$\partial_{\nu}[R^\mu] = \partial^\mu R^{\nu}/\partial^\nu = \Lambda_{\mu}^{\nu}$

$\Lambda_{\mu}^{\nu} = \Lambda_{\mu}^{\nu} = (\Lambda^{-1})_{\alpha}^{\mu} : \Lambda_{\mu}^{\alpha}\Lambda_{\alpha}^{\nu} = \eta_{\mu}^{\nu} = \delta_{\mu}^{\nu}$

$\eta_{\mu}^{\nu} = \delta_{\mu}^{\nu}$

$\Lambda_{\mu\nu} = 4$

$\det[\Lambda_{\mu}^{\nu}] = \pm 1$

### CPT Symmetry

- Normal Matter
- Antimatter
- This side follows its own time-arrow with “matter” acting “properly”.

This solves the: Baryon (Matter-AntiMatter) Asymmetry Problem & Arrow(s)-of-Time Problem (+ / -)
**SRQM Study:**

**Model SpaceTimes**

<table>
<thead>
<tr>
<th>Model SpaceTimes</th>
<th>$\Lambda &lt; 0$</th>
<th>$\Lambda = 0$</th>
<th>$\Lambda &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Klein Geometry G/H</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lorentzian pseudo-Riemannian</td>
<td>Anti de Sitter SO(3,2)/SO(3,1)</td>
<td>Minkowski ISO(3,1)/SO(3,1) $ds^2 = (cdt)^2 - dx \cdot dx$</td>
<td>De Sitter SO(4,1)/SO(3,1)</td>
</tr>
<tr>
<td>Riemannian</td>
<td>Hyperbolic SO(4,1)/SO(4)</td>
<td>Euclidean ISO(4)/SO(4) $ds^2 = (cdt)^2 + dx \cdot dx$</td>
<td>Spherical SO(5)/SO(4)</td>
</tr>
</tbody>
</table>

### Lie Groups

**de Sitter Group SO(1,4)**
- de Sitter invariant relativity (maybe?)

**Poincaré Group ISO(1,3)**
- $\{ r \ll r_{dS} = \text{de Sitter Radius} \}$
  - $r_{dS} = \sqrt{3/\Lambda} = L_{\mu}/\sqrt{\Omega_{\Lambda}}$

**SR & GR Physics**
- (** currently thought correct **)

**Galilei Group**
- $\{ |v| \ll c \}$
- Classical Physics

**Motion:Shear**

**Boost**

\[ \Lambda^{\mu}_{\nu} \rightarrow B^{\mu}_{\nu} = \begin{pmatrix} 1 & x & y & z \\ \frac{x}{|v|} & 1 & -\beta & 0 \\ \frac{y}{|v|} & \beta & 1 & 0 \\ \frac{z}{|v|} & 0 & 0 & 1 \end{pmatrix} \]
### SRQM Transforms: Venn Diagram

**Poincaré = Lorentz + Translations**

(10) (6) (4)

#### Transformations

<table>
<thead>
<tr>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete</td>
<td>Continuous</td>
</tr>
</tbody>
</table>

#### Lorentz Transform

- **4-Tensor** \( \Lambda^\mu_\nu \) \( (3+3=6) \)
  - \( \Lambda^\mu_\nu \rightarrow T^\mu_\nu \) (0)
  - Time-reversal \( t \rightarrow -t^* \) time parity anti-unitary
  - \( \Lambda^\mu_\nu \rightarrow R^\mu_\nu \) (0) SpatialFlipCombos \( \Lambda^\mu_\nu \rightarrow \eta^\mu_\nu \) unitary
  - Identity \( I^{(4)} \)
  - \( \Lambda^\mu_\nu \rightarrow \eta^\mu_\nu \) no mixing unitary
  - Charge-Conjugation \( \Lambda^\mu_\nu \rightarrow C^\mu_\nu \) (0) \( R \rightarrow -R^* \), \( q \rightarrow -q \) charge parity anti-unitary
- **Translation Transform** \( \Delta X^\mu \) \( (1+3=4) \)
  - Temporal \( \Delta X^t \rightarrow (c\Delta t, 0) \) (1) \( \Delta t \)
  - Spatial \( \Delta X^\mu \rightarrow (0, \Delta x) \) (3) \( \Delta x | \Delta y | \Delta z \)
  - Isotropy \( \{ \text{same all directions} \} \)
- **4-Zero** \( \Delta X^\mu \rightarrow (0,0) \) (0) no motion

#### Discrete Lorentz Matrices

- \( \Lambda^\mu_\nu \) = Lorentz + Translations
- \( \Lambda^\mu_\nu \rightarrow \Lambda^{-1,\lambda}_\nu, r^\mu_\nu, \delta^\mu_\nu \)
- Parameters \( \text{no motion} \) \( \Delta X^\mu \rightarrow (0,0) \) (0)
- Parameters \( \Delta X^\mu \rightarrow (0, \Delta x) \) (3)
- Parameters \( \Delta X^t \rightarrow (c\Delta t, 0) \) (1)

#### Lorentz Matrices can be generated by a matrix \( M \) with \( Tr[M]=0 \) which gives:

- \( \Lambda^\mu_\nu = e^M \)
- \( \Lambda^{-1,\lambda}_\nu = e^{-M} \)
- \( 4-\text{Angular Momentum} \)
- \( M = e^{+\theta J - \xi K} \)
- \( \Delta \Lambda^\mu_\nu = \delta^\mu_\nu \)
- \( \Lambda^\mu_\nu \rightarrow \Lambda^{-1,\lambda}_\nu, r^\mu_\nu, \delta^\mu_\nu \)
- \( \text{Det}[\Lambda^\mu_\nu] = +1 \) for Proper Lorentz Transforms
- \( \text{Det}[\Lambda^\mu_\nu] = -1 \) for Improper Lorentz Transforms

#### Feynman-Stueckelberg Interpretation

John B. Wilson  
SciRealm.org

Amusingly, Inhomogeneous Lorentz adds homogeneity.
Classical Transforms: Venn Diagram

**Full Galilean = Galilean + Translations**

(10) (6) (4)

### Transformations

- **Discrete**
  - Time-reversal: $\Lambda^\nu_\mu \to T^\nu_\mu$
  - Parity-Inversion: $\Lambda^\nu_\mu \to P^\nu_\mu$
  - SpatialFlipCombos: $\Lambda^\nu_\mu \to P^\nu_\mu$
  - Identity: $\Lambda^\nu_\mu \to \eta^\nu_\mu = \delta^\nu_\mu$
  - Motion:Shear: $\Lambda^\nu_\mu \to S^\nu_\mu$

- **Continuous**
  - Rotation: $\Lambda^\nu_\mu \to R^\nu_\mu$
  - Temporal: $\Delta X^\nu \to (c\Delta t, 0)$
  - Spatial: $\Delta X^\nu \to (0, \Delta x)$

- **Isotropy**
  - (same all directions)

- **Homogeneity**
  - (same all points)

### Lie Groups

- **de Sitter Group SO(1,4)**
  - de Sitter invariant relativity (maybe?)

- **Poincaré Group ISO(1,3)**
  - $r << r_{ds} = \text{de Sitter Radius}$
  - $r_{ds} = \sqrt{3/\Lambda} = L_{\mu}/\sqrt{\Omega_{\Lambda}}$

- **SR & GR Physics** *(currently thought correct)*

- **Galilei Group**
  - $\{ |v| << c \}$
  - Classical Physics

---

**Galilean Transform** $\Lambda^\nu_\mu$

- 4-Tensor {mixed type-(1,1)}

- Discrete
  - Time-reversal: $t \to -t^*$
  - Parity-Inversion: $r \to -r$

- Continuous
  - Rotation: $\Lambda^\nu_\mu \to R^\nu_\mu$
  - Temporal: $\Delta X^\nu \to (c\Delta t, 0)$
  - Spatial: $\Delta X^\nu \to (0, \Delta x)$

- Motion:Shear: $\Lambda^\nu_\mu \to S^\nu_\mu$

- Isotropy: (same all directions)

---

**Translation Transform** $\Delta X^\mu$

- 4-Vector

- Discrete
  - 4-Zero: $\Delta X^\mu \to (0,0)$

- Continuous
  - Spatial: $\Delta X^\mu \to (0, \Delta x)$

- Motion:Shear
  - $t:x | t:y | t:z$

---

**Lie Group of all affine isometries of Classical:Euclidean Time + Space**

- (preserve quadratic form $\delta_{ij}$)

- General Linear,Affine Transform
  - $X'_{\mu'} = \Lambda_{\mu'}^{\nu'} X^{\nu'} + \Delta X_{\mu'}$

- Translation Transform
  - $\Delta X_{\mu'} = 1+3=4$

- 4-Vector

- 4-Tensor {mixed type-(1,1)}

- Discrete
  - Identity: $\Lambda^\nu_\mu \to \eta^\nu_\mu = \delta^\nu_\mu$
  - no mixing

- Continuous
  - Identity: $\Lambda^\nu_\mu \to \eta^\nu_\mu = \delta^\nu_\mu$
  - no mixing

- Motion:Shear
  - $t:x | t:y | t:z$

---

**Full Galilean**

- $\{ \text{Galilean} + \text{Translations} \}$

- (10) (6) (4)
Review of SR Transforms

10 Poincaré Symmetries, 10 Conservation Laws

10 Generators: Noether's Theorem

A Tensor Study of Physical 4-Vectors

4-Vector SRQM Interpretation of QM

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John B. Wilson

10 Generators = 10 Conservation Laws: Noether's Theorem

Lagrange "Shift Operator" version of Taylor's Theorem: $e^{αdx/Δx}(x) = f(x+α)

Bloch Theorem: Translation Operator: $e^{ik\cdot x} = \psi(x+R)$, with $K$ as reciprocal lattice

Conservation of 4-Momentum $(1 + 3) = (4)$ Laws

Conservation of Linear 3-momentum $(3 + 3) = (6)$ Laws

Conservation of Angular 3-momentum $(3 + 3) = (6)$ Laws

Conservation of relativistic 3-mass-moment (temporal-spatial)

Conservation Laws

of Translation Transformations $(4)$

of Lorentz Transformations $(6)$

of Rotations $(3)$

Jacobi's Formula for Complex Square Matrix $A$: $\text{Det}([A]_0) = \text{Exp}([\text{Tr}(A)]_0)$

$\text{Det}([A]_0) = ([\text{Tr}(A)]_0 - 6 \text{tr}(A^2)_0 + 3 \text{tr}(A^3)_0 + 8 \text{tr}(A^4)_0) \text{tr}A - 6 \text{tr}(A^4)_0)/24$
The (10) one-parameter groups can be expressed directly as exponentials of the generators:

\[ U[\Lambda(\lambda \phi)], \quad \Lambda \in \mathbb{R} \]

These are the commutators of the Poincaré Algebra:

\[ [M^{\mu \nu} P^\rho, P_\sigma] = i \epsilon^{\mu \sigma \rho \nu} P^\nu \]

These are the Casimir Invariants = Operators that commute with all of the Poincaré Generators

\[ M^{00} = -c^2, \quad M^{0i} = c i \theta^i, \quad M^{ij} = \delta^{ij} c^2 - \epsilon^{ijlm} p_l p_m \]

Poincaré Algebra is the Lie Algebra of the Poincaré Group.

The set of all Lorentz Generators \( V = \{ \zeta \cdot K + \theta \cdot J \} \) forms a vector space over the real numbers. The components of the axis-angle vector and rapidity vector \( \{ \theta_x, \theta_y, \theta_z, \zeta_x, \zeta_y, \zeta_z \} \) are the coordinates of a Lorentz generator wrt. this basis.

Very importantly, the Poincaré group has Casimir Invariant Eigenvalues = \{ Mass \ mx, Spin \ jm \}, hence Mass *and* Spin are purely SR phenomena, no QM axioms required!

This Representation of the Poincaré Group or Representation of the Lorentz Group is known as Wigner’s Classification in Representation Theory of Particle Physics

Poincaré Algebra has 2 Casimir Invariants = Operators that commute with all of the Poincaré Generators

These are \( \{ P^2, P^i \} = (m c)^2, \quad W^2 = W^\mu W_\mu = -(m c)^2 (j + 1) \}, \) with \( W^\mu = (-1/2) \epsilon^{\nu \rho \sigma \mu} J_\rho p_\sigma \) as the Pauli-Lubanski Pseudovector

\[ P^2 = P^i P_i = (m c)^2, \quad W^2 = W^\mu W_\mu = -(m c)^2 (j + 1) \}

Hence the 4-Momentum Magnitude squared commutes with all Poincaré Generators

\[ W^2, P^2 \]
10 Poincaré Symmetry Invariances
Noether’s Theorem: 10 SR Conservation Laws

A Tensor Study
of Physical 4-Vectors

SR → QM

4-Vector SRQM Interpretation of QM

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SR 4-Tensor
(2,0)-Tensor $T^i_{\mu}$
(1,1)-Tensor $T^\nu_\mu$, or $T_{\mu}^\nu$
(0,2)-Tensor $T_{\mu \nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V^\mu_\nu v_\nu$

SR 4-CoVector
(0,1)-Tensor $V_\nu = (v_\nu, v^\mu_\nu)$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar

---

d’Alembertian Invariant Wave Equation: $\partial^2 = (\partial/c)^2 - \nabla \cdot \nabla = (\partial_t/c)^2$

Time Translation:
Let $X_t = (ct + c\Delta t, x)$, then $\partial[X_t] = (\partial/c, \nabla)(ct + c\Delta t, x) = \text{Diag}[1, -1] = \partial[X] = \eta^{\mu\nu}$

Lorentz Time-Space Boost:
Let $L = \gamma(\beta \cdot x, \beta c + x)$, then $\partial[L] = \gamma(\beta/c, -\nabla)(\beta \cdot x, \beta c + x)$

Space Translation:
Let $X_s = (ct, x + \Delta x)$, then $\partial[X_s] = (\partial/c, \nabla)(ct, x + \Delta x) = \text{Diag}[1, -1] = \partial[X] = \eta^{\mu\nu}$

Lorentz Space-Space Rotation:
Let $T = \gamma(\beta \cdot x, \beta c + x)$, then $\partial[T] = \gamma(\beta/c, -\nabla)(\beta \cdot x, \beta c + x)$

Lorentz Time-Space Boost:
Let $X_b = \gamma(\beta \cdot x, -\beta ct + x)$, then $\partial[X_b] = (\partial/c, \nabla)(\beta \cdot x, -\beta ct + x) = [\beta, -\beta \gamma \beta, -\beta \gamma \beta] = \Lambda^{\mu\nu}$

$d$-Alembertian Invariant Wave Equation: $\partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla = (\partial_t/c)^2$

Temporal part of $P^\mu\nu = (E/c, p)$

Spatial part of $P^\mu_\nu = (E/c, p)$

Spatial-Spatial part of $M^{\mu\nu} = X^P$

Temporal-Spatial part of $M^{\mu\nu} = X^P$

see Wikipedia: Relativistic Angular Momentum

Conservation of Relativistic Mass-Momentum
Conservation of Angular (Spatial) Momentum
Conservation of Linear (Spatial) Momentum

Total of $1 + 3 + 3 + 3 = 10$ Invariances from Poincaré Symmetry

Lorentz Space-Space Rotation Invariances (3)

Temporal-Spatial part of $M^{\mu\nu} = X^P$

---

\[
\text{Temporal-Spatial part of } M^{\mu\nu} = X^P
\]

---

\[
\text{Conservation of Linear (Spatial) Momentum}
\]

\[
\text{Temporal-Spatial part of } M^{\mu\nu} = X^P
\]

Temporal-Spatial part of $M^{\mu\nu} = X^P$

Lorentz Space-Space Rotation Invariances (3)

Conservation of Angular (Spatial) Momentum
Conservation of Linear (Spatial) Momentum

Total of $1 + 3 + 3 + 3 = 10$ Invariances from Poincaré Symmetry
SR 4-Vector Magnitudes

Dot Product, Lorentz Scalar Product

Einstein Summation Convention

An example of the magnitude of a 3-vector is the length of a 3-displacement \( \Delta r = (r_x - r_x) \).
Examine 3-position \( r_x \rightarrow r = (x,y,z) \), which is a 3-displacement with the base at the origin \( r_x \rightarrow 0 = (0,0,0) \).
The Dot Product of \( r \), \( r \cdot r = x^2 + y^2 + z^2 = r^2 \) is the Pythagorean Theorem.
The Kronecker Delta \( \delta_{jk} = \text{Diag}[1,1,1] = I_{(3)} \).
The magnitude is \( \sqrt{r \cdot r} = \sqrt{r^2} = |r| \). 3D magnitudes are always positive.

The magnitude of a 4-Vector is very similar to the magnitude of a 3-vector, but there are some interesting differences.
One uses the Lorentz Scalar Product, a 4D Dot Product, which includes a time component, and is based on the SR:Minkowski Metric Tensor. I typically use the “Particle Physics” convention of the Minkowski Metric \( \eta_{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1] \) (Cartesian form), with the other entries zero.

\[ A' \cdot A = A' A = A' \eta_{\mu\nu} A = A' A = \Sigma_{\mu=0..3}[a_\mu a_\nu] = (a_0 - a_1 + a_2 a_2 + a_3 a_3) = \Sigma_{\mu=0..3}[a_\mu a_\nu] \]

using the Einstein summation convention where upper-lower paired indices are summed over.

\[ R \cdot R = (c t)^2 - r \cdot r = (c t)^2 - (x^2 + y^2 + z^2) = (c \Delta t)^2 \]

for 4-Position \( R = (c t, r) \)

4D magnitudes can be negative, zero, positive

The 4-Vector version has the Pythagorean element in the spatial components, the temporal component is of opposite sign.
This gives a “causality condition”, where SpaceTime intervals (in the \([+,+,-,-]\) metric) can be:

\[ \Delta r \cdot \Delta r = [(c \Delta t)^2 - \Delta r \cdot \Delta r] \]

\( (c \Delta t)^2 \) Time-like:Temporal \( (\pm) \text{ causal = temporally-ordered} \)

\( 0 \) Light-like:Null:Photonic \( (0) \text{ causal, maximum signal speed (|} \Delta r / \Delta t | = c) \)

\(- (\Delta r)^2 \) Space-like:Temporal \( (\sim) \text{ non-causal, spatially-extended} \)
Lorentz Scalar Product $A \cdot B = A_\mu B^\mu$

Exterior Product $A^\wedge B = A^\mu B^\nu - A^\nu B^\mu$

4-Gradient
$$\partial = (\partial/c, \nabla)$$

Minkowski Metric
$$\partial[R] = \delta^{\mu}[R^\nu] = \eta^{\mu\nu}$$

Lorentz Transform
$$\partial[R^\nu] = \Lambda^{\nu\prime}_\mu = 4$$

SpaceTime Dimension
$$\partial[R] = \partial^\mu R^\nu = 4$$

4-Position
$$R = (ct, r)$$

4-Velocity
$$U = \gamma(c, u)$$

4-Momentum
$$P = (mc, p) = \left(\frac{E}{c}, p\right)$$

4-Gradient
$$\partial \cdot A = \partial_\mu A^\mu = 0$$: the Lorenz Gauge, a conservation of 4-EMVectorPotential

4-Position
$$R \cdot P = -S_{\text{action, free}}$$: the Action Scalar

4-Vector SRQM Interpretation of QM

There are at least three 4-Vector relations which use the Exterior Product.

$$\partial^A A = \partial^\mu \wedge A^\nu = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}$$: the Faraday EM 4-Tensor

$$R^\wedge P = R^\mu P^\nu = R^\mu P^\nu - R^\nu P^\mu = M^{\mu\nu}$$: the 4-Angular Momentum

$$R^\wedge F = R^\mu F^\nu = R^\mu F^\nu - R^\nu F^\mu = \Gamma^{\mu\nu}$$: the 4-Angular Torque

This gives the components of each remarkably similar properties.

Likewise, each of these has a physical Dot Product relation as well.

$$\partial \cdot A = \partial_\mu A^\mu = 0$$: the Lorenz Gauge, a conservation of 4-EMVectorPotential

$$R \cdot P = R^\mu P^\nu = -S_{\text{action, free}}$$: the Action Scalar

$$R \cdot F = R^\mu F^\nu = ???$$: probably something important

SR 4-Vector
$$(2,0)$$-Tensor $T^{\mu\nu}$
$$(1,1)$$-Tensor $V_\mu = V = \left(\begin{array}{cc} \phi & \mathbf{a} \end{array}\right)$
$$(0,2)$$-Tensor $T_{\mu\nu}$

SR 4-CoVector
$$(1,0)$$-Tensor $V^\mu = V = \left(\begin{array}{c} \phi \\mathbf{a} \end{array}\right)$
$$(0,1)$$-Tensor $V_\mu = \left(\begin{array}{c} \phi \mathbf{a} \end{array}\right)$

SR 4-Scalar
$$(0,0)$$-Tensor $S$

Lorentz Scalar
$$\eta_{\mu\nu} T^{\mu\nu} = \eta_{\mu\nu} T^{\nu\mu} = T$$

$$V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = \left(\begin{array}{c} \phi \\mathbf{a} \end{array}\right) \cdot \left(\begin{array}{c} \phi \mathbf{a} \end{array}\right) = \left(\begin{array}{c} \phi \mathbf{a} \end{array}\right)^2$$

Lorentz Scalar
SRQM Study: 4-Momentum, 4-Force, 4-AngularMomentum, 4-Torque

Linear:
4-Force is the ProperTime Derivative of 4-Momentum.

Angular:
4-Torque is the ProperTime Derivative of 4-AngularMomentum.

**4-Displacement**
\[ \Delta X = (c \Delta t, \Delta x) \]

**4-Position**
\[ X = (ct, x) \]

**4-Velocity**
\[ U = \gamma (c, u) = \frac{dX}{d\tau} \]

**4-Momentum**
\[ P = (m c, p) = (E/c, p) \]

**4-AngMomentum Tensor**
\[ M^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu = X \wedge P \]

**4-Force Tensor**
\[ F = \gamma (E/c, f) = dP/d\tau \]

**4-Torque Tensor**
\[ \Gamma^{\mu\nu} = X^\mu F^\nu - X^\nu F^\mu = X \wedge F \]

**Trace**
\[ \text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = T_{\mu}^{\mu} = T \]

**SpaceTime**
\[ \partial X = \partial_\mu X^\mu = \eta^{\mu\nu} \]

**Minkowski Metric**
\[ \partial_\nu [X^\mu] = \Lambda^{\mu}_{\nu} \]

**Lorentz Transform**
\[ \gamma (\partial/c, -V) \]

**4-Gradient**
\[ \partial \cdot X = \partial_\mu X^\mu = 4 \text{ Dimension} \]

**4-AngularMomentum**
\[ \text{4-AngMomentum, 4-Torque} \]

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SR Minkowski SpaceTime

4-Vectors, 4-CoVectors, Scalars, Tensors

Invariant Lorentz Scalar Product

4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers, meaning that they may have different components for different observers, but describe the same physical object. (like viewing a sculpture from different angles – snapshots look different but it's actually the same object)

There are also 4-CoVectors, or One-Forms, which are (0,1)-Tensors and dual to 4-Vectors. (like viewing a sculpture from different angles – snapshots look different but it's actually the same object)

4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers, meaning that they may have different components for different observers, but describe the same physical object. (like viewing a sculpture from different angles – snapshots look different but it's actually the same object)

Einstein & Lorentz “saw” the physics of SR, Minkowski & Poincaré “saw” the mathematics of SR. We are indebted to all of them for the simplicity, beauty, and power of how SR and 4-vectors work...

SR Minkowski SpaceTime

4-Vectors, 4-CoVectors, Scalars, Tensors

Invariant Lorentz Scalar Product

4-Vectors are actually tensorial entities of Minkowski SpaceTime, (1,0)-Tensors, which maintain covariance for inertial observers, meaning that they may have different components for different observers, but describe the same physical object. (like viewing a sculpture from different angles – snapshots look different but it's actually the same object)

There are also 4-CoVectors, or One-Forms, which are (0,1)-Tensors and dual to 4-Vectors. (like viewing a sculpture from different angles – snapshots look different but it's actually the same object)
SR 4-Vectors & Lorentz Scalars

Rest Values ("naughts") = 0 are Lorentz Scalars

A Tensor Study of Physical 4-Vectors

\[ \mathbf{A} \cdot \mathbf{A} = (a^0 a^0 - a \cdot a) = (a^0_0)^2 \], where \((a^0_0)\) is the rest-value, the value of the temporal coordinate when the spatial coordinate is zero.
The "rest-values" of several physical properties are all Lorentz scalars.

\[
\begin{align*}
\mathbf{P} &= (mc, p) \\
\mathbf{P} \cdot \mathbf{P} &= (mc)^2 - p \cdot p \\
\mathbf{K} &= (\omega/c, k) \\
\mathbf{K} \cdot \mathbf{K} &= (\omega/c)^2 - k \cdot k
\end{align*}
\]

\((\mathbf{P} \cdot \mathbf{P})\) and \((\mathbf{K} \cdot \mathbf{K})\) are Lorentz Scalars. We can choose a frame that may simplify the expressions.

Choose a frame in which the spatial component is zero.
This is known as the "rest-frame" of the 4-Vector. It is not moving spatially.

\[
\begin{align*}
\mathbf{P} \cdot \mathbf{P} &= (mc)^2 - p \cdot p = (mc)^2 \\
\mathbf{K} \cdot \mathbf{K} &= (\omega/c)^2 - k \cdot k = (\omega/c)^2
\end{align*}
\]
The resulting simpler expressions then give the "rest values", indicated by \((0,0)\).
RestMass \((m_o)\) and RestAngularFrequency \((\omega_o)\) are Invariant Lorentz Scalars by construction.

This leads to simple relations between 4-Vectors.
\[
\begin{align*}
\mathbf{P} &= (m_o) \mathbf{U} = (E/c, p) \\
\mathbf{K} &= (\omega/c)^2 \mathbf{U}
\end{align*}
\]
And gives nice Scalar Product relations between 4-Vectors as well.
\[
\begin{align*}
\mathbf{P} \cdot \mathbf{U} &= (m_o) (m_o) \mathbf{U} = (m_o)^2 \mathbf{U} = (E_o) \mathbf{U} \\
\mathbf{K} \cdot \mathbf{U} &= (\omega/c)^2 (m_o)^2 \mathbf{U} = (\omega/c)^2 (E_o) \mathbf{U} = (\omega/c)^2 \mathbf{U} = (E_o) \mathbf{U}
\end{align*}
\]
This property of SR equations is a very good reason to use the "naught" convention for specifying the difference between relativistic component values which can vary, like \((m)\), versus Rest Value Invariant Scalars, like \((m_o)\), which do not vary.
They are usually related via a Lorentz Factor: \(\{ m = \gamma m_o \} \) and \(\{ E = \gamma E_o \}\), as seen in the relation of \(\mathbf{P}\) and \(\mathbf{U}\).

\[
\begin{align*}
\mathbf{P} &= (mc, p) \\
\mathbf{P} \cdot \mathbf{U} &= (mc, p) \cdot (E_o/c^2) \mathbf{U} = (mc, p) \cdot (E_o/c^2) \mathbf{U} = (mc, p) \cdot (E_o/c^2) \mathbf{U} = (mc, p) \cdot (E_o/c^2) \mathbf{U} = (E/c, p)
\end{align*}
\]
SR 4-Vectors & 4-Tensors

Lorentz Scalar Product & Tensor Trace

Invariants: Similarities

All \{4-Vectors:4-Tensors\} have an associated \{Lorentz Scalar Product:Trace\}

Each 4-Vector has a “magnitude” given by taking the Lorentz Scalar Product of itself.
\[
V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = \left( v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3 \right) = \left( v^0 v^0 - \mathbf{v} \cdot \mathbf{v} \right) = \left( v^0 \right)^2
\]
The absolute magnitude of \( V \) is \( \sqrt{|V \cdot V|} \)

Each 4-Tensor has a “magnitude” given by taking the Tensor Trace of itself.
\[
\text{Trace}\left[ T^{\mu\nu} \right] = T_{\mu\mu} = \left( T_{00} - T_{11} - T_{22} - T_{33} \right) = T
\]
Note that the Trace runs down the diagonal of the 4-Tensor.

Notice the similarities. In both cases there is a tensor contraction with the Minkowski Metric Tensor \( \eta_{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1] \) \{Cartesian basis\}

ex. \( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2 = (m_o c)^2 \)
which says that the “magnitude” of the 4-Momentum is the RestEnergy/c = RestMass*c

ex. \( \text{Trace}[\eta^{\mu\nu}] = (\eta_{00} - \eta_{11} - \eta_{22} - \eta_{33}) = 1 -(-1) -(-1) -(-1) = 1+1+1+1 = 4 \)
which says that the “magnitude” of the Minkowski Metric = SpaceTime Dimension = 4

SR 4-Tensor
(2,0)-Tensor \( T^{\mu\nu} \)
(1,1)-Tensor \( T^\mu \), or \( T^\nu \)
(0,2)-Tensor \( T_{\mu\nu} \)

SR 4-Vector
(1,0)-Tensor \( V^\mu = V = (\mathbf{v}^0, \mathbf{v}) \)

SR 4-CoVector
(0,1)-Tensor \( V_\mu = (\mathbf{v}_0,-\mathbf{v}) \)

SR 4-Scalar
(0,0)-Tensor \( S = \text{Lorentz Scalar} \)
Some 4-Vectors have an alternate form of Tensor Invariant: \( \frac{d{v'}}{v' \circ} = \frac{dv}{v \circ} \), in addition to the standard Lorentz Invariant \( V \cdot V = V^\mu V_\mu = (v_0^2 - v \cdot v) = (v_0^0)^2 \)

If \( V \cdot V = (\text{constant}) \), with \( V = (v_0, v) \)
then \( d(V \cdot V) = 2 (V \cdot dv) = d(\text{constant}) = 0 \)

\( dv_0 = v \cdot dv/v_0 \)

Generally: with \( \Lambda = \Lambda^{\mu'}_{\nu'} \), = Lorentz Boost Transform in the \( \beta \)-direction
\( V' = \Lambda V \): from which the temporal component \( v_0' = (\gamma v_0 - \gamma \beta \cdot v) \)
\( dv' = \Lambda dv \): from which the spatial component \( dv' = (\gamma dv - \gamma \beta dv^2) \)

Combining:
\( dv' = (\gamma dv - \gamma \beta (v \cdot dv/v_0)) \)
\( dv' = (1/v_0^0)(\gamma v_0^0 dv - \gamma \beta (v \cdot dv)) \)
\( dv' = (1/v_0^0)(\gamma v_0^0 - \gamma \beta v) dv \)
\( dv' = (\gamma v_0^0 - \gamma \beta \cdot v)^*(1/v_0^0)*dv \)
\( dv' = (v'/v_0) dv \)

\( dv'/v_0 = dv/v_0 \) = Invariant of \( V = (v_0, v) \) for \( V \cdot V = (\text{constant}) \)

So, for example:
\( P \cdot P = (mc)^2 \) = (constant)

Thus, \( dp'/E' = dp/E \rightarrow d^3p'/E = d^3p/E \) = Invariant, usually seen as \( \int F(\text{various invariants}) d^3p/E \) = Invariant

An alternate approach is:
\( \int d^4p \delta[p^2-(m_0c)^2] \) = \( \int d^4p (1/2|m_0c|) (\delta[p+m_0c] + \delta[p-m_0c]) \)
\( = \frac{c d^3p}{2E} \)
\( = \text{Invariant} \)
The 4D Position coords that are integrated to give a 4D volume: SI units [m^4]

\[ d^4X = -(V_o) \cdot dT \cdot dX = -(dV_o) \cdot cdt \cdot dx \cdot dy \cdot dz \]

The 4D Position coords that are integrated to give a 4D volume: SI units [m^4]

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The 4D Position coords that are integrated to give a 4D volume: SI units [m^4]
\[ \rho \, d^3x = \rho' \, d^3x' = (-V/c) d^4T \cdot J = \text{Lorentz Scalar Invariant} \]

\[ \eta_{\mu\nu} \cdot N = (-V/c) \int d^4T \cdot N = \text{Lorentz Scalar Invariant} \]

\[ 4-(\text{Dust})\text{NumberFlux} N = (cn, nu) = n(c, u) = n_0 U \]

\[ 4-\text{ChargeFlux} Q = (-V/c) \int d^4T \cdot J = [pd^3x] = [\rho d^3x] = \rho_0 U = q n_0 U = q N \]

\[ V_o = \int \gamma d^3x = -c N/\int d^4T \cdot N = \text{Lorentz Scalar Invariant} \]

\[ \text{Total # Particles} \; N \; \text{is a Lorentz Scalar Invariant} \]

\[ \text{Total EM Charge} \; Q \; \text{is a Lorentz Scalar Invariant} \]

\[ \int V_o \cdot \eta_{\mu\nu} = \int \gamma d^3x = [\gamma n_0 d^3x] = \int n_0 U \]

\[ \int U = \gamma (c, u) \]

\[ \int \rho_0 U = q n_0 U = q N \]

\[ \int N = (cn, nu) = n(c, u) = n_0 U \]

\[ \int Q = (-V/c) \int d^4T \cdot J = [pd^3x] = [\rho d^3x] = \rho_0 U = q n_0 U = q N \]

\[ \int V_o = \int \gamma d^3x = -c N/\int d^4T \cdot N = \text{Lorentz Scalar Invariant} \]

\[ \int U = \gamma (c, u) \]

\[ \int \rho_0 U = q n_0 U = q N \]
SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

\[ d^4P = (dE/c) d\gamma \]  
\[ d^4K = (d\omega/c) d\gamma \]

4-Momentum Differential 
\[ dP = dP^\mu = (dE/c, d\gamma) \]

4-Unit Temporal Differential 
\[ dT = (d[\gamma], d[\gamma\beta]) \]

\[ (V_{Po})dT\cdot dP = \text{Invariant, because Rest Scalar} \times \text{Lorentz Scalar Product} \]

\[ dP = (dE/c) dp^\mu dp^\nu dp^\rho dp^\sigma \]

\[ d^4P = (dE/c) dp^x dp^y dp^z = (dE/c) d^3p \]

\[ \int F[\text{various Invariants}] d^4P \]

\[ \int F[\text{various Invariants}] d^4K \]
SR 4-Vectors & 4-Tensors
More 4-Vector-based Invariants
Phase Space Integration
SRQM Study: SR 4-Tensors

General → Symmetric & Anti-Symmetric

Any SR Tensor $T^\mu_\nu = (S^\mu_\nu + A^\mu_\nu)$ can be decomposed into parts:

Symmetric $S^\mu_\nu = (T^\mu_\nu + T^\nu_\mu)/2$ with $S^\mu_\nu = +S^\nu_\mu$

Anti-Symmetric $A^\mu_\nu = (T^\mu_\nu - T^\nu_\mu)/2$ with $A^\mu_\nu = -A^\nu_\mu$

$S^\mu_\nu + A^\mu_\nu = (T^\mu_\nu + T^\nu_\mu)/2 + (T^\mu_\nu - T^\nu_\mu)/2 = T^\mu_\nu + T^\nu_\mu - T^\nu_\mu = T^\mu_\nu + 0 = T^\mu_\nu$

Independent components: $\{4^2 = 16 = 10 + 6\}$

Max 16 possible
Max 10 possible
Max 6 possible

Physically, the anti-symmetric part contains rotational information and the symmetric part contains information about isotropic scaling and anisotropic shear.

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

*Note* These don’t have to be composed from a single general tensor.

$S^\mu_\nu A^\mu_\nu = 0$

Proof:

$S^\mu_\nu A^\mu_\nu = S^\nu_\mu A^\nu_\mu$: because we can switch dummy indices

$= (+S^\mu_\nu)A^\nu_\mu$: because of symmetry

$= S^\mu_\nu (-A^\mu_\nu)$: because of anti-symmetry

$= -S^\mu_\nu A^\mu_\nu$

$= 0$: because the only solution of $\{c = -c\}$ is 0

Traces:

$\text{Tr}[S^\mu_\nu] = S^\mu_\mu$

$\text{Tr}[A^\mu_\nu] = 0$

$a_{\text{skew}}$ aka Skew-Symmetric

$\text{Trace}[T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T^\mu_\mu$
SRQM Study: SR 4-Tensors

Symmetric $\rightarrow$ Isotropic & Anisotropic

Any Symmetric SR Tensor $S^{\mu\nu} = (T_{\text{iso}}^{\mu\nu} + T_{\text{aniso}}^{\mu\nu})$ can be decomposed into parts:

- **Isotropic** $T_{\text{iso}}^{\mu\nu} = (1/4)\text{Trace}[S^{\mu\nu}] \eta^{\mu\nu} = (T) \eta^{\mu\nu}$
- **Anisotropic** $T_{\text{aniso}}^{\mu\nu} = S^{\mu\nu} - T_{\text{iso}}^{\mu\nu}$

The Anisotropic part is Traceless by construction, and the Isotropic part has the same Trace as the original Symmetric Tensor. The Minkowski Metric is a symmetric, isotropic 4-tensor with $T=1$.

**Independent components:**

**Max 10 possible**

**Symmetric 4-Tensor**

$S^{\mu\nu} = \begin{bmatrix} S^{00}_{00}, S^{01}_{01}, S^{02}_{02}, S^{03}_{03} \\ S^{10}_{10}, S^{11}_{11}, S^{12}_{12}, S^{13}_{13} \\ S^{20}_{20}, S^{21}_{21}, S^{22}_{22}, S^{23}_{23} \\ S^{30}_{30}, S^{31}_{31}, S^{32}_{32}, S^{33}_{33} \end{bmatrix}$

$\text{Tr}[S^{\mu\nu}] = 4T$

**Max 1 possible**

**Symmetric Isotropic 4-Tensor**

$T_{\text{iso}}^{\mu\nu} = \begin{bmatrix} T, 0, 0, 0 \\ 0, -T, 0, 0 \\ 0, 0, -T, 0 \\ 0, 0, 0, -T \end{bmatrix}$

$\text{Tr}[T_{\text{iso}}^{\mu\nu}] = 4T$

**Max 9 possible**

**Symmetric Anisotropic 4-Tensor**

$T_{\text{aniso}}^{\mu\nu} = \begin{bmatrix} S^{00}_{00}-T, S^{01}_{01}, S^{02}_{02}, S^{03}_{03} \\ S^{10}_{10}+T, S^{11}_{11}, S^{12}_{12}, S^{13}_{13} \\ S^{20}_{20}, S^{21}_{21}, S^{22}+T, S^{23}_{23} \\ S^{30}_{30}, S^{31}_{31}, S^{32}, S^{33}+T \end{bmatrix}$

$\text{Tr}[T_{\text{aniso}}^{\mu\nu}] = 0$

**Note** These don’t have to be composed from a single general tensor.

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

$S^{\mu\nu} A_{\mu\nu} = 0$

**Proof:**

$S^{\mu\nu} A_{\mu\nu} = S^{\mu\nu}(A_{\nu\mu})$ because of symmetry

$= -S^{\mu\nu} A_{\mu\nu}$ because of anti-symmetry

$= 0$: because the only solution of $\{c = -c\}$ is $0$

Physically, the isotropic part represents a direction-independent transformation (e.g., a uniform scaling or uniform pressure); the deviatoric part represents the distortion.
SRQM Study: SR 4-Tensors

SR Tensor Invariants

(0,0)-Tensor = Lorentz Scalar S: Has either (0) or (1) Tensor Invariant, depending on exact meaning (S) itself is Invariant

(1,0)-Tensor = 4-Vector Vμ: Has (1) Tensor Invariant = The Lorentz Scalar Product

V·V = VμτνVμτν = Tr[VμVμ] = V02 + V12 + V22 + V32 = (V0)2

(2,0)-Tensor = Tμν: Has (4+) Tensor Invariants (though not all independent)

a) Tμν = Trace = Sum of EigenValues for (1,1)-Tensors (mixed)
b) Tμ[ν] = Asymm Bi-Product → Inner Product
c) TμνTνμ = Asymm Tri-Product → ?Name?
d) TμνTνμ = Asymm Quad-Product → 4D Determinant = Product of EigenValues for (1,1)-Tensors

eg. TμνTμμ = Tμ0Tμ0 - Tμ1Tμ1 = (T0)2 - T1T1 = Tr[Tμμ] = (1/4)Tr[TTμμ]

and, bending tensor rules slightly: = (T0)2 - T1T1 = (1/4)Tr[Tμμ] = (1/4)(Tr[Tμμ]) = Tr[Tμμ]

and, since linear combinations of invariants are invariant:

Examine just the (T00T00) part, which for symm/asymm is (1/4)(Tr(T00)) ie. the InnerProduct Invariant

a): Trace[Tμν] = Tr[Tμν] = ημτμτν = Tμντ = (T0)2 + T1 + T2 + T3 = (T00 - T11 - T22 - T33) = (T)

for anti-symmetric: = 0

b): InnerProduct TμνTνμ = T0000 + T1001 + T2010 + T3012 = (T00)2 - Σ[Tμμ]2 = (T00)2 - Σ[Tr[Tμμ]]2 = Tr[Tμμ]2 = 4-Tr[Tμμ]2

for symmetric | anti-symmetric: = (T00)2 - 2Σ[Tμμ]2 = (1/4)Tr[Tμμ]2 - 2Σ[Tr[Tμμ]]2 = 2Σ[Tr[Tμμ]]2 = 2Σ[Tr[Tμμ]]2

c): Antisymmetric Triple Product TμνTμνTμν = Tr[Tμν]3 - 3(Tr[Tμν])Tμν + TμνTμνTμν = TμνTμνTμν

for anti-symmetric: = 0

d): Determinant Det[Tμν] = ? = -(1/2)εμνρσεμνρσTτστVτσTτστVτσ

det for antisymmetric: Det[Tμν] = Pfaffian[TτστVτστVτστVτστ]2 (The Pfaffian is a special polynomial of the matrix entries)

SR 4-Vector

(1,0)-Tensor Vμ = V = (V0, V1, V2, V3)

SR 4-Scalar

(0,0)-Tensor S = Lorentz Scalar

SR 4-Vector

(1,0)-Tensor Vμ = V = (V0, V1, V2, V3)

SR 4-Scalar

(0,0)-Tensor S = Lorentz Scalar

Det[Tμν] = ΠIi(λi); with {λi} = Eigenvalues

Characteristic Eqns: Det[Tμν - λiI4] = 0

Trace[Tμν] = μν | Tμν = (T00 - T11 - T22 - T33) = T

V·V = VμτνVμτν = (v0)2 + (v1)2 + (v2)2 + (v3)2 = (v0)2

4-Vector SRQM Interpretation of QM

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SRQM Study: SR 4-Tensors

SR Tensor Invariants

Tensor Gymnastics

### Some Tensor Gymnastics:

Matrix $\mathbf{A} = \text{Tensor } A^c_c$ with rows denoted by “r”, columns by “c”

Example with dim=4: $r,c \in \{0..3\}$

Matrix $\mathbf{A} = 
\begin{bmatrix}
A^0_{00} & A^0_{01} & A^0_{02} & A^0_{03} \\
A^1_{00} & A^1_{01} & A^1_{02} & A^1_{03} \\
A^2_{00} & A^2_{01} & A^2_{02} & A^2_{03} \\
A^3_{00} & A^3_{01} & A^3_{02} & A^3_{03}
\end{bmatrix}$

with the rows of $\mathbf{A}$ multiplied by the columns of $\mathbf{B}$ due to the summation over index “c”

If we have sums over both indices:

$A^c_d B^d_a = M^c_a$ = Trace[$\mathbf{M}$]

The sum over “c” gives the matrix multiplication and then the sum over “d” gives the Trace of the resulting matrix $\mathbf{M}$

$A^c_d = (\mathbf{A} \mathbf{A})^c_d = (\mathbf{N})^c_d = \text{Trace}[\mathbf{N}] = \text{Trace}[\mathbf{A}^2] = \text{Tr}[\mathbf{A}^2]$

$A^c_d = (\eta^c_d A^c_d = \eta^c_d N^c_d) = \delta^c_d (N^c_d) = \text{Tr}[\mathbf{N}] = \text{Tr}[\mathbf{A}^2]$

$A^c_d A^d_c = A^c_c - A^c_a A^d_a = (\text{Tr}[\mathbf{A}])^2 - \text{Tr}[\mathbf{A}^2]$ with brackets $[.]$ around the indices indicating anti-symmetric product

<table>
<thead>
<tr>
<th>$A^{\mu}_{a}$ = Tr[$\mathbf{A}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{\mu}<em>{a} A^{\mu}</em>{b} = A^{a}<em>{a} A^{b}</em>{b} - A^{a}<em>{a} A^{b}</em>{b} = (\text{Tr}[\mathbf{A}])^2 - \text{Tr}[\mathbf{A}^2]$</td>
</tr>
</tbody>
</table>

For a 4-Vector $\mathbf{V}$ = $v^\alpha \mathbf{\delta}_\alpha = (v^0, v^1, v^2, v^3)$

Trace[$\mathbf{V}^\alpha$] = $\eta^\mu_{\nu} (T^\mu_{\nu} = T^\nu_{\mu} = T$ = Lorentz Scalar

$\mathbf{V} \cdot \mathbf{V} = V^\nu (V^\nu = ((v^0)^2 - v^1 v^1 - v^2 v^2 - v^3 v^3)^2 = \text{Lorentz Scalar}$

$\text{Det} [\lambda_{\alpha}] = \Pi_{\alpha} \lambda_{\alpha}$; with $\{\lambda_{\alpha}\} = \text{Eigenvalues}$

Characteristic Eqns: $\text{Det}[\mathbf{T}^\alpha_{\alpha} - \lambda_{\alpha} I_{\alpha\beta}] = 0$
SRQM Study: SR 4-Tensors

SR Tensor Invariants

Cayley-Hamilton Theorem

General Cayley-Hamilton Theorem

\[ A^4 + c_3 A^3 + \ldots + c_0 A^0 = 0 \]

Characteristic Polynomial: \( p(\lambda) = \det(A - \lambda I_d) \)

The following are the Principle Tensor Invariants for dimensions 1..4

**Dimension 1**

\[ \dim = 1: \quad A^1 = \sum_{\lambda_j} \lambda_j \]

**Dimension 2**

\[ \dim = 2: \quad A^2 = \sum_{\lambda_j, \lambda_k} \lambda_j \lambda_k \]

**Dimension 3**

\[ \dim = 3: \quad A^3 = \sum_{\lambda_j, \lambda_k, \lambda_m} \lambda_j \lambda_k \lambda_m \]

**Dimension 4**

\[ \dim = 4: \quad A^4 = \sum_{\lambda_j, \lambda_k, \lambda_m, \lambda_n} \lambda_j \lambda_k \lambda_m \lambda_n \]

The characteristic equation for a tensor of dimension 1 is:

\[ \text{Trace}(T) = 0 \]

For a tensor of dimension 2:

\[ \text{Trace}(T) = 0 \quad \text{and} \quad \det(T) = 0 \]

For a tensor of dimension 3:

\[ \text{Trace}(T) = 0, \quad \det(T) = 0, \quad \text{and} \quad \text{Tr}(T^2) = 0 \]

For a tensor of dimension 4:

\[ \text{Trace}(T) = 0, \quad \det(T) = 0, \quad \text{Tr}(T^2) = 0, \quad \text{and} \quad \text{Tr}(T^3) = 0 \]

\[ \text{Tr}(T^4) = 0 \]

**Trace**

\[ \text{Trace}(T^r) = \eta_{\nu_1 \nu_2 \ldots \nu_r} T^\nu_1 T^\nu_2 \ldots T^\nu_r = T \]

**Lorentz Scalar**

\[ \eta_{\nu_1 \nu_2 \ldots \nu_r} V^\nu_1 V^\nu_2 \ldots V^\nu_r = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = \text{Lorentz Scalar} \]
# SRQM Study: SR 4-Tensors

## SR Tensor Invariants

### Cayley-Hamilton Theorem

**General Cayley-Hamilton Theorem**

\[ A^d + c_{d-1}A^{d-1} + \ldots + c_0 A^0 = 0 \]

with \( A \) = square matrix, 
\( d \) = dimension, \( A^0 = \text{Identity}(d) = I_{(d)} \)

\[ I_0 A^d - I_1 A^{d-1} + I_2 A^{d-2} - I_3 A^{d-3} + I_4 A^0 = 0 \] for 4D

**Characteristic Polynomial:**

\[ p(\lambda) = \text{Det}[A - \lambda I_{(d)}] \]

### Tensor Invariants \( I_n \)

- **\( I_0 = 1/0! = 1 \)**
- **\( I_1 = \text{tr}[A]/1! \)**
  \[ = A^\alpha _{\alpha} \]
  \[ = \Sigma \text{[Eigenvalues]} \]
  \[ = \text{Det}_0[A] \]
  \[ = \Pi[\text{Eigenvalues}] \]
- **\( I_2 = (\text{tr}[A]^2 - \text{tr}[A^2])/2! \)**
  \[ = A^\alpha _{\alpha} A^\beta _{\beta} /2 \]
  \[ = \Sigma \text{[Unique Eigenvalue Doubles]} \]
- **\( I_3 = [\text{tr}(A)^3 - 3 \text{tr}(A^2)(\text{tr} A) + 2 \text{tr}(A^3)])/3! \)**
  \[ = A^\alpha _{\alpha} A^\beta _{\alpha} A^\gamma _{\beta} /6 \]
  \[ = \Sigma \text{[Unique Eigenvalue Triples]} \]
- **\( I_4 = ([\text{tr}(A)^4 - 6 \text{tr}(A^3)(\text{tr} A)^2 + 3 \text{tr}(A^2))^2 + 8 \text{tr}(A^3) \text{ tr} A - 6 \text{ tr}(A^4))/4! \)**
  \[ = A^\alpha _{\alpha} A^\beta _{\alpha} A^\gamma _{\gamma} A^\delta _{\gamma} /24 \]
  \[ = \Sigma \text{[Unique Eigenvalue Quadruples]} \]
The Faraday EM Tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \partial^\alpha \wedge A$ is an anti-symmetric tensor that contains the Electric and Magnetic Fields, defined by the Exterior Product ($\wedge$). The 3-electric components ($e = e^i$) are in the temporal-spatial sections. The 3-magnetic components ($b = b^i$) are in the only-spatial section.

(2,0)-Tensor = 4-Tensor $T^{\mu\nu}$; Has (4+) Tensor Invariants (though not all independent)

a) $T^\alpha_\alpha = \text{Trace} = \sum_i \lambda_i$, $\lambda_i$ are EigenValues for (1,1)-Tensors (mixed)
b) $T^\alpha_\beta T^\beta_\alpha = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$
c) $T^\alpha_\beta T^\beta_\gamma T^\gamma_\alpha = \text{Asymm Tri-Product} \rightarrow \text{?Name?}$
d) $T^\alpha_\beta T^\beta_\gamma T^\gamma_\delta T^\delta_\alpha = \text{Asymm Quad-Product} \rightarrow 4$ Determinant = Product of EigenValues for (1,1)-Tensors

4-Gradient $\partial = \partial^\nu(\partial/\partial x^\nu, \nabla)$

<table>
<thead>
<tr>
<th>Invariant</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F^{\mu\nu}$</td>
<td>$\mathbf{V} \cdot \mathbf{V}$</td>
</tr>
<tr>
<td>$F^{\mu\nu} F_{\mu\nu}$</td>
<td>$\mathbf{V} \times \mathbf{V}$</td>
</tr>
<tr>
<td>$F^{\mu\nu} F^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$</td>
<td>$\mathbf{V} \times \mathbf{V} = \mathbf{V} \times \mathbf{V}$</td>
</tr>
<tr>
<td>$F^{\mu\nu} F^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$</td>
<td>$\mathbf{V} \times \mathbf{V} = \mathbf{V} \times \mathbf{V}$</td>
</tr>
</tbody>
</table>

Importantly, the Faraday EM Tensor has only (2) linearly-independent invariants:

- Faraday Trace $F^{\mu\nu} = V^{\mu} = (F^{01}F^{23} - F^{02}F^{13}) = (0 - 0 - 0) = 0$
- Faraday Inner Product $F_{\mu\nu} F^{\mu\nu} = (F^{00} + F^{11} + F^{22} + F^{33})^2 = 0$

The 4-Gradient and 4-EMVectorPotential have (4) independent components each, for total of (8). Subtract the (2) invariants which provide constraints to get a total of (6) independent EM field components

The 3-electric components ($e = e^i$) are in the temporal-spatial sections. The 3-magnetic components ($b = b^i$) are in the only-spatial section.

$4\text{-Form Vector Potential} A = A^\mu = (\phi/c, \mathbf{a})$

The Faraday EM Tensor $F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \partial^\mu \wedge A$

**SR Tensor Invariants for Faraday EM Tensor**

4-EM Vector Potential

$A = A^\mu = (\phi/c, \mathbf{a})$
The 4-AngularMomentum Tensor \( M^{\alpha \beta} = X^\alpha P^\beta - X^\beta P^\alpha \) is an anti-symmetric tensor. The 3-mass-moment components \( (n = n) \) are in the temporal-spatial sections. The 3-angular-moment components \( (I = I) \) are in the only spatial section.

The 4-AngMom Trace\(^{[\text{M}]}\) = \( M_{\alpha \alpha}^{\alpha} = 0 \) is another classical conserved vector. The invariance is shown here to be relativistic in origin.

\[
3\text{-mass-moment} = (3) \text{3-mass-momentum}
\]

Subtract the (2) invariants which provide constraints to get a total of (6) independent components:

\[
(2) \text{invariants} \rightarrow (6) \text{independent components}
\]

\[\begin{align*}
\text{a): 4-AngMom Trace}^{[\text{M}]} &= \text{M}_{\alpha \beta}^{\alpha \beta} = (M^{00} - M^{11} - M^{22} - M^{33}) = (0 - 0 - 0 - 0) = 0 \\
\text{b): 4-AngMom Inner Product}^{[\text{M}]} &= \sum_{\alpha=0}^{3} (M^{\alpha \beta})^2 + 2\sum_{\alpha<\beta} (M^{\alpha \beta})^2 = (0) - 2(c^2 \cdot n) + 2((I-H) - (c^2 \cdot n)) = 0 \\
\text{c): 4-AngMom Asymmetric Tri}^{[\text{M}]} &= \text{Tr}^{[\text{M}]} - 3(\text{Tr}^{[\text{M}]})(M^{00}M^{00} + M^{11}M^{11} + M^{22}M^{22} + M^{33}M^{33}) = 0 \\
\text{d): 4-AngMom Asymmetric Tri}^{[\text{M}]} &= \text{Paffian}^{[\text{M}]} = [(\text{cn}(+I)) - (\text{cn}(+I)) - (\text{cn}(+I)) - (\text{cn}(+I))] = (c(n-I))^2
\end{align*}
\]

Importantly, the 4-AngMom Tensor has only (2) linearly-independent invariants:

\[\begin{align*}
\text{b): 2((I-H) - (c^2 \cdot n))} &: \text{see Wikipedia Laplace–Runge–Lenz vector, sec. Casimir Invariants} \\
\text{a): (c(n))} &\text{ and c): give } 0 = 0, \text{ and do not provide additional constraints} \\
\text{The 4-Position and Momentum have (4) independent components each, for total of (8).}
\end{align*}\]

Subtract the (2) invariants which provide constraints to get a total of (6) independent components:

\[\begin{align*}
(6) \text{ independent components of a 4x4 anti-symmetric tensor} \\
(3) \text{ 3-mass-moment } n = 3 \text{ independent components of a 4x4 tensor} \\
(3) \text{ 3-angular-momentum } I = 6 \text{ independent components of a 4x4 tensor}
\end{align*}\]

The 4-AngMom Tensor has only (2) linearly-independent invariants:

\[\begin{align*}
\text{b): } 2((I-H) - (c^2 \cdot n)) \\
\text{a): (c(n))} &\text{ and c): give } 0 = 0, \text{ and do not provide additional constraints}
\end{align*}\]

4-D Determinant = \[\begin{bmatrix}
0 & x^p x^p & x^p x^p & x^p x^p \\
x^p x^p & 0 & x^p x^p & x^p x^p \\
x^p x^p & x^p x^p & 0 & x^p x^p \\
x^p x^p & x^p x^p & x^p x^p & 0
\end{bmatrix} = 0 \]

\[\begin{align*}
\text{Asymmetric Tri-Product Tensor Invariant}^{[\text{M}]} &= 0 \\
\text{Determinant Tensor Invariant}^{[\text{M}]} &= (c(n-I))^2
\end{align*}\]
The Minkowski Metric Tensor $\eta^{\mu\nu}$ is the tensor all SR 4-Vectors are measured by.

$(2,0)$-Tensor = 4-Tensor $T^{\mu\nu}$: Has $(4+)$ Tensor Invariants (though not all independent)

a) $T^{\alpha\alpha} = \text{Trace} = \text{Sum of EigenValues for (1,1)-Tensors (mixed)}$

b) $T^{[\alpha\beta}T^{\gamma\beta]} = \text{Asymm Bi-Product}$ → Inner Product

c) $T^{[\alpha\beta\gamma\delta]} = \text{Asymm Tri-Product}$ → $\text{?Name?}$

d) $T^{[\alpha\beta\gamma\delta]} = \text{Asymm Quad-Product}$ → 4D Determinant = Product of EigenValues for (1,1)-Tensors

$\eta^{\mu\nu}$ is not defined for the standard Minkowski Metric Tensor since it is a type $(2,0)$-Tensor, all upper indices. However, they are defined for the mixed form $(1,1)$-Tensor, mixed indices.
SRQM Study: SR 4-Tensors

SR Tensor Invariants for Continuous Lorentz Transform Tensors

The Lorentz Transform Tensor \( \{ \Lambda^\mu_\nu \} \) is the tensor all SR 4-Vectors must transform by.

(2,0)-Tensor = 4-Tensor \( T^{\mu\nu} \); Has (4+) Tensor Invariants (though not all independent)

a) Lorentz Trace[\( \Lambda^\mu_\nu \)] = \{0..4..Infinity\} Lorentz Boost meets Rotation at Identity of 4
b) Lorentz Inner Product \( \Lambda^\mu_\nu \Lambda^\nu_\mu = 4 \) from \( \{\eta_{\mu\nu} \Lambda^\mu_\nu = \eta_{\delta\mu} \} \) and \( \{\eta_{\mu\nu}\}^{\nu\mu} = 4 \)
c) Lorentz AsymmTri[\( \Lambda^\mu_\nu \)] =
d) Lorentz Det[\( \Lambda^\mu_\nu \)] = +1 for Proper Transforms, Continuous Transforms Proper

An even more general version would be with a \& b as arbitrary complex values:

- SR: Lorentz Transform
  \( \delta[R^\mu_\nu] = \partial R^\mu_\nu / \partial R^\nu_\mu = \Lambda^\mu_\nu \)
  \( \Lambda^\mu_\nu = (\Lambda^{-1})_\mu^n \cdot \Lambda^\mu_\nu \cdot \eta_{\delta\mu} = \delta^\mu_\nu \)

- Product of EigenValues \( \Lambda^\mu_\nu \)
  \( \det[\Lambda^\mu_\nu] = +1 \)
  \( \partial[R^\mu_\nu] / \partial R^\nu_\mu = \Lambda^\mu_\nu \)
  \( \Lambda^\mu_\nu = (\Lambda^{-1})_\mu^n \cdot \Lambda^\mu_\nu \cdot \eta_{\delta\mu} = \delta^\mu_\nu \)

- Trace Tensor Invariant
  \( \text{Tr}[\Lambda^\mu_\nu] = \{0..4..Infinity\} \text{ Depends on "rotation" amount} \)
  \( \text{Det}[\Lambda^\mu_\nu] = +1 \)

- Inner Product Tensor Invariant
  \( \Lambda^\mu_\nu \Lambda^\nu_\rho = 4 \)
  \( \text{Asymm Tri-Product Tensor Invariant} \)

- Determinant Tensor Invariant
  \( \text{det}[\Lambda^\mu_\nu] = +1 \)
  \( \text{Proper Transform always +1} \)

EigenValues[\( R^\mu_\nu \)] = \{e^a, e^b, e^c, e^d\}

- Trace[\( R^\mu_\nu \)] = \{0..4..Infinity\}
  \( \text{Sum of EigenValues[\( R^\mu_\nu \)]} = \text{Tr}[R^\mu_\nu] = R^\mu_\nu = 1+e^a+e^b+e^c+e^d = 4 \)

- Product of EigenValues[\( R^\mu_\nu \)]
  \( \text{Det}[R^\mu_\nu] = +1 \)

- Lorentz SR Identity Tensor \( \Lambda^\nu_\mu \rightarrow R^\nu_\mu \)
  \( \text{Identity} \)

- Lorentz SR Boost Tensor \( \Lambda^\nu_\mu \rightarrow B^\nu_\mu \)

- Lorentz SR Identity
  \( \text{Lorentz SR Identity Tensor} \)
  \( \text{Lorentz SR Boost} \)

- Lorentz SR Identity
  \( \text{Lorentz SR Boost} \)

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SRQM Study: SR 4-Tensors

SR Tensor Invariants for Discrete Lorentz Transform Tensors

The Trace of various discrete Lorentz transforms varies in steps from \{-4, -2, 0, 2, 4\}.

This includes Mirror Flips, Time Reversal, and Parity Inversion – essentially taking all combinations of \pm 1 on the diagonal of the transform.

The Trace Tensor Invariant

The determinant of a Lorentz Transform \( A' \) with diagonal \( \Lambda' \) yields:

- \( \det(A') = \pm 1 \)
- Proper Transform: \( +1 \)
- Improper Transform: \( -1 \)

For proper Lorentz transforms, the determinant is always 1, and for improper, it is always -1.

Trace Tensor Invariant

The trace of a Lorentz Transform \( A' \) with diagonal \( \Lambda' \) yields:

- \( \text{Tr}(A') = \pm 1 \)
- Proper Transform: \( +1 \)
- Improper Transform: \( -1 \)

For proper Lorentz transforms, the trace is always 1, and for improper, it is always -1.

Inner Product Tensor Invariant

Lorentz SR T-Combo Tensor \( \Lambda''_{\mu\nu} \rightarrow \Lambda''_{\nu\mu} \)

- \( \partial_{\nu} \partial_{\mu} \Lambda''_{\rho\sigma} = \delta_{\rho\sigma} \Lambda''_{\mu\nu} \)

Asymmetry Tri-Product Tensor Invariant

- \( \Lambda''_{\mu\nu} = \pm \Lambda_{\mu\nu} \)
- \( \det(\Lambda'') = \pm 1 \)
- Proper Transform: \( +1 \)
- Improper Transform: \( -1 \)

Trace Tensor Invariant

The determinant of a Lorentz Transform \( A' \) with diagonal \( \Lambda' \) yields:

- \( \det(A') = \pm 1 \)
- Proper Transform: \( +1 \)
- Improper Transform: \( -1 \)

For proper Lorentz transforms, the determinant is always 1, and for improper, it is always -1.

Trace Tensor Invariant

The trace of a Lorentz Transform \( A' \) with diagonal \( \Lambda' \) yields:

- \( \text{Tr}(A') = \pm 1 \)
- Proper Transform: \( +1 \)
- Improper Transform: \( -1 \)

For proper Lorentz transforms, the trace is always 1, and for improper, it is always -1.
SRQM Study: SR 4-Tensors

More SR Tensor Invariants for Discrete Lorentz Transform Tensors

A Tensor Study of Physical 4-Vectors

SR: Lorentz Transform
\[ \partial[R^\nu] = \partial R^\nu / \partial r^\nu = \Lambda^\nu_\nu \]
\[ \Lambda^\nu_\mu = (\Lambda^\nu_\mu) = \Lambda^\mu_\nu \]
\[ \eta_{\nu\mu} = \delta_{\nu\mu} \]
\[ \text{Det} [\Lambda^\nu_\mu] = \pm 1 \]
\[ \Lambda^\mu_\nu \Lambda^\nu_\mu = 4 \]

Note:
The Flip-xy-Combo is the equivalent of a π-Rotation-z.

I suspect that this may be related to exchange symmetry and the Spin-Statistics idea that a particle-exchange is the equivalent of a spin-rotation.

A single Flip would not be an exchange because it leaves a mirror-inversion of <right-|-left>.

But the extra Flip along an orthogonal axis corrects the mirror-inversion, and would be an overall exchange because the particle is in a different location.

EigenValues[R^\nu_\nu] = Set{1, e\^\i\pi, e\^\-\i\pi, 1}

EigenValues[F^\nu_\nu] = Set{1, 1, 1, 1}

EigenValues[\eta^\nu_\mu] = Set{-1, 1, 1, 1}

EigenValues[\mu\nu] = Minkowski Delta

Product of EigenValues[\mu\nu] = Det[R^\nu_\nu] = 1 \cdot e\^\i\pi \cdot e\^\-\i\pi \cdot 1 = 1

Product of EigenValues[\eta^\nu_\mu] = Det[\eta^\nu_\mu] = 1 \cdot 1 \cdot 1 \cdot 1 = 1

Trace[T^\nu_\nu] = \eta_{\nu\mu} T^\nu_\mu = T

\[ V \cdot V = V^\nu \eta_{\nu\mu} V^\mu = (v^2)^2 - (v^0)^2 = (v^2)^2 \]

= Lorentz Scalar

SR 4-Vector
(2,0)-Tensor T^\nu_\nu = V = (v^2, v^\nu)

SR 4-Vector
(1,0)-Tensor V^\nu = V = (v^0, v^\nu)

SR 4-Scalar
(0,0)-Tensor S = Lorentz Scalar

Det[T^\nu_\nu] = \Pi_{\nu}(\lambda_\nu); \text{ with } \{\lambda_\nu\} \text{ = Eigenvalues}

Characteristic Eqns: Det[T^\nu_\nu - \lambda_k I_{4\times4}] = 0
SR 4-Scalars, 4-Vectors, 4-Tensors beautifully and elegantly display the relations between lots of different physical properties and relations. Their notation makes navigation through the physics very simple.

They also devolve very nicely into the limiting/approximate Newtonian cases of \(|v| \ll c\) by letting \(\gamma \to 1\) and \(\gamma' = \frac{d\gamma}{dt} \to 0\).

SR tells us that several different physical properties are actually dual aspects of the same thing, with the only real difference being one's point of view, or reference frame.

Examples of 4-Vectors = (1,0)-Tensors include:

\((\text{Time, Space}), (\text{Energy, Momentum}), (\text{Power, Force}), (\text{Frequency, WaveNumber}), (\text{Time Differential, Spatial Gradient}), (\text{ChargeDensity, CurrentDensity}), (\text{EM-ScalarPotential, EM-VectorPotential}), \text{etc.}\)

One can also examine 4-Tensors, which are type (2,0)-Tensors.

The Faraday EM Tensor similarly combines EM fields:

\[
\begin{array}{c}
F^{0i} = 0 & -e^i/c \\
+e^i/c & -(\epsilon^i_k b^k)
\end{array}
\]

Also, things are even more related than that.

The 4-Momentum is just a constant times 4-Velocity. The 4-WaveVector is just a constant times 4-Velocity.

In addition, the very important conservation/continuity equations seem to just fall out of the notation. The universe apparently has some simple laws which can be easy to write down by using a little math and a super notation.

\[
\text{SR 4-Vector } V = V^\alpha = (v^0, v^\alpha) = (v^0, v^x, v^y, v^z) \text{ where } (v^0, v^\alpha) = \frac{m_o E}{c^2} \quad \omega/c^2 \quad \omega/c^2
\]
SRQM Diagram: SR 4-Vectors and Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
\[ dR = (c dt, dr) \]

4-Position
\[ R = (ct, r) = \langle \text{Event} \rangle \]

4-Unit Temporal
\[ T = (1, 0) \]

4-Unit Spatial
\[ S = (n\cdot\beta, n) \]

Heart of SR

4-Dimension
\[ \partial R = 4 \]

Space Time

4-Acceleration
\[ A = \gamma(c\beta, u) = dU/d\tau \]

4-Velocity
\[ U = \gamma(c, u) = dR/d\tau \]

4-Charge Flux
\[ N = (nc, n) = n(c, u) \]

4-Current Density
\[ J = (pc, j) = \rho(c, u) \]

4-EM Vector Potential
\[ A = (\phi/c, a) \]

4-EM Potential Momentum
\[ Q = (U/c, q) = qA \]

4-Momentum
\[ P = (mc, p) = (E/c, p) \]

4-Mass Density
\[ \rho_o = \rho \]

4-Charge Density
\[ n_o = n \]

4-Coupling
\[ \phi_o = \phi/c^2 \]

4-Momentum Inc Field
\[ P_T = (E/\gamma, p_T) = (H/c, p_T) \]

4-Force
\[ F = \gamma(\dot{E}/c, \dot{f}) = dP/d\tau \]

4-Polarization
\[ E(\varepsilon, \beta, \varepsilon) \]

4-Gradient
\[ \partial[R] = \partial[R^u] = \eta^{uv} \]

4-Metric
\[ \delta_\alpha[R] = \delta[ R^\mu ] = \Lambda^\nu_\mu \]

Minkowski Metric

4-Unit Spatial
\[ (0, 2) \]

4-Unit Temporal
\[ (2, 0) \]

4-Unit Spatial
\[ (1, 1) \]

4-Unit Temporal
\[ (0, 1) \]

Lorentz Scalars

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
\[ dR = (c dt, dr) \]

4-Position
\[ R = (ct, r) = \langle \text{Event} \rangle \]

4-Unit Temporal
\[ T = (1, 0) \]

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\[ \partial R = 4 \]

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Lorentz Scalars

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Heart of SR

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\[ \partial[R] = \partial[R^u] = \eta^{uv} \]

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Minkowski Metric

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\[ (0, 2) \]

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\[ (2, 0) \]

4-Unit Spatial
\[ (1, 1) \]

4-Unit Temporal
\[ (0, 1) \]
SRQM Diagram: SRQM 4-Vectors and Lorentz Scalars / Physical Constants

A Tensor Study of Physical 4-Vectors

---

4-Displacement \( \Delta R = (c \Delta t, \Delta r) \)

4-Position \( R = (ct, r) \)

\( R' = (ct', r') \)

4-UnitTemporal \( T = \gamma(1, \beta) \)

\( T \cdot T = 1 \)

4-UnitSpatial \( S = \gamma(\hat{n}, \hat{r}) \)

\( S \cdot S = -1 \)

4-NumberFlux \( N = (nc, n) = n(c, u) \)

4-ProbCurrDensity \( J = (pc, j) = (p, j) \)

4-ProbabilityFlux \( J_{prob} = (\rho_{prob}, \rho) \)

4-Displacement \( \Delta R = (c \Delta t, \Delta r) \)

4-Position \( R = (ct, r) \)

4-UnitTemporal \( T = \gamma(1, \beta) \)

\( T \cdot T = 1 \)

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4-ProbabilityFlux \( J_{prob} = (\rho_{prob}, \rho) \)

---

4-Derivative \( \frac{\partial R}{\partial t} = 4 \) SpaceTime Dimension

4-Acceleration \( A = \gamma(\epsilon, \gamma u + \gamma a, u) = dU/dt \)

4-Oscillation \( E = (\epsilon, \gamma u + \gamma a, u) = dU/dt \)

4-Spin \( S = (s, u) = (s \cdot \beta, \beta) \)

Spin is actually an outcome of Pauli\'s Invariance, not QM

4-ProbabilityFlux \( J_{prob} = (\rho_{prob}, \rho) \)

---

4-Displacement \( \Delta R = (c \Delta t, \Delta r) \)

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4-Spin \( S = (s, u) = (s \cdot \beta, \beta) \)

Spin is actually an outcome of Pauli\'s Invariance, not QM

4-ProbabilityFlux \( J_{prob} = (\rho_{prob}, \rho) \)
SR Gradient 4-Vectors = (1,0)-Tensors
SR Gradient One-Forms = (0,1)-Tensors

4-Vector = Type (1,0)-Tensor
4-Position $\mathbf{R} = R^\mu = (ct,r)$
4-Gradient $\partial \mathbf{R} = \partial = \partial^\mu = \partial/\partial R_\mu = (\partial/\partial c, -\nabla)$

**Standard 4-Vector**
4-Position $\mathbf{R} = R^\mu = (ct,r)$
4-Velocity $\mathbf{U} = U^\mu = \gamma(c,u)$
4-Momentum $\mathbf{P} = P^\mu = (E/c,p)$
4-WaveVector $\mathbf{K} = K^\mu = (\omega/c,k)$

**Related Gradient 4-Vector (from index-raised Gradient One-Form)**
4-Position Gradient $\partial \mathbf{R} = \partial R^\mu = \partial/\partial R_\mu = (\partial/\partial c, -\nabla)$
4-Velocity Gradient $\partial \mathbf{U} = \partial U^\mu = \partial/\partial U_\mu = (\partial/\partial c, -\nabla)$
4-Momentum Gradient $\partial \mathbf{P} = \partial P^\mu = \partial/\partial P_\mu = (\partial/\partial c, -\nabla)$
4-Wave Gradient $\partial \mathbf{K} = \partial K^\mu = \partial/\partial K_\mu = (\partial/\partial c, -\nabla)$

In each case, the (Whichever)Gradient 4-Vector is derived from an SR One-Form or 4-CoVector, which is a type (0,1)-Tensor
ex. One-Form Position Gradient $\partial R^\nu = \partial/\partial R^\nu = (\partial/\partial c, -\nabla)$

The (Whichever)Gradient 4-Vector is the index-raised version of the SR One-Form (Whichever)Gradient ex. 4-Position Gradient $\partial R^\nu = \partial/\partial R_\mu = (\partial/\partial c, -\nabla)$ = $\eta^{\mu\nu}(\partial R^\nu/c, -\nabla)$

This is why the 4-Gradient is commonly seen with a minus sign in the spatial component, unlike the other regular 4-Vectors, which have all positive components.

4-Tensors can be constructed from the Tensor Outer Product of 4-Vectors
Some Basic 4-Vectors

Minkowski SpaceTime Diagram

Events & Dimensions

past

future

now · here

elsewhere

\( c \)

\(-c\)

\( \Delta t \) time-like interval

\( \Delta r \) space-like interval

\( \Delta R_{CM} = (c\Delta t, \Delta r) \)

Note the matching dimensional units: (4D SpaceTime)

\( \Delta t \) is [time], \( |\Delta r| \) is [length]

Classical Mechanics

3-displacement

\( \Delta r = \Delta r^\rightarrow (\Delta x, \Delta y, \Delta z) \)

Note the separate dimensional units: (time + 3D space)

\( \Delta t \) is [time], \( |\Delta r| \) is [length]

Special Relativity

\( \Delta t \) time-like interval (+)

\( c \) light-like interval (0) = null

\( \Delta r \) space-like interval (-)

\( \Delta R \cdot \Delta R = [(c\Delta t)^2 - \Delta r \cdot \Delta r] = 0 \)

\( (c\Delta t)^2 \) Time-Like (+)

\( -(|\Delta r|)^2 \) Space-like (-)

Note the matching dimensional units: (4D SpaceTime)

\( (c\Delta t) \) is [length/time]*[time] = [length], \( |\Delta r| \) is [length], \( |\Delta R| \) is [length]

\( \tau \) is the Proper Time = “rest-time”, time as measured by something not moving spatially

The Minkowski Diagram provides a great visual representation of SpaceTime

SR 4-Vector

\((2,0)\)-Tensor \( T_{\mu}^{\nu} \)

\((1,1)\)-Tensor \( T^\nu_\mu \), or \( T_{\nu}^\mu \)

\((0,2)\)-Tensor \( T_{\mu \nu} \)

SR 4-CoVector

\((0,1)\)-Tensor \( V_{\mu} = (v_0, -v) \)

SR 4-Scalar

\((0,0)\)-Tensor \( S \)

Lorentz Scalar

Classical (scalar)

3-vector

Galilean Invariant

Not Lorentz Invariant

Lorentz Scalar

Classical (scalar)

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson
Some Basic 4-Vectors

Minkowski SpaceTime Diagram, WorldLines, LightSpeed to the Future!

\[ \Delta R = (c \Delta t, \Delta r) \]

4-Position
\[ R = (ct, r) = \langle \text{Event} \rangle \]

\[ \Delta R \cdot \Delta R = [(c \Delta t)^2 - \Delta r \cdot \Delta r] = 0 \]

\[ (c \Delta t)^2 \text{ for time-like (+) } \]
\[ - (\Delta r)^2 \text{ for space-like (−) } \]

\[ \Delta R \cdot \Delta R = [(c \Delta t)^2 - \Delta r \cdot \Delta r] = 0 \]

\[ U = \gamma (c, u) = dR/d\tau \]
\[ U = \gamma (c, u) \]
\[ U \cdot U = c^2 \]
\[ U \cdot U = \gamma (c, u) \cdot \gamma (c, u) = \gamma (c^2 - u \cdot u) = (c^2) \]
\[ \gamma = 1/\sqrt{1-(u/c)^2} = 1/\sqrt{1-(\beta)^2} \]

Massive particles move temporally into future at the speed-of-light (c) in their own rest-frame.

Massless particles (photonic) move nullly into the future at the speed-of-light (c), and have no rest-frame.

\[ \text{Trace}[T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T \]
\[ V \cdot V = V^\mu \eta^\mu_\nu V_\nu = [(v^0)^2 - v \cdot v] = (v^0)^2 \]
\[ = \text{Lorentz Scalar} \]
**SR Invariant Intervals**

**Minkowski Diagram: Lorentz Transform**

Since the SpaceTime magnitude of \( U \) is a constant (c), changes in the components of \( U \) are like rotating the 4-Vector without changing its length. It keeps the same magnitude. Rotations, purely spatial changes, (eg. along x,y) result in circular displacements. Boosts, or temporal-spatial changes, (eg. along x,t) result in hyperbolic displacements. The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

\[ U \cdot U = \gamma(c,u) - \gamma(c,u) = \gamma^2(c^2 - u \cdot u) = (c^2) \]

The Minkowski Diagram provides a great visual representation of SpaceTime.
SR Invariant Intervals

Minkowski Diagram

Since the SpaceTime magnitude of $U$ is a constant (c), changes in the components of $U$ are like rotating the 4-Vector without changing its length. It keeps the same magnitude. Rotations, purely spatial changes, {eg. along x,y} result in circular displacements. Boosts, or temporal-spatial changes, {eg. along x,t} result in hyperbolic displacements. The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

The Minkowski Diagram provides a great visual representation of SpaceTime.
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration

SpaceTime Kinematics

4-Velocity

\[ U = \gamma (c, u) \]

4-Position

\[ R = (ct, r) \]

4-Acceleration

\[ A = \gamma (c \gamma', u + \gamma a) \]

Special Relativity

|v| = |u| = 0 → c
\[ \gamma = 1 / \sqrt{1 - (v/c)^2} \]

Classical Mechanics

|v| = |u| << c
\[ \gamma' \rightarrow 0 \]

ProperTime

\[ R \cdot U / U \cdot U = (ct, r) \cdot \gamma (c, u) c^2 = \gamma (c^2 t - r \cdot u) / c^2 = (c t_o) / c^2 \]
\[ t_o = \tau \]

ProperTime Derivative

\[ U \cdot \partial \left[ \ldots \right] \frac{d}{d\tau} \left[ \ldots \right] \gamma \frac{d}{dt} \left[ \ldots \right] = d/d\tau \]

The relativistic Gamma factor
\[ \gamma = 1 / \sqrt{1 - (v/c)^2} \]

The 1st order Newtonian Limit gives
\[ \gamma \sim 1 + O[(v/c)^2] \]

The 2nd order Newtonian Limit gives
\[ \gamma \sim 1 + (v/c)^2/2 + O[(v/c)^4] \]

For historical reasons, velocity can be represented by either (v) or (u)

SR 4-Tensor

(2,0)-Tensor \( T^{\mu \nu} \)
(1,1)-Tensor \( T^{\mu} \), or \( T' \)
(0,2)-Tensor \( T_{\mu \nu} \)

SR 4-Vector

(1,0)-Tensor \( V^{\mu} = (v^0, v) \)

SR 4-CoVector

(0,1)-Tensor \( V_{\mu} = (v_0, -v) \)

SR 4-Scalar

(0,0)-Tensor \( S \)

Galilean Invariant

Not Lorentz Invariant

Classical (scalar)

3-vector

Lorentz Scalar

Lorentz Scalar

\[ \text{Trace}[T^{\mu \nu}] = \eta_{\mu \nu} T^{\mu \nu} = T^{\mu \nu} = T \]
\[ V \cdot V = V^{\mu} \eta_{\mu \nu} V^{\nu} = (v^0)^2 - V \cdot V = (v^0)^2 \]

SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SciRealm.org

John B. Wilson

4-Vectors:

\[ R = \langle \text{Event} \rangle \]
\[ U = dR/d\tau \]
\[ A = dU/dt \]

Newtonian/Classical Limit

4-Gradient

\[ \partial = (\partial_t / c, \nabla) \rightarrow (\partial_t / c, -\partial_x, -\partial_y, -\partial_z) \]

Trace[\( T^{\mu \nu} \)] = \( \eta_{\mu \nu} T^{\mu \nu} = T^{\mu \nu} = T \)

Lorentz Scalar

Lorentz Scalar

\[ V \cdot V = V^{\mu} \eta_{\mu \nu} V^{\nu} = (v^0)^2 - V \cdot V = (v^0)^2 \]

Lorentz Scalar

\[ \text{Trace}[T^{\mu \nu}] = \eta_{\mu \nu} T^{\mu \nu} = T^{\mu \nu} = T \]
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Lorentz Scalar

\[ \text{Trace}[T^{\mu \nu}] = \eta_{\mu \nu} T^{\mu \nu} = T^{\mu \nu} = T \]
\[ V \cdot V = V^{\mu} \eta_{\mu \nu} V^{\nu} = (v^0)^2 - V \cdot V = (v^0)^2 \]

Lorentz Scalar
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration, 4-Momentum, 4-Force

SpaceTime Dynamics

4-Vectors:

- \( 
  R = (c t, r) 
  \)

- \( 
  U = \frac{d R}{d \tau} = c t, \mathbf{r} \)/c^2 = (c^2 t^2)/c^2 \)

- \( 
  A = \frac{d U}{d \tau} = \gamma (c u) \)

- \( 
  P = m \gamma (c u) \)

- \( 
  F = \frac{d P}{d \tau} = \gamma (c u) \)

4-Gradient

\[ 
\partial = (\partial \tau, \mathbf{\nabla}) \rightarrow (\partial \tau, -\partial x, -\partial y, -\partial z) 
\]

ProperTime Derivative

\[ 
U \cdot \partial = \gamma (c u) \cdot (\partial \tau, -\mathbf{\nabla}) = \gamma (\partial \tau + u \cdot \mathbf{\nabla}) = \gamma d/dt 
\]

This group of 4-Vectors are the main ones that are connected by the ProperTime Derivative.

The classical part of it, the convective derivative, \( (\partial \tau + u \cdot \mathbf{\nabla}) \), is known by many different names:

The convective derivative is a derivative taken with respect to a moving coordinate system. It is also called the advective derivative, derivative following the motion, hydrodynamic derivative, Lagrangian derivative, material derivative, particle derivative, substantial derivative, substantival derivative, Stokes derivative, or total derivative.

Special Relativity

\(|v| = |u| = 0 \rightarrow c \)
\( \gamma = 1/\sqrt{1-(v/c)^2} \)

This group of 4-Vectors are the main ones that are connected by the ProperTime Derivative.
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Momentum, \( E=mc^2 \)

**Special Relativity**

| \[ \gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \] |
| \( \gamma \) = 1/\sqrt{1-(v/c)^2} |
| The 1st order Newtonian Limit gives \( \gamma \sim 1 + O((v/c)^2) \) |
| The 2nd order Newtonian Limit gives \( \gamma \sim 1 + (v/c)^2/2 + O((v/c)^4) \) |

Since time:space don’t mix in CM, typically use energy \( E \) & 3-momentum \( p \) separately

The relativistic Gamma factor \( \gamma = 1/\sqrt{1-(v/c)^2} \)

**Classical Mechanics**

| \[ \gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \] |
| \( \gamma \) = 1/\sqrt{1-(v/c)^2} |
| The 1st order Newtonian Limit gives \( \gamma \sim 1 + O((v/c)^2) \) |
| The 2nd order Newtonian Limit gives \( \gamma \sim 1 + (v/c)^2/2 + O((v/c)^4) \) |

Typically use energy \( E \) & 3-momentum \( p \) separately

Temporal part: \( E = \gamma m_c^2 = mc^2 \)

\( E = E_0 + (\gamma - 1)m_0c^2 \)

\( E = m_0c^2 + (\gamma - 1)m_0c^2 \)

\( E = E_0 + (\gamma - 1)E_0 \)

(rest) + (kinetic)

Spatial part: \( p = \gamma m_0 u = mu \)

\( p \sim (1)m_0 u = m_0 u \rightarrow mu \)

For historical reasons, velocity can be represented by either \( v \) or \( u \)

\( V \cdot V = V^\mu V_\mu = (v^\mu - v_\mu) = (v^2)^2 \)

\( = \) Lorentz Scalar

\( \text{Trace}[T_{\mu\nu}] = \eta_{\mu\nu} T_{\mu\nu} = T_{\mu\nu} = T \)

\( V \cdot V = V^\mu \eta_{\mu\nu} V_\nu = (v^\mu - v_\mu) = (v^2)^2 \)

Lorentz Scalar
SRQM: Some Basic 4-Vectors
4-Velocity, 4-Acceleration, SpaceTime Orthogonality

4-Velocity
\[ U = \gamma (c, u) \]

4-Acceleration
\[ A = \gamma (c\gamma', c\gamma' + u + a) \]

Proper Time Derivative
\[ \gamma \frac{d}{d\tau} \]

Proper Time Derivative of 4-Velocity
\[ U \cdot \partial = \gamma (\partial_t/c, -\nabla ) \]

Proper Time Derivative of 4-Acceleration
\[ A \cdot \partial = \gamma (\partial_t/c, -\nabla ) \]

4-Gradient
\[ \partial = \left( \frac{\partial_t}{c}, \nabla \right) \]

蜕变公式
\[ \delta R = 4 \]

空间时度

4-Position
\[ R = (ct, r) \]

4-Dimension

SR 4-Tensor
(2,0)-Tensor \( T^\mu_{\nu} \)
(1,1)-Tensor \( T^\mu_{\nu} \), or \( T_\mu^\nu \)
(0,2)-Tensor \( T_{\mu\nu} \)

SR 4-Vector
(1,0)-Tensor \( V^\mu = V = (v^0, v) \)

SR 4-CoVector
(0,1)-Tensor \( V_\mu = (v_0, -v) \)

SR 4-Scalar
(0,0)-Tensor \( S \)

Lorentz Scalar

Trace\( [T] = \eta_{\mu\nu} T^{\mu\nu} = T^\mu_{\mu} = T \)

\[ V \cdot V = V^\mu \eta_{\mu\nu} V^\nu = \left( v^0 \right)^2 - \left( v^\mu \right)^2 = \left( v^0 \right)^2 \]

Lorentz Scalar

The Lorentz Scalar Product can be used to show SpaceTime orthogonality when the result is zero.

\[ U \cdot U = c^2 \]

\[ d/\tau[U \cdot U] = d/\tau[c^2] = 0 \]

\[ d/\tau[U \cdot U] = d/\tau[U] \cdot U + U \cdot d/\tau[U] = A \cdot U + U \cdot A = 2(U \cdot A) = 0 \]

\[ U \cdot A = U' U'' = 0 : The 4-Velocity is SpaceTime orthogonal to it's 4-Acceleration. \]

4-Velocity is the direction along a WorldLine.
4-Acceleration is the thing which causes a WorldLine to bend/curve.
SRQM: Some Basic 4-Vectors

4-Displacement, 4-Velocity, Relativity of Simultaneity

4-Velocity
\[ U = \gamma (c, u) \]

4-Gradient
\[ \partial = (\partial_t / c, -\nabla) \]

4-Acceleration
\[ A = \gamma (c \gamma', \gamma' u + \gamma a) \]

4-Displacement
\[ \Delta X = (c \Delta t, \Delta x) \]
\[ dX = (ct, dx) \]

4-Position
\[ X = (ct, x) \]

If Lorentz Scalar \( \Delta X \cdot U = 0 = c^2 \Delta t \), then the ProperTime displacement \( \Delta \tau \) is zero, and the event separation \( \Delta X = X_2 - X_1 \) is orthogonal to the worldline \( U \).

\( X_1 \) and \( X_2 \) are therefore simultaneous for the observer on this worldline \( U \).

Examining the equation we get \( \gamma (c^2 \Delta t - u \cdot \Delta x) = 0 \). The coordinate time difference between the events is \( \Delta t = u \cdot \Delta x / c^2 \).

The condition for simultaneity in an alternate frame (moving at 3-velocity \( u \) wrt. the worldline \( U \)) is \( \Delta t = 0 \), which implies \( u \cdot \Delta x = 0 \).

This can be met by:
- \( |u| = 0 \), the alternate observer is not moving wrt. the events, i.e. is on worldline \( U \) or on a worldline parallel to \( U \).
- \( |\Delta x| = 0 \), the events are at the same spatial location (co-local).
- \( u \cdot \Delta x = 0 \), the alternate observer's motion is perpendicular (orthogonal) to the spatial separation \( \Delta x \) of the events in that frame.

If none of these conditions is met, then the events will not be simultaneous in the alternate reference frame.

This is the mathematics behind the concept of Relativity of Simultaneity.
SR Diagram:  
SR Motion * Lorentz Scalar = Interesting Physical 4-Vector
SRQM Diagram:  
**SRQM Motion * Lorentz Scalar**

= Interesting Physical 4-Vector

### SRQM Interpretation of QM

- **SR → QM**
- **SR 4-Tensor** (2,0)-Tensor $T^{iv}$ (1,1)-Tensor $T^v$, or $T_{iv}$  (0,2)-Tensor $T_{px}$  
- **SR 4-Vector** (1,0)-Tensor $V^i = V (\hat{v}^i, v)$  
- **SR 4-CoVector** (0,1)-Tensor $V_\nu = (v_\nu, -\nu)$
- **SR 4-Scalar** (0,0)-Tensor $S$

#### Lorentz Scalar

- $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

#### Maxwell EM Wave Eqn

- $\Box A = \frac{4\pi}{c^2} \beta \cdot j$  

---

**Interesting note:**

- Most 4-Vectors have 4 independent components. (1 temporal, 3 spatial)

- The 4-Velocity has only the 3 spatial however, due to its invariant magnitude $\mathbf{U} \cdot \mathbf{U} = c^2$.

- This fact allows one to multiply it by a Lorentz Scalar to make a new 4-Vector with 4 independent components, as shown in the diagram.

- Proof of non-varying $(c)$
SRQM Diagram: ProperTime Derivative

**Very Fundamental Results**

- **4-Displacement**: $\Delta R = (c\Delta t, \Delta r)$
- **4-Position**: $R = (ct, r)$
- **4-Gradient**: $\partial \cdot R = 4$ SpaceTime Dimension

**SRQM Diagram**

- **4-Vector**: $U = \gamma(c, u)$
- **4-Momentum**: $P = (E/c, p)$
- **4-Force**: $F = \gamma(E/c, f)$

**SR 4-Tensor**

- $(2,0)$-Tensor $T^{\mu\nu}$
- $(1,1)$-Tensor $T^\mu\nu$ or $T\nu^\mu$
- $(0,2)$-Tensor $T_{\mu\nu}$

**4-Vector SRQM Interpretation of QM**

- **SR 4-Vector**: $V^\nu = V = (\sqrt{\gamma}, 0, 0, 0)$
- **SR 4-CoVector**: $V_\mu = (V_\mu, 0, 0, 0)$
- **SR 4-Scalar**: $S = 0$

4-Vectors:

- $R = <Event>$
- $U = dR/d\tau$
- $A = dU/d\tau$

**SRQM Diagram Productions**

- **ProperTime Derivative**: $\partial \cdot \tau = \gamma(c, u)\cdot(\partial/c, -\nabla) = \gamma(\partial + u \cdot \nabla)$
- **ProperTime Derivative**: $\gamma d/d\tau = d/d\tau$

**Continuity of 4-Velocity Flow**: $\partial \cdot U = 0$

- **Acceleration of Event**: $A = \gamma(c, u)\cdot(\partial/c, -\nabla) = \gamma(\partial + u \cdot \nabla)$

4-Vectors:

- $R = <Event>$
- $U = dR/d\tau$
- $A = dU/d\tau$

**SR 4-Tensor**

- $(2,0)$-Tensor $T^{\mu\nu}$
- $(1,1)$-Tensor $T^\mu\nu$ or $T\nu^\mu$
- $(0,2)$-Tensor $T_{\mu\nu}$

**SR 4-Scalar**

- $(0,0)$-Tensor $S$
- Lorentz Scalar

**4-Gradient**: $\partial = (\partial/c, -\nabla)$

**4-Displacement**: $\Delta R = (c\Delta t, \Delta r)$

**SR 4-Tensor**

- $(2,0)$-Tensor $T^{\mu\nu}$
- $(1,1)$-Tensor $T^\mu\nu$ or $T\nu^\mu$
- $(0,2)$-Tensor $T_{\mu\nu}$

**SR 4-Scalar**

- $(0,0)$-Tensor $S$
- Lorentz Scalar

**4-Vectors**

- $R = <Event>$
- $U = dR/d\tau$
- $A = dU/d\tau$

**SRQM Diagram Productions**

- **ProperTime Derivative**: $\partial \cdot \tau = \gamma(c, u)\cdot(\partial/c, -\nabla) = \gamma(\partial + u \cdot \nabla)$
- **ProperTime Derivative**: $\gamma d/d\tau = d/d\tau$
Local Continuity of 4-Velocity leads to all the Conservation Laws

Conservation Laws:

- **Charge Conservation**
  \[ \partial \cdot J = 0 \]
  \[ \partial \cdot J = (\partial_t \rho + \mathbf{j} \cdot \nabla) = 0 \]

- **Conservation of Proper Time Derivative**
  \[ \partial \cdot U = 0 \]
  \[ \partial \cdot (\text{Lorentz Scalar}) U = 0 \]
  \[ \partial \cdot (\text{Interesting 4-Vector}) = 0 \]

Example:

\[ \partial \cdot (\rho c) U = 0 \]
\[ \partial \cdot J = 0 \]
\[ (\partial / c \ p c + \nabla \cdot j) = 0 \]
\[ (\partial / c + \nabla \cdot j) = 0 \]

= Conservation of Charge
= A Continuity Equation

**Conservation of Proper Time Derivative**

\[ \partial \cdot U = \gamma (\mathbf{c}_0, \mathbf{u}) \]

4-Acceleration

\[ \mathbf{A} = \gamma (c', \mathbf{u} + \gamma \mathbf{a}) \]

SRQM Diagram:

- **Partial Derivative**
  \[ \partial \cdot R = 0 \]
  \[ \partial \cdot U = 0 \]

- **Total Derivative**
  \[ dR = c d\Delta t, d\Delta r \]
  \[ dR = c d\Delta t, d\Delta r \]

- **Minkowski Metric**
  \[ \partial [\cdots] = \gamma \frac{d}{dt} [\cdots] \]
  \[ \partial [\cdots] = \gamma \frac{d}{d\tau} [\cdots] \]

- **Lorentz Scalar**
  \[ V \cdot V = V^\mu \eta_{\mu \nu} V^\nu = (v^0)^2 - \mathbf{v} \cdot \mathbf{v} = (v^0)^2 \]
  \[ = \text{Lorentz Scalar} \]

SR 4-Tensor

- **(2,0)-Tensor** \( T^{\mu \nu} \)
- **(1,1)-Tensor** \( T^\nu \), or \( T^\nu \)
- **(0,2)-Tensor** \( T_{\mu \nu} \)

SR 4-CoVector

- **(1,0)-Tensor** \( V^\mu = (v^\nu, v) \)
- **(0,1)-Tensor** \( V_\mu = (v^\nu, v) \)

SR 4-Scalar

- **(0,0)-Tensor** \( S \)
- **Lorentz Scalar**
These are Fluid or Density -type Conservation/Continuity Laws

These are Individual Particle/Wave/Delta-function Conservation/Continuity Laws

SR 4-Tensor
(2,0)-Tensor $T^{ix}$
(1,1)-Tensor $T^{i}$, or $T^{i}_{o}$
(0,2)-Tensor $T^{ij}_{px}$

SR 4-Vector
(1,0)-Tensor $V^{i} = (\nabla^{i} \psi)$
SR 4-CoVector
(0,1)-Tensor $V_{i} = (\nabla_{i} \psi)$

SR 4-Scalar
(0,0)-Tensor $S$

Lorentz Scalar

Existing SR Rules
Quantum Principles

Conservation Laws, Continuity Eqns

SRQM Diagram:
SRQM Motion * Lorentz Scalar

Conservation Laws:

All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local spatial region.

Conservation of Charge:
$\partial \cdot J = (\partial \mu + \nabla \cdot j) = 0$

Conservation of Mass:
$\partial \cdot G = 0$

Conservation of Charge:
$\partial \cdot J = 0$

Conservation of Mass:
$\partial \cdot m = 0$

SR → QM

4-Vector SRQM Interpretation of QM

SciRealm.org
John B. Wilson
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Gradient, Time Dilation

The Minkowski Diagram provides a great visual representation of SpaceTime.

at-rest
worldline \( U_0 \)
\((u=0)\)
fully temporal

const inertial motion
worldline \( U \)
\((0<u<c)\)
trades some time for space

Everything moves into future (+t)
at the speed-of-light (c)
in its own spatial rest-frame

Since the SpaceTime magnitude of \( U \) is a constant, changes in the components of \( U \) are like “rotating” the 4-Vector without changing its length. However, as \( U \) gains some spatial velocity, it loses some “relative” temporal velocity. Objects that move in some reference frame “age” more slowly relative to those at rest in the same reference frame.

Time Dilation!
\( \Delta t = \gamma \Delta \tau = \gamma \Delta t_0 \)
\( \frac{d}{d\tau} = \gamma \frac{d}{dt} \)

Each observer will see the other as aging more slowly; similarly to two people moving oppositely along a train track, seeing the other as appearing smaller in the distance.
SRQM: Some Basic 4-Vectors

SR 4-WaveVector K

4-WaveVector, aka. Wave 4-Vector, solution of d’Alembertian Wave Eqn.

\[ K = (\omega/c, k) = (\omega/c, \omega n/\text{phase}) \]

There are multiple ways of writing out the components of the 4-WaveVector, with each one giving an interesting take on what the 4-WaveVector means.

An SR wave \( \Psi \) is actually composed of two tensors:
1. 4-Vector propagation part = \( K^\alpha \), (the engine)
2. Variable amplitude part = A (the load), depends on what is waving...

4-Vector Propagation Part = \( K^\alpha \)
- \( K^\alpha \cdot \pd{t} = (\omega t - k \cdot r) \)
- \( (1/c \pd{\tau} - \lambda \cdot r / \lambda) \)
- \( = -\Phi_{\text{phase,plane}} \)

4-Space Tensor: \( A_{\mu\nu} = \Lambda^\mu_{\nu} e^{i k^\alpha x^\alpha} \)

The wave tensor-type will match the tensor-type A as the propagation part \( e^{i k^\alpha x^\alpha} \) is overall dimensionless.

One comparison I find very interesting is:
- \( R^\cdot R = (c t)^2 - r \cdot r \)
- \( K^\cdot K = (\lambda^2 - \omega^2) \)
- \( \partial \partial = (\partial/c \partial \tau)^2 \)

I believe the last one is correct: \( (\partial/c \partial \tau)^2 \) = 0. The 4-Acceleration seen in the ProperTime Frame = RestFrame = 0

Normally \( (\partial/c \partial \tau)^2 \) = \( A_\tau \), which could be non-zero. But that is for the total derivative, not the partial derivative.

An SR wave \( \Psi \) is: \[ \Psi(X) = A e^{-i(K \cdot X)} \]

- Explicit form of an SR plane wave
- Complete wave is a superposition of multiple plane waves:

\[ \bar{\Psi}(X) = [A e^{-i(K \cdot X)} - iK \bar{A} e^{-i(K \cdot X)}] = -iK \bar{A} e^{-i(K \cdot X)} \]

\( \bar{\Psi} \) = -iK as the condition for a complex-valued plane wave.
SRQM: Some Basic 4-Vectors

4-Velocity, 4-WaveVector

Wave Properties, Relativistic Doppler Effect

Relativistic SR Doppler Effect

(\( \hat{n} \)) here is the unit-directional 3-vector of the photon

Choose an observer frame for which:

\[ K = (\omega/c, k), \text{ with } k, \hat{n} \text{ pointing toward observer} \]

\[ \mathbf{U}_{\text{obs}} = (c, 0) \quad K \cdot \mathbf{U}_{\text{obs}} = (\omega/c, k) \cdot (c, 0) = \omega = \omega_{\text{obs}}^{\circ} \]

\[ \mathbf{U}_{\text{emit}} = (\gamma(c, u)) \quad K \cdot \mathbf{U}_{\text{emit}} = (\omega/c, k) \cdot (\gamma(c, u)) = (\omega - k \cdot u) = \omega_{\text{emit}}^{\circ} \]

\[ K \cdot \mathbf{U}_{\text{obs}} / K \cdot \mathbf{U}_{\text{emit}} = \omega_{\text{obs}}^{\circ} / \omega_{\text{emit}}^{\circ} = \omega/\gamma(\omega - k \cdot u) \]

For photons, \( K \) is null \( \rightarrow K \cdot K = 0 \rightarrow k = (\omega/c)\hat{n} \)

\[ \omega_{\text{obs}}/\omega_{\text{emit}}^{\circ} = \omega/[\gamma(\omega - (\omega/c)\hat{n} \cdot u)] = 1/[\gamma(1 - |\hat{n} \cdot \beta|)] = 1/[\gamma(1 - |\beta| |\cos(\theta_{\text{obs}})|)] \]

\[ \omega_{\text{obs}}/\omega_{\text{emit}} = \gamma \omega_{\text{obs}}^{\circ} ([\gamma \omega_{\text{emit}}^{\circ}]) = \omega_{\text{obs}}^{\circ} / \omega_{\text{emit}}^{\circ} \]

\[ \omega_{\text{obs}} = \omega_{\text{emit}}/[\gamma(1 - \hat{n} \cdot \beta)] = \omega_{\text{emit}}^* [1 + |\beta|] * [1 - |\beta|] / (1 - \hat{n} \cdot \beta) \]

with \( \gamma = 1/\sqrt{1 - \beta^2} = 1/(\sqrt{[1 + |\beta|] * [1 - |\beta|]})) \)

For motion of emitter \( \beta \): (in observer frame of reference)

Away from obs, \( (\hat{n} \cdot \beta) = -\beta \), \( \omega_{\text{obs}} = \omega_{\text{emit}}^* [1 + |\beta|] / (1 + |\beta|) = \text{Red Shift} \)

Toward obs, \( (\hat{n} \cdot \beta) = +\beta \), \( \omega_{\text{obs}} = \omega_{\text{emit}}^* [1 + |\beta|] / (1 - |\beta|) = \text{Blue Shift} \)

Transverse, \( (\hat{n} \cdot \beta) = 0, \omega_{\text{obs}} = \omega_{\text{emit}}^* \gamma = \text{Transverse Doppler Shift} \)

The Phase Velocity of a Photon \( \{v_{\text{phase}} = c\} \) equals the Particle Velocity of a Photon \( \{u = c\} \)

The Phase Velocity of a Massive Particle \( \{v_{\text{phase}} > c\} \) is greater than the Velocity of a Massive Particle \( \{u < c\} \)
SRQM: Some Basic 4-Vectors

4-Velocity, 4-WaveVector

Wave Properties, Relativistic Aberration

\[ \mathbf{K} \cdot \mathbf{U} = \gamma (\omega - \mathbf{k} \cdot \mathbf{u}) = \omega_o \]

\[ \omega_o/c^2 \]

\[ \mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{n}/\nu_{\text{phase}}) \]

\( \omega/c^2 \mathbf{U} \)

\[ \mathbf{K} \cdot \mathbf{K} = (\omega_o/c)^2 \]

Taking just the spatial components of the 4-WaveVector:

\[ \omega n/\nu_{\text{phase}} = (\omega/c^2) \mathbf{u} \]

\[ \mathbf{n}/\nu_{\text{phase}} = (\mathbf{u}/c^2) \]

\[ \mathbf{u} \cdot \nu_{\text{phase}} = c^2 \]

\[ \nu_{\text{group}} \cdot \nu_{\text{phase}} = c^2, \text{ with } \mathbf{u} = \nu_{\text{group}} \]

Wave Group velocity (\( \nu_{\text{group}} \)) is mathematically the same as Particle velocity (\( \mathbf{u} \)).

Wave Phase velocity (\( \nu_{\text{phase}} \)) is the speed of an individual plane-wave.

The Phase Velocity of a Photon \( \{ \nu_{\text{phase}} = c \} \) equals the Particle Velocity of a Photon \( \{ \mathbf{u} = c \} \)

The Phase Velocity of a Massive Particle \( \{ \nu_{\text{phase}} > c \} \) is greater than the Velocity of a Massive Particle \( \{ \mathbf{u} < c \} \)

\[ \text{Relativistic SR Doppler Effect} \]

\[ \omega_{\text{obs}} = \omega_{\text{emit}}/[\gamma (1 - \mathbf{n} \cdot \mathbf{\beta})] = \omega_{\text{emit}}/[\gamma (1 - |\mathbf{\beta}|\cos[\theta_{\text{obs}}])] \]

Change reference frames with \( \{ \text{obs} \rightarrow \text{emit} \} \) & \( \{ \mathbf{\beta} \rightarrow -\mathbf{\beta} \} \)

\[ \omega_{\text{emit}} = \omega_{\text{obs}}/[\gamma (1 + \mathbf{n} \cdot \mathbf{\beta})] = \omega_{\text{obs}}/[\gamma (1 + |\mathbf{\beta}|\cos[\theta_{\text{emit}}])] \]

\[ (\omega_{\text{obs}})^*(\omega_{\text{emit}}) = (\omega_{\text{emit}}/[\gamma (1 - |\mathbf{\beta}|\cos[\theta_{\text{obs}}])]^*(\omega_{\text{obs}}/[\gamma (1 + |\mathbf{\beta}|\cos[\theta_{\text{emit}}])]) \]

\[ 1 = (1/[\gamma (1 - |\mathbf{\beta}|\cos[\theta_{\text{obs}}])]^*(1/[\gamma (1 + |\mathbf{\beta}|\cos[\theta_{\text{emit}}])]) \]

\[ 1 = (\gamma (1 - |\mathbf{\beta}|\cos[\theta_{\text{obs}}])^*(\gamma (1 + |\mathbf{\beta}|\cos[\theta_{\text{emit}}]) \]

\[ 1 = \gamma^2 (1 - |\mathbf{\beta}|\cos[\theta_{\text{obs}}])^*(1 + |\mathbf{\beta}|\cos[\theta_{\text{emit}}]) \]

Solve for \( |\mathbf{\beta}|\cos[\theta_{\text{obs}}] \) and use \( \{ (\gamma^2 - 1) = \beta^2 \gamma^2 \} \)

\[ \text{Relativistic SR Aberration Effect} \]

\[ \cos[\theta_{\text{obs}}] = (\cos[\theta_{\text{emit}}] + |\mathbf{\beta}|) / (1 + |\mathbf{\beta}|\cos[\theta_{\text{emit}}]) \]
SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

{ P = (E/c, p) = -\partial[S] = (\vec{\partial}/c\partial[t][S], \vec{\nabla}[S]) } 
{ temporal component } E = -\partial[S] = -\partial[V][S] 
{ spatial component } p = \vec{V}[S] 
*Note* This is the Action (S_{action}) for a free particle. Generally Action is for the 4-TotalMomentum P of a system.

See SR Wave Definition for info on the Lorentz Scalar Invariant SR WavePhase.
{ K = (\omega/c, k) = -\partial[\Phi] = (\vec{\partial}/c\partial[t][\Phi], \vec{\nabla}[\Phi]) } 
{ temporal component } \omega = -\partial[S] = -\partial[V][\Phi] 
{ spatial component } k = \vec{V}[\Phi] 
*Note* This is the Phase (\Phi) for a single plane-wave. Generally WavePhase is for the 4-TotalWaveVector K of a system.

Rest Mass: Energy
E=mc²

4-Position
4-Velocity, 4-Gradient, Vectors

4-Vector SRQM Interpretation of QM

4-Momentum
P=(mc,p)=(E/c,p)
P=-\partial[S]

4-WaveVector
K=(\omega/c,k)=(\omega/c,\omega\vec{n}/v_{phase})
K=-\partial[\Phi_{phase,plan}]

Treating motion like a particle 
Moving particles have a 4-Velocity
4-Momentum is the negative 4-Gradient of the SR Action (S)

Treating motion like a wave
Moving waves have a 4-WaveVector 
4-WaveVector is the negative 4-Gradient of the SR Phase (\Phi)

SR 4-Tensor
(2,0)-Tensor T^{\mu\nu}_{\eta\gamma} 
(1,1)-Tensor T^{\mu}_{\eta}, or T^\mu
(0,2)-Tensor T^{\mu}_{\nu\chi}

SR 4-Vector
(1,0)-Tensor V = (\vec{v},v)

SR 4-CoVector 
(0,1)-Tensor V = (v_0,\vec{v})

SR 4-Scalar 
(0,0)-Tensor S Lorentz Scalar

Diag[T^{\mu\nu}_{\eta\gamma}] = \eta^{\mu\nu}T^{\eta\gamma}_{\mu\nu} = T^{\eta\gamma}_{\eta\gamma} = T^{\eta\gamma}_{\nu\nu} = T 

V\cdot V = V^\alpha\eta_{\alpha\beta}V^\beta = (v_0)^2 - \vec{v}\cdot\vec{v} = (v_0)^2 

Lorentz Scalar
Some Cool Minkowski Metric Tensor Tricks

4-Gradient, 4-Position, 4-Velocity

SpaceTime is 4D

\[ \mathbf{U} \cdot \partial \equiv \gamma(c, \mathbf{u}) \left( \partial/c, -\nabla \right) = \gamma \left( \partial + \mathbf{u} \cdot \nabla \right) \]

\[ \mathbf{U} \cdot \partial \left[ \ldots \right] = d/d\tau = \gamma d/dt \]

4-Velocity \n
\[ \mathbf{U} = \gamma(c, \mathbf{u}) \]

4-Position \n
\[ \mathbf{R} = (ct, \mathbf{r}) \]

4-Gradient \n
\[ \partial = (\partial/c, -\nabla) \rightarrow (\partial/c, -\partial_x, -\partial_y, -\partial_z) \]

\[ \partial \mathbf{R} = \eta^{\mu \nu} \]

\[ \rightarrow \text{Diag}[1,-1,-1,-1] \]

Minkowski Metric

\[ \eta_{\mu \nu} = \delta_{\mu \nu} = \text{Diag}[1,1,1,1] \]

Single Index-Lowering the Minkowski Metric \((\eta^{\mu \nu})\) gives the Kronecker Delta

\[ \eta_{\mu \nu} = \delta_{\mu \nu} = \text{Diag}[1,1,1,1] \]

The Divergence of 4-Position \((\partial \cdot \mathbf{R})\) = “Magnitude” of the Minkowski Metric \(\text{Tr}[\eta^{\mu \nu}] = \text{Dimension of SpaceTime} (4)\)

\[ \left( \mathbf{U} \cdot \partial \right) \left[ \mathbf{R} \right] = \left( U^\mu \partial_{\mu} \right) \left[ \mathbf{R} \right] = \left( U^\mu \eta_{\mu \rho} \partial_{\rho} \right) \left[ \mathbf{R} \right] = \left( U^\mu \right) \partial_{\mu} \left[ \mathbf{R} \right] = (U_{\mu}) \eta^{\mu \nu} = \mathbf{U} \cdot \mathbf{U} = \mathbf{U} = (d/d\tau)[\mathbf{R}] \]

Lorentz Scalar Product \((\mathbf{U} \cdot \partial)\) = Derivative wrt. ProperTime \((d/d\tau) = \text{Relativistic Factor} \ast \text{Derivative wrt. CoordinateTime} \gamma(d/dt)\):

\[ \text{Trace}[\mathbf{T}^{\mu \nu}] = \eta_{\mu \nu} T^{\mu \nu} = T_{\mu}^{\mu} = T \]

\[ \mathbf{V} \cdot \mathbf{V} = V^\mu \eta_{\mu \nu} V^\nu = \left( |\mathbf{V}|^2 - \mathbf{V} \cdot \mathbf{V} \right) = \left( |\mathbf{V}|^2 \right)^2 = \text{Lorentz Scalar} \]
SRQM+EM Diagram: 4-Vectors, 4-Tensors

A Tensor Study of Physical 4-Vectors

SR Perfect Fluid
\[ T^{\mu\nu} = \frac{(\rho + p)}{c^2} U^\mu U^\nu - \frac{p}{c^2} \eta^{\mu\nu} \]

Stress-Energy 4-Tensor

EB Faraday
\[ F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{bmatrix} 0 & -e \end{bmatrix} \]

4-Tensor

Existing SR Rules

Quantum Principles

4-Vector SRQM Interpretation of QM

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[Image of SRQM+EM Diagram: 4-Vectors, 4-Tensors]
SRQM+EM Diagram: 4-Vectors, 4-Tensors
Lorentz Scalars / Physical Constants
SRQM Diagram: Physical Constants Emphasized

Notice that all the main “Universal” or “Fundamental” Physical Constants are here: G,c,ħ,ε₀,μ₀.

Some depend on the actual particle type: q,m₀,ω₀
Some depend on regional conditions: τ,ρeo,p₀,φ₀,ψ*,ψ
Some depend on interaction:Φ phase, S action
Some are mathematical: 0,4,π,i,Diag[1,-1,-1,-1],d/dτ
Conservation Laws are also a type of “zero” constant in this regard.

The majority of the constants are Lorentz Scalars, but some are 4-Vector or 4-Tensor, and all are valid for all inertial observers.

Fundamental Physical Constants are SR Lorentz Scalars

The fact that these “tie together” a network of 4-Vectors is a good argument for why their values are constant. Changing even one would change the relationship properties among all of the 4-Vectors.
SRQM Diagram: Projection Tensors & Perfect-Fluid Stress-Energy Tensor

**SR 4-Tensor**

(2,0)-Tensor \( T^{\mu
\nu}_{\text{rest}} = (p_{eo}, 0, 0, 0) \)

(1,1)-Tensor \( T^{\nu\mu} = (\rho_{eo}, \rho, \rho, \rho) \)

(0,2)-Tensor \( T^{\nu\mu} \)

**SR 4-Vector**

(1,0)-Tensor \( \mathbf{V}^\nu = (\mathbf{v}^\nu) \)

(0,1)-Tensor \( \mathbf{V}_\nu = (v_\nu) \)

**SR 4-Scalar**

(0,0)-Tensor \( S = \) Lorentz Scalar

**Perfect-Fluid Stress-Energy 4-Tensor**

\( T^{\mu
\nu}_{\text{rest}} \rightarrow \text{Diag}[p_{eo}, p_0, p_0, p_0] \)

\( T^{\rho_{eo}}_{\nu\mu} = \rho_{eo}\mathbf{V}^{\nu\mu} + (-p_0)H^{\nu\mu} \)

Projection Tensors act as follows:

\( A^\nu = (a^0, a^1, a^2, a^3) \)

\( V^\mu = \eta_{\nu\mu}V^{\nu\mu} \rightarrow \text{Diag}[1, 0, 0, 0] \)

\( V^\mu, A^\nu = (a^0, 0, 0, 0) \)

\( H^\mu, A^\nu = \eta^{\mu\nu}A^\nu = (a^0, a^1, a^2, a^3) \)

\( V^\nu + H^\nu = \eta_{\nu\nu} \)

\( V^{\nu\mu} + H^{\nu\mu} = \eta^{\nu\mu} \)

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

The rest-energy-density \( \rho_{eo} \) is the Temporal Projection.

The neg rest-pressure \(-p_0\) is the Spatial Projection.

\( T^{\nu\mu}_{\text{rest}} \rightarrow \text{Diag}[p_{eo}, p_0, p_0, p_0] \)

Conservation of StressEnergy

\( \partial_{\nu}T^{\nu\mu} = 0 \)

Light-like Interval (0)

Time-like Interval (\(-\))

Space-like Interval (\(+\))

Null Projection

"Vertical" Temporal Projection

"SpaceTime" Spatial Projection

LightCone

\( \Delta t \)

\( \Delta r \)

Perfect-Fluid StressEnergy 4-Tensor:

\( T^{\mu\nu} = \left((p_{eo} + p_0)c^2\right)\mathbf{u}^\nu \mathbf{u}^\mu - (p_0)\eta^{\mu\nu} \)

The neg rest-pressure \(-p_0\) can be written in much simpler form using Projection Tensors:

\( T^{\mu\nu} = \left(p_{eo}\right)\mathbf{V}^{\nu\mu} - (p_0)H^{\nu\mu} \)

4-Position \( R = (ct, r) \)

4-UnitTemporal \( T^\gamma = (1, \beta) \)

4-Velocity \( U^\gamma = (c, u) \)

ProperTime \( U \cdot \partial = d\gamma/dt = \gamma d/dt \)

4-Gradient \( \partial = (\partial_{/\gamma}, -\nabla) \)

Diag[1, 0, 0, 0] Temporal "Vertical" Projection Tensor

Diag[0, -1, -1, -1] Spatial "Horizontal" Projection Tensor

\( \Delta [\eta^{\mu\nu} = \eta^{\nu\mu} \) Interval (0)

\( \Delta [\eta^{\mu\nu} = \eta^{\nu\mu} \) Interval (1)

\( \Delta [\eta^{\mu\nu} = \eta^{\nu\mu} \) Interval (2)
SRQM Study:

4D Gauss’ Theorem

Gauss’ Theorem in SR:

\[ \int_{\Omega} d^{4}X \left( \partial_{\mu} V^\mu \right) = \oint_{\partial \Omega} dS \left( V^\mu N_\mu \right) \]

\[ \int_{\Omega} d^{4}X \left( \partial \cdot V \right) = \oint_{\partial \Omega} dS \left( V \cdot N \right) \]

where:

- \( V = V^\mu \) is a 4-Vector field defined in \( \Omega \)
- \( \partial \cdot V \) is the 4-Divergence of \( V \)
- \( V \cdot N \) is the component of \( V \) along the \( N \)-direction
- \( \Omega \) is a 4D simply-connected region of Minkowski SpaceTime
- \( \partial \Omega = S \) is its 3D boundary with its own 3D Volume element \( dS \) and outward pointing normal \( N \).
- \( N = N^\mu \) is the outward-pointing normal

\( d^{4}X = (c \, dt)(d^{3}x) = (c \, dt)(dx \, dy \, dz) \) is the 4D differential volume element

In vector calculus, the divergence theorem, also known as Gauss’s theorem or Ostrogradsky’s theorem, is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

In vector calculus, and more generally in differential geometry, the generalized Stokes' theorem is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus.
SRQM Diagram:
Minimal Coupling = Potential Interaction
 Conservation of 4-TotalMomentum

4-Vector SRQM Interpretation of QM
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SR → QM

4-Vector 4-Vectors

A Tensor Study of Physical 4-Vectors

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu}_{\nu}$,
or $T_{\nu}^{\mu}$
(0,2)-Tensor $T^\nu_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\nu = V = (\vec{v},v)$

SR 4-Scalar
(0,0)-Tensor $S$ Lorentz Scalar

Minkowski Metric

4-Displacement
$\Delta R = (c \Delta t, \Delta r)$

4-Position
$R = (ct, r)$

4-Velocity
$U = \gamma(c, u)$

ProperTime Derivative
$\frac{d}{d\tau} [\ldots]$

4-Momentum
$P = (E/c, p) = (E/c, p)$

Rest Energy
$E = mc^2$

Minimal Coupling Relation

Conservation of 4-TotalMomentum
$P = \Sigma_n [P_n]$

EM Force
$F = qA$

Hamilton-Jacobi
$H = -\partial S / \partial \tau$

$P = \Sigma_n [P_n]$

4-TotalMomentum
$P = (E/c, p) = (H/c, p)$

SRQM Diagram:
Interpretation of QM

$\delta[R] = \eta^{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1]$

Minkowski Metric

$Lorentz$ $Scalar$

$\{\ldots\} = \{v_0, v\}$

Rest Scalar Potential
$\phi/c^2$

4-Displacement ε-$\Delta R = (c \Delta t, \Delta r)$

4-Position $R = (ct, r)$

4-Velocity $U = \gamma(c, u)$

ProperTime Derivative $\frac{d}{d\tau} [\ldots]$

4-Momentum $P = (E/c, p) = (E/c, p)$

Rest Mass:Energy $E = mc^2$

Minimal Coupling $P = P + qA$

EM Charge $q$

4-EMVectorPotential $A = (\phi/c, a)$

4-TotalMomentum $Q = (U/c, q) = qA$

4-Gradient $\nabla = (\partial/c, - \nabla)$

4-Displacement $\delta[R]$

4-Velocity $\gamma(c, u)$

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Rest Mass:Energy $E = mc^2$

Minimal Coupling $P = P + qA$

EM Charge $q$

4-EMVectorPotential $A = (\phi/c, a)$

4-TotalMomentum $Q = (U/c, q) = qA$
SRQM Hamiltonian: Lagrangian Connection

\[ H + L = (p_T \cdot u) = \gamma (P_T \cdot U) - (P_T \cdot U)/\gamma \]

4-Momentum \( \mathbf{P} = m_0 \mathbf{U} = (E_0/c^2)\mathbf{U} \); 4-Vector Potential \( \mathbf{A} = (\varphi_0/c^2)\mathbf{U} \)

4-Total Momentum \( \mathbf{P}_T = (\mathbf{P} + q\mathbf{A}) = (H/c, \mathbf{p}_T) \)

\[ \mathbf{P} \cdot \mathbf{U} = \gamma (E - p \cdot u) = E_0 = m_0 c^2 \; ; \; \mathbf{A} \cdot \mathbf{U} = \gamma (\varphi - a \cdot u) = \varphi_0 \]

\[ \mathbf{P}_T \cdot \mathbf{U} = (\mathbf{P} \cdot \mathbf{U} + q\mathbf{A} \cdot \mathbf{U}) = E_0 + q\varphi_0 = m_0 c^2 + q\varphi_0 \]

\[ \gamma = 1/\sqrt{1 - \gamma^2} = (\gamma \mathbf{\beta} \cdot \mathbf{\beta}) \]: Relativistic Gamma Identity

(\( \gamma - 1/\gamma \) = (\( \gamma \mathbf{\beta} \cdot \mathbf{\beta} \)): Manipulate into this form... still an identity

(\( \gamma - 1/\gamma \))(\( \mathbf{P}_T \cdot \mathbf{U} \)) = (\( \gamma \mathbf{\beta} \cdot \mathbf{\beta} \))(\( \mathbf{P} \cdot \mathbf{U} \)): Still covariant with Lorentz Scalar

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \mathbf{\beta} \cdot \mathbf{\beta}) (\mathbf{P} \cdot \mathbf{U}) \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \mathbf{\beta} \cdot \mathbf{\beta}) (E_0 + q\varphi_0) \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma \mathbf{u} \cdot \mathbf{u}) (E_0 + q\varphi_0)/c^2 \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\gamma (E_0/c^2 + q\varphi_0/c^2) \cdot \mathbf{u}) \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = ((\gamma E_0/c^2 + q\varphi_0/c^2) \cdot \mathbf{u}) \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = ((\mathbf{E}u/c^2 + q\varphi_0/c^2) \cdot \mathbf{u}) \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = ((\mathbf{p} + qa) \cdot \mathbf{u}) \]

\[ \gamma (\mathbf{P}_T \cdot \mathbf{U}) + -(\mathbf{P}_T \cdot \mathbf{U})/\gamma = (\mathbf{p} \cdot \mathbf{u}) \]

\(
\{ H \} + \{ L \} = (\mathbf{p}_T \cdot \mathbf{u}) \): The Hamiltonian/Lagrangian connection

\[ H = \gamma (P_T \cdot U) = \gamma (\mathbf{P} + qA) \cdot \mathbf{U} \] = The Hamiltonian with minimal coupling

\[ L = -(P_T \cdot U)/\gamma = -(\mathbf{P} + qA) \cdot \mathbf{U} \] \(/\gamma\) = The Lagrangian with minimal coupling

\[ H_o = (P_T \cdot U) \]

\[ L_o = -(P_T \cdot U) \]

Poynting Vector \( |s| = |\mathbf{g}|c^2 \rightarrow c\varepsilon_0 \mathbf{e} \cdot \mathbf{e} \)

\[ \mathbf{H}_o + \mathbf{L}_o = 0 \] Calculating the Rest Values

4-Vector notation gives a very nice way to find the Hamiltonian/Lagrangian connection:

\( H = \gamma (P_T \cdot U) \) & \( L = -(P_T \cdot U)/\gamma \)
SRQM Study: 
SR Lagrangian, Lagrangian Density, and Relativistic Action (S) 

Relativistic Action (S) is Lorentz Scalar Invariant 
\[ S = \frac{1}{2} (L_0 \gamma) \gamma dt = \frac{1}{2} (L_0) (d^4 x) + \int \mathcal{L}(n) dt = \int \mathcal{L}(d^3 x) dt = \int (\mathcal{L}/c) (d^3 x) (cdt) = \int (\mathcal{L}/c) (d^4 x) \]

Explicitly-Covariant Relativistic Action (S) 
**Particle Form** 
**Density Form \( \neq n_\ast \)**
- \[ S = \int L_0 dt = \int n_0 L_0 (d^4 x) = -(1/c) \int n_0 H_0 (d^4 x) \]
- \[ S = \int (P_T \cdot U) dt \]
- \[ S = \int (P_T \cdot dR/dt) dt \]
- \[ S = \int (P_T \cdot dR) \]

with \( V = q \phi_0 \)

for an EM field = no rest frame

Lagrangian \( L = (p_T \cdot u) - H \) is *not* Lorentz Scalar Invariant

Rest Lagrangian \( L_0 = \gamma L = -(P_T \cdot U) \) is Lorentz Scalar Invariant

Lagrangian Density \( \mathcal{L} = n L = (\gamma n_0)(L_0/\gamma) = n_0 L_0 \) is Lorentz Scalar Invariant

\[ n = \gamma n_0 \Rightarrow \#/(dx)(dy)(dz) = \text{number density} \]
\[ dt = \gamma dt \]
\[ cdn = n_0 (cdt)(dx)(dy)(dz) = n_0 (d^4 x) \]
\[ dv = (n_0/c) (d^4 x) \]

**H-L Connection in Density Format for Photonic System (no rest-frame)**
- \[ H + L = (p_T \cdot u) \]
- \[ nH + nL = n(p_T \cdot u), \text{ with number density } n = n_0 \]
- \[ H + L = (g_T \cdot u), \text{ with momentum density } \{ g_T = np_T \} \]
- \[ \mathcal{H} = nH \]
- \[ \text{Hamiltonian density } \{ \mathcal{H} = nH \} \]
- Lagrangian Density \( \mathcal{L} = nL = (\gamma n_0)(L_0/\gamma) = n_0 L_0 \)
- Lagrangian Density is Lorentz Scalar

for an EM field (photonic):
- \[ \mathcal{H} = (1/2) [\varepsilon_0 e \cdot e + b \cdot b/\mu_0] = n_0 E_0 = \rho_{E_0} = \text{EM Field Energy Density} \]
- \[ \mathcal{L} = (1/2) [\varepsilon_0 e \cdot e + b \cdot b/\mu_0] = (-1/4 \mu_0) F_{\mu\nu} F^{\mu\nu} = (-1/4 \mu_0)*\text{Faraday EM Tensor Inner Product} \]
- \[ \|u\| = c \]
- \[ |g_T| = \varepsilon_0 e \cdot e/c \]

Poynting Vector \[ |s| = |g|c^2 \rightarrow c\varepsilon_0 e \cdot e \]

\[ \varepsilon_0 \mu_0 = 1/c^2 \] : Electric : Magnetic Constant Eqn
Lagrangian \{ L = (p_T \cdot u) - H \} is *not* a Lorentz Scalar
Rest Lagrangian \{ L_o = \gamma L = -(P_T \cdot U) \} is a Lorentz Scalar

Relativistic Action (S) is Lorentz Scalar
\[ S = \int L \, dt \]
\[ S = \int (L_o/\gamma)(\gamma d\tau) \]
\[ S = \int (L_o)(d\tau) \]

Explicitly Covariant
Relativistic Action (S)
\[ S = \int L_o d\tau = -\int H_o d\tau \]
\[ S = -\int (P_T \cdot dR)/d\tau d\tau \]
\[ S = -\int (P_T \cdot dR) \]
\[ S = -\int [(P + qA) \cdot U] d\tau \]
\[ S = \int (E_o + q\phi_o) d\tau \]
\[ S = \int (m_c^2 + V) d\tau \]
\[ S = \int (H_o) d\tau \]

Hamilton-Jacobi Equation
\[ \partial [-S] = -\partial [S] = P_T \]
\[ S = -\int (E_o + q\phi_o)(\tau + const) d\tau \]
\[ \partial [-S] = (E_o + q\phi_o)\partial [(\tau + const)] \]

4-Scalars
Relativistic Action Eqn
Integral Format
\[ S_{\text{action}} = -\int [P_T \cdot dR] \]
\[ = -\int [P_T \cdot U] d\tau \]
\[ = -\int [(H/c, p_T) \cdot \gamma (c, u)] d\tau \]
\[ = -\int (H/c, p_T) \cdot \gamma (c, u) d\tau \]

4-Vectors
Relativistic Hamilton-Jacobi Eqn
Differential Format
\[ P_T = \frac{(E_t/c, p_T) = (H/c, p_T)}{-\partial [S_{\text{action}}]} \]
\[ \partial [-S] = \partial [S_{\text{action}}] \]
\[ \partial [-S] = (m_o + q\phi_o)c^2 \]
\[ \partial [-S] = P_T + qA \]

The Hamilton-Jacobi Equation is incredibly simple in 4-Vector form
SRQM Diagram: 
Relativistic Hamilton-Jacobi Equation
\((P_T = -\partial[S])\) Differential Format : 4-Vectors

Relativistic Action \(S\) is Lorentz Scalar Invariant
\[ S = \int L dt = \int (L_o/\gamma)(\gamma dt) = \int (L_o)(dt) = \int L_o dt \]

Explicitly-Covariant Relativistic Action \(S\):
\[ d\tau = (1/c) \sqrt{\gamma} \left[ d\mathbf{R} \cdot d\mathbf{R} \right] \]

Proper Time Derivative
\[ d\tau = (1/c) \sqrt{\gamma} \left[ d\mathbf{R} \cdot d\mathbf{R} \right] \]

4-Velocity
\[ \mathbf{U} = \gamma(c,u) \]

4-Force
\[ F = \gamma(E/c,f) \]

4-EMPotentialMomentum
\[ Q = (U/c,q) = qa \]

SR 4-Tensor
\[(2,0)\] Tensor \(T_{\mu \nu}^{\rho \sigma}\)
\[(1,1)\] Tensor \(T^\alpha_\mu\), or \(T^\mu_\alpha\)
\[(0,2)\] Tensor \(T_{\mu \nu}^{\rho} \)

SR 4-Vector
\[(1,0)\] Tensor \(V^\mu = (\gamma, V)\)
\[\mathbf{V} = (v_0, V)\] Lorentz Scalar

SR 4-Scalar
\[(0,0)\] Tensor \(S\)
Trace\([\mathbf{T}^\mu_\nu] = \eta_{\mu \nu} T^{\rho \sigma} = T^\rho_\rho = T\)
\[\mathbf{V} \cdot \mathbf{V} = \gamma v_0^2 = (\gamma v_0^2) \]
\[= \text{Lorentz Scalar}\]
SRQM Diagram: Relativistic Action Equation

\[
S = -\int (P_T \cdot dR)
\]

Integral Format: 4-Scalars

\[
T = \frac{\gamma^2}{1-\beta \cdot \beta}
\]

\[
\gamma = \frac{1}{\sqrt{1-\beta \cdot \beta}}
\]

\[
\rho = \frac{\gamma}{c}
\]

\[
\phi = \frac{\gamma}{c^2}
\]

4-Displacement
\[
\Delta \mathbf{R} = (c\Delta t, \Delta r)
\]

4-Position
\[
\mathbf{R} = (ct, \mathbf{r})
\]

\[
\delta \mathbf{R} = \delta (\mathbf{R}) = \delta (ct, \mathbf{r})
\]

\[
\mathbf{V} \cdot \mathbf{V} = \mathbf{V}^2 = (\mathbf{V}^0)^2 = (\mathbf{V}^0)^2
\]

\[
\mathbf{V} \cdot \mathbf{V} = \gamma (\mathbf{V}^0)^2 - \mathbf{V}^2 = (\mathbf{V}^0)^2
\]

\[
\mathbf{V} \cdot \mathbf{V} = \gamma (\mathbf{V}^0)^2 - \mathbf{V}^2 = (\mathbf{V}^0)^2
\]

\[
\mathbf{V} \cdot \mathbf{V} = \gamma (\mathbf{V}^0)^2 = (\mathbf{V}^0)^2
\]

\[
\mathbf{L} = (\mathbf{P}_T \cdot \mathbf{U})/\gamma = \mathbf{L} = \gamma \mathbf{P}_T \cdot \mathbf{U}
\]

\[
\mathbf{H} = \gamma (\mathbf{P}_T \cdot \mathbf{U}) = \gamma \mathbf{H}
\]

\[
\mathbf{F} = \gamma (\mathbf{E}/c, \mathbf{f})
\]

\[
\mathbf{Q} = (\mathbf{U}/c, q) = q \mathbf{A}
\]

\[
\mathbf{P}_T = (E/c, \mathbf{p}) = (H/c, \mathbf{p})
\]

\[
\mathbf{4-EMPotentialMomentum}
\]

\[
\mathbf{4-MomentumIncField}
\]

\[
\mathbf{4-Force}
\]

\[
\mathbf{4-Momentum}
\]

\[
\mathbf{4-CurrentDensity}
\]

\[
\mathbf{4-ChargeFlux}
\]

\[
\mathbf{4-NumberFlux}
\]

\[
\mathbf{4-Gradient}
\]

\[
\mathbf{4-Displacement}
\]

\[
\mathbf{4-Position}
\]

\[
\mathbf{4-Number}
\]

\[
\mathbf{4-Charge}
\]

\[
\mathbf{4-Current}
\]

\[
\mathbf{4-Vector}
\]

\[
\mathbf{4-Tensor}
\]
SRQM Diagram:

Relativistic Euler-Lagrange Equation

The Easy Derivation \((U=(d/d\tau)R)\rightarrow(\partial_R=(d/d\tau)\partial_U)\)

Note Similarity:

- 4-Velocity is ProperTime Derivative of 4-Position
  \(U = (d/d\tau)R \ [m/s] = [1/s][m]\)

Relativistic Euler-Lagrange Eqn
\(\partial_x = (d/d\tau)\partial_x \ [1/m] = [1/s][s/m]\)

The differential form just inverses the dimensional units, so the placement of the \(R\) and \(U\) switch.

That is it: so simple! Much, much easier than how I was taught in Grad School.

To complete the process and create the Equations of Motion, one just applies the base form to a Lagrangian.

This can be: a classical Lagrangian or a Lorentz scalar.

SR 4-Tensor
\((2,0)\)-Tensor \(T^{\mu\nu}\) or \(T^{\nu\mu}\)
\((1,1)\)-Tensor \(T^\nu\)
\((0,2)\)-Tensor \(T_{\mu\nu}\)

SR 4-Vector
\((1,0)\)-Tensor \(V^\nu = V = (\vec{v},\nu)\)
\((0,1)\)-Tensor \(V_\nu = (\nu\vec{v})\)

SR 4-Scalar
\((0,0)\)-Tensor \(S\)
Lorentz Scalar
SRQM Diagram:

Relativistic Euler-Lagrange Equation

Alternate Forms: Particle vs. Density

4-Velocity $U$ is ProperTime Derivative of 4-Position $R$. The Euler-Lagrange Eqn can be generated by taking the differential form of the same equation.

Relativistic 4-Vector Kinematical Eqn
$U = (d/d\tau)R$
$U \cdot K = (d/d\tau)R \cdot K$

Relativistic Euler-Lagrange Eqns (uses gradient-type 4-Vectors)
$\partial R = (d/d\tau) \partial_U$: {particle format}

$\partial_R: \text{SpaceTime Derivative}$

$\partial_U: \text{4-Position} R=(ct, r)$

$4$-Velocity Gradient
$\partial_U^\mu = \partial U^\mu = \partial U = (\partial U/c, -\nabla U)$
$\rightarrow (\partial \partial_{\gamma c}, -\partial \partial_{\gamma u_x}, -\partial \partial_{\gamma u_y}, -\partial \partial_{\gamma u_z})$

$\partial U \cdot \partial R = (d/d\tau)\partial_U R = \partial U = (d/d\tau)\partial R$

Kirchhoff Lagrangian
$L = (1/2) \{ \partial_\mu \Phi \cdot \partial^\mu \Phi - (m_0c/h)^2 \Phi^2 \}$: KG Lagrangian Density

Euler-Lagrange Eqn {density format}

$-m_0c/h)^2 \Phi = (\partial R) \cdot \partial_\mu \Phi$

$\partial_\mu \partial_R \Phi = - (m_0c/h)^2 \Phi$

$\partial_\mu \partial_R \Phi = - (m_0c/h)^2 \Phi$: KG Eqn of Motion

Klein-Gordon Relativistic Quantum Wave Eqn

SR 4-Tensor
(2,0)-Tensor $T^{iv}_{iv}$
(1,1)-Tensor $T^i_\mu$, or $T^\mu_\mu$, or $T^i_{ij}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V = (\dot{\gamma}, \dot{\nu})$

SR 4-CoVector
(0,1)-Tensor $V_\mu = (\dot{\gamma}, \dot{\nu})$

SR 4-Scalar
(0,0)-Tensor S
Lorentz Scalar
SRQM Diagram:

Relativistic Euler-Lagrange Equation
Equation of Motion (EoM) for EM particle

\[ \delta_R[R] = \eta^{\mu \nu} \rightarrow \text{Diag}[1, -1, -1, -1] \]

\[ \text{Minkowski Metric} \]

\[ \delta \text{r} = \frac{\partial}{\partial t} \]

\[ \text{Proper Time Derivative} \]

\[ U = \sigma \left[ \text{d/dt} \right] \]

\[ \text{d/dt} \left[ \right] \]

\[ U \cdot \sigma \left[ \right] = \delta \text{r} \]

\[ \text{Euler-Lagrange Eqn: } (d/dt)[U] = \delta \text{r} \]

\[ -(F + q \cdot U \sigma[A]) = -q U \cdot \sigma[A] \]

\[ F = q U \cdot \sigma[A] - q U \cdot \sigma[A] \]

\[ F = q U \cdot \sigma[A] - q U \cdot \sigma[A] \]

\[ F = q U \cdot \sigma[A] - q U \cdot \sigma[A] \]

\[ \text{Lorentz Force Equation} \]

\[ P_t \cdot U = (P + q A) \cdot U \]

\[ \gamma(H - p_t \cdot U) = H_L = 0 \]

\[ 4 \text{- Potential Momentum} \]

\[ Q = \left( U \text{c}, q \right) = q A \]

\[ 4 \text{- Force} \]

\[ F = \gamma \left( \frac{E \text{c}}{c}, f \right) \]

\[ 4 \text{- Velocity} \]

\[ U = \gamma \left( c, u \right) \]

\[ 4 \text{- Displacement} \]

\[ \Delta R = \left( c \Delta t, \Delta r \right) \]

\[ \Delta x = \left( c \Delta t, \Delta r \right) \]

\[ \Delta \text{r} \]

\[ 4 \text{- Position} \]

\[ R = (c t, r) \]

\[ \text{SR 4-Tensor} \]

\[ (2,0) \text{- Tensor} \]

\[ T^{\mu \nu} = \left( \frac{\partial \phi}{\partial \tau}, \frac{\partial \phi}{\partial \tau} \right) \]

\[ \text{SR 4-Vector} \]

\[ (1,0) \text{- Tensor} \]

\[ V = \left( \frac{v}{c}, \frac{v}{c} \right) \]

\[ \text{SR 4-Scalar} \]

\[ (0,1) \text{- Tensor} \]

\[ S = \left( \frac{\partial \phi}{\partial \tau}, \frac{\partial \phi}{\partial \tau} \right) \]

\[ \text{Trace} \left[ T^{\mu \nu} \right] = \eta_\mu \eta_\nu T^{\mu \nu} = T_{\mu \nu} = T \]

\[ V \cdot V = \left( \frac{v}{c}, \frac{v}{c} \right) \cdot \left( \frac{v}{c}, \frac{v}{c} \right) = \left( \frac{v}{c} \right)^2 = \text{Lorentz Scalar} \]

\[ \left( dP^\mu / dt \right) = F^\mu = q U \cdot (F^\mu) \]

\[ \text{Equation of Motion for charged particle} \]

\[ 4 \text{- Momentum} \]

\[ P = (m \text{c}, p) = (E/c, p) \]

\[ \text{EM Faraday} \]

\[ \beta = \frac{e}{c} \]
SRQM Diagram:
Relativistic Euler-Lagrange Equation
Equation of Motion (EoM) for EM particle

\[ \gamma = 1/\sqrt{1 - \beta \cdot \beta}; \text{ Relativistic Gamma Identity} \]
\[ (\gamma - 1)/\gamma = (\gamma - 1)/(\gamma + 1) \]
\[ \gamma = (\gamma + 1)/(\gamma - 1) \]
\[ \gamma = (\gamma + 1)/(\gamma - 1) \]

Classical limit, 
\[ \partial/\partial \tau \]
Relativistic Lagrangian 
\[ U = \gamma(c, u) \]

ProperTime 
\[ U \cdot \partial = d/d\tau = \gamma d/dt \]
Derivative

4-Velocity
\[ U = \gamma(c, u) \]

Rest Lagrangian 
\[ L_0 = -(P \cdot U) \]
\[ -q \cdot U \cdot (P + qA) \]
\[ \gamma \cdot \partial L_0/\partial U = -q \cdot U \cdot (P + qA) \]
Euler-Lagrange Equations of Motion
\[ (d/d\tau) \partial L_0/\partial U = \partial L_0/\partial \tau \]

Lorentz Force Equation
\[ -(F^a + qU_\beta \partial_\beta [A^a]) = -qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]

ProperTime 
\[ U \cdot \partial = d/d\tau = \gamma d/dt \]
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Euler-Lagrange Equations of Motion
\[ (d/d\tau) \partial L_0/\partial U = \partial L_0/\partial \tau \]

Lorentz Force Equation
\[ -(F^a + qU_\beta \partial_\beta [A^a]) = -qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]

ProperTime 
\[ U \cdot \partial = d/d\tau = \gamma d/dt \]
Derivative

4-Velocity
\[ U = \gamma(c, u) \]

Rest Lagrangian 
\[ L_0 = -(P \cdot U) \]
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\[ \gamma \cdot \partial L_0/\partial U = -q \cdot U \cdot (P + qA) \]
Euler-Lagrange Equations of Motion
\[ (d/d\tau) \partial L_0/\partial U = \partial L_0/\partial \tau \]

Lorentz Force Equation
\[ -(F^a + qU_\beta \partial_\beta [A^a]) = -qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]

ProperTime 
\[ U \cdot \partial = d/d\tau = \gamma d/dt \]
Derivative

4-Velocity
\[ U = \gamma(c, u) \]

Rest Lagrangian 
\[ L_0 = -(P \cdot U) \]
\[ -q \cdot U \cdot (P + qA) \]
\[ \gamma \cdot \partial L_0/\partial U = -q \cdot U \cdot (P + qA) \]
Euler-Lagrange Equations of Motion
\[ (d/d\tau) \partial L_0/\partial U = \partial L_0/\partial \tau \]

Lorentz Force Equation
\[ -(F^a + qU_\beta \partial_\beta [A^a]) = -qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]

ProperTime 
\[ U \cdot \partial = d/d\tau = \gamma d/dt \]
Derivative

4-Velocity
\[ U = \gamma(c, u) \]

Rest Lagrangian 
\[ L_0 = -(P \cdot U) \]
\[ -q \cdot U \cdot (P + qA) \]
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Euler-Lagrange Equations of Motion
\[ (d/d\tau) \partial L_0/\partial U = \partial L_0/\partial \tau \]

Lorentz Force Equation
\[ -(F^a + qU_\beta \partial_\beta [A^a]) = -qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
\[ F^a = qU_\beta \partial_\beta [A^a] \]
Relativistic Hamilton’s Equations of Motion (EoM) for EM particle

\[ \gamma = 1/\sqrt{1 - \beta c^2} \] Relativistic Gamma Identity

\[ (\gamma - 1/\gamma) = (\gamma - 1)/\gamma \] Manipulate into this form... still an identity

\[ \gamma (\mathbf{p} + \mathbf{u}) \gamma = (\gamma \mathbf{p}) \gamma \]

\[ \gamma (\mathbf{p} + \mathbf{u}) \gamma = (\gamma \mathbf{p}) \gamma \]

\[ H = \gamma H_0 = \gamma (\mathbf{p} + q \mathbf{A}) \mathbf{u} \]

The Hamiltonian with minimal coupling

\[ L = L_0 = (\mathbf{p} - \mathbf{u}) \mathbf{u} \]

Rest Hamiltonian = Total RestEnergy

\[ P_x = \frac{d}{dt} X \]

\[ U = \gamma (c, u) \]

\[ \mathbf{T} = \mathbf{P} \cdot \mathbf{U} + q \mathbf{A} \cdot \mathbf{U} \]

\[ \mathbf{P} \mathbf{T} = (E/c, P_T) = (H/c, P_T) \]

Rest Hamiltonian Hamilton’s Equations of Motion

\[ \frac{d}{dt} [\mathbf{X}] = \left( \frac{\partial}{\partial \mathbf{P}_T} \right) [H_0] \]

\[ \frac{d}{dt} [\mathbf{P}_T] = \left( \frac{\partial}{\partial \mathbf{X}} \right) [H_0] \]

\[ \left[ \begin{array}{c} \dot{X} \\ \dot{P}_x \\ \dot{P}_y \\ \dot{P}_z \\ \dot{\gamma} \end{array} \right] = \left[ \begin{array}{c} \mathbf{F} + q (\mathbf{d}/\mathbf{d}t) \mathbf{A} \\ \left( \frac{\partial}{\partial \mathbf{P}_T} \right) [H_0] \\ \left( \frac{\partial}{\partial \mathbf{X}} \right) [H_0] \\ \left( \frac{\partial}{\partial \mathbf{P}_T} \right) [H_0] \\ \left( \frac{\partial}{\partial \mathbf{X}} \right) [H_0] \end{array} \right] \]

Classical Hamilton’s Equations (3-vector):

\[ (d/dt)\mathbf{x} = \mathbf{F} + q(d/dt)\mathbf{A} \]

\[ (d/dt)\mathbf{p} = (d/dt)(\gamma \mathbf{p}) \]

\[ (d/dt)\mathbf{p} = (d/dt)(\gamma \mathbf{p}) \]

Sign-flip difference is interaction of \((d/dt)\mathbf{p}\) with \([1/\gamma]\)

SR 4-Tensor

\[(1,0)-\text{Tensor} \quad T^\mu_\nu = V \cdot V = (V^\mu \cdot V) \]

SR 4-Vector

\[(0,1)-\text{Tensor} \quad V^\mu = (V_\mu) \]

SR 4-Scalar

\[(0,0)-\text{Tensor} \quad S \quad \text{Lorentz Scalar} \]

Trace\[ T^{\mu}_\nu = n_\mu T^{\mu}_\nu = T^\nu_\nu = T \]

\[ V \cdot V = V^\mu_n V_n = (V^\mu_\nu - V^\mu_\nu) = (V_\mu)^2 = \text{Lorentz Scalar} \]
SRQM Diagram: EM Lorentz Force Eqn

→ Force = - Grad[Potential]

Lorentz EM Force Equation:

\[ F^\alpha = q(F^{\alpha \beta}U_\beta) \]
\[ F^\alpha = q(\partial A^\beta - \partial A^\beta)U_\beta \]

Examine just the spatial components of 4-Force \( F \):

\[ F^i = q(\partial A^0 - \partial A^0)U_0 + q(\partial A^1 - \partial A^1)U \]

\[ \gamma f = q(-\nabla[\phi/c] - (\partial/c)a)(yc) + q(-\nabla[a\cdot u] - u\cdot\nabla[a])\gamma \]
\[ f = q(-\nabla[\phi/c] - (\partial/c)a(c) + q(u\cdot\nabla[a]-\nabla[a\cdot u]) \]
\[ f = q(-\nabla[\phi] - \partial^\alpha a + u\cdot\nabla[a] - \nabla[a\cdot u]) \]
\[ f = q(-\nabla[\phi] - \partial^\alpha a + u \times b) \]

Take the limit of \(|\nabla[\phi]| >> |\partial a| + |u \times b|\)

\[ f \sim q(-\nabla[\phi]) = -\nabla[q\phi] = -\nabla[U] \]

The Classical Force = -Grad[Potential]

when \(|\nabla[\phi]| >> |\partial a| + |u \times b|\) or when \(a = 0\)

The majority of non-gravity, non-nuclear potentials dealt with in CM are those mediated by the EM potential.

ex. Spring Potential \( U = \frac{kx^2}{2} \), then \( f = -\nabla[\frac{kx^2}{2}] = -kx \) Hooke’s Law
SRQM: The Speed-of-Light (c)

Invariant Relations (part 1)

The Speed-of-Light (c) is THE connection between Time and Space: dR = (cdt,dr)

This physical constant appears in several seemingly unrelated places. You don’t notice these cool relations when you set c→1. Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

\[ U-U = \gamma^2 (c^2-u-u) = c^2 \]

Speed of all things into the Future

\[ (E/m_c) = (\gamma E/\gamma m_c) = (E/m) = c^2 \] Mass is concentrated Energy, \( E = mc^2 \)

\[ |u * v_{phase}| = |v_{group} * \gamma_{phase}| = c^2 \]

Particle-Wave “Duality” Correlation

\[ \lambda^2 (\omega^2 - \omega_0^2) = \lambda^2 (\omega^2 - c^2) \]

Wavelength-Frequency Relation: \( \lambda f = c \) for photons

\[ (1/\epsilon_0 \mu_0) = c^2 \]

Electric (\( \epsilon_0 \)) and Magnetic (\( \mu_0 \)) EM Field Constants

\[ -(h/m_c)^2 (\partial^2 - \gamma) = c^2 \]

Relativistic Quantum Wave Equation

\[ (h/\hbar \_c m_c)^2 = c^2 \]

Reduced Compton Wavelength: \( \lambda_c = (h/m_c) \)

\[ 2GM/R_s = c^2 \]

GR Black Hole Equation

\[ 8\pi G/\kappa = c^2 \]

GR Einstein Curvature Constant: \( \kappa = 8\pi G/c^2 \)

\[ (c_3^2 * \text{scalar, 3-vector}) = 4\text{-vector} \]

Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector

4-Vector SRQM Interpretation of QM

SciRealm.org
John B. Wilson
SRQM: The Speed-of-Light (c)

\( c^2 \) Invariant Relations (part 2)

The Speed-of-Light (c) is THE connection between Time and Space: \( dR = (c dt, dr) \)

This physical constant appears in several seemingly unrelated places. You don’t notice these cool relations when you set \( c \to 1 \). Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

\[ \eta_{\mu\nu} \]

Minkowski Metric

\[ \delta^{\mu}[R] = \eta^{\mu\nu} \]

4-Vector Scalar Product

\[ \eta_{\mu\nu} \cdot \]

\[ U \cdot U = c^2 \]

Speed of all things into the Future

\[ (E/m_o) = (h/\lambda C m_o)^2 \]

Energy:Mass

\[ 1/(\epsilon_o \mu_o) = c^2 \]

Electric:Magnetic

\[ 1/(c \epsilon_o) = c^2 \]

Relativistic Quantum Wave Equation

\[ (h/\lambda C m_o)^2 = c^2 \]

Reduced Compton Wavelength: \( \lambda_C = (h/\lambda m) \)

\[ 2GM/R_s = c^2 \]

GR Black Hole Equation

\[ R_s = \text{Schwarzschild Radius} \]

\[ G = \text{GR Gravitational Const}, M = \text{BH Mass} \]

\[ 8\pi G/\kappa = c^2 \]

GR Einstein Curvature Constant

\[ (c^{a*}, \text{scalar, 3-vector}) = 4-Vector \]

Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector.

SR 4-Tensor

\[ (2,0)-Tensor \]

\[ (1,1)-Tensor \]

SR 4-Vector

\[ (0,1)-Tensor \]

SR 4-CoVector

\[ (0,0)-Tensor \]

SR 4-Scalar

\[ \text{Lorentz Scalar} \]
SRQM 4-Vector Study:

4-ThermalVector

Relativistic Thermodynamics

The 4-ThermalVector is used in Relativistic Thermodynamics.

My prime motivation for the form of this 4-Vector is that the probability distributions calculated by statistical mechanics ought to be covariant functions since they are based on counting arguments.

F(state) ∼ e^(-E/kT), with this β = 1/kT, (not v/c)

A covariant way to get this is the Lorentz Scalar Product of the 4-Momentum P with the 4-ThermalVector Θ.

R(state) ∼ e^(-P·Θ) = e^(-E/kT)

This also gets Boltzmann’s constant (kT) out there with the other Lorentz Scalars like (c) and (h)

see (Relativistic) Maxwell-Jüttner distribution

f[P] = N/(2c(mo)² K_0(mo/2π^2)²) * e^(-P·Θ)

f[P] = (Θo/N)/(4πm0c² K_0(mo/2π)^2) * e^(-P·Θ)

E = mc²

It is possible to find this distribution written in multiple ways because many authors don’t show constants, which is quite annoying. Show the damn constants people! (kT),(c),(h) deserve at least that much respect.

Be careful not to confuse (unfortunate symbol clash):

Thermal β = 1/kT
Relativistic β = v/c

These are totally separate uses of (β)
The 4-ThermalVector is used in Relativistic Thermodynamics. It can be used in a partial derivation of Unruh-Hawking Radiation (up to a numerical constant).

Let a “Unruh-DeWitt thermal detector” be in the Momentarily-Comoving-Rest-Frame (MCRF) of a constant spatial acceleration \( \mathbf{a} \), in which \( |\mathbf{u}| \rightarrow 0 \), \( \gamma \rightarrow 1 \), and \( \gamma \rightarrow 0 \).

The 4-Acceleration in MCRF \( \mathbf{A}_{\text{MCRF}} = \mathbf{A} = (0, \mathbf{a}) \) is given by:

\[
\mathbf{A} = \mathbf{a} = \gamma (c \mathbf{v}' + \gamma \mathbf{a})
\]

where \( \mathbf{v}' \) is the velocity of the detector in the MCRF, \( \mathbf{a} \) is the acceleration of the detector, and \( \gamma \) is the Lorentz factor. The 4-Acceleration is given by:

\[
\mathbf{A} = \gamma (c \mathbf{v}' + \gamma \mathbf{a})
\]

where \( \mathbf{v}' \) is the velocity of the detector in the MCRF, \( \mathbf{a} \) is the acceleration of the detector, and \( \gamma \) is the Lorentz factor.

The 4-Acceleration in MCRF is given by:

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\mathbf{A} = \gamma (c \mathbf{v}' + \gamma \mathbf{a})
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The 4-Acceleration in MCRF is given by:

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The 4-Acceleration in MCRF is given by:

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\mathbf{A} = \gamma (c \mathbf{v}' + \gamma \mathbf{a})
\]

where \( \mathbf{v}' \) is the velocity of the detector in the MCRF, \( \mathbf{a} \) is the acceleration of the detector, and \( \gamma \) is the Lorentz factor.

Further methods give the constant of proportionality (1/2):

\[
\frac{T_{\text{Unruh}}}{k_B} = \frac{\hbar}{2 \pi c} \quad \text{[due to constant Minkowski-hyperbolic acceleration]}
\]

\[
T_{\text{Hawking}} = \frac{\hbar g}{2 \pi k_B c} \quad \text{[due to gravitational acceleration \( a \) = \( \gamma \gamma \)]}
\]

The temperature \( T \) is given by:

\[
T = h / 2 \pi c \quad \text{[correct version from 4-Vector derivation \( A_{\text{MCRF}} \cdot \Theta = 2 \pi c / h \)]}
\]

Further methods give the constant of proportionality (1/2):

\[
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The temperature \( T \) is given by:

\[
T = h / 2 \pi c \quad \text{[correct version from 4-Vector derivation \( A_{\text{MCRF}} \cdot \Theta = 2 \pi c / h \)]}
\]
The 4-EntropyVector is used in Relativistic Thermodynamics.

Pure Entropy is a Lorentz Scalar in all frames.

not finished yet…

Page under construction
Up to this point, we have basically been exploring the SR aspects of 4-Vectors. It is now time to show how RQM and QM fit into the works...

This is SRQM, [ SR → QM ]

RQM & QM are derivable from SR

SRQM: A treatise by John B. Wilson (SciRealm@aol.com)
The basic idea is to show that Special Relativity plus a few empirical facts lead to Relativistic Wave Equations, and thus RQM, without using any assumptions or axioms from Quantum Mechanics.

Start only with the concepts of SR, no concepts from QM
(1) SR provides the ideas of Invariant Intervals and (c) as a Physical Constant, as well as: Poincaré Invariance, Minkowski 4D SpaceTime, ProperTime, and Physical SR 4-Vectors

Note empirical facts which can relate the SR 4-Vectors from the following:
(2a) Elementary matter particles each have RestMass, \( m_0 \), which can be measured by experiment: eg. collision, cyclotrons, Compton Scattering, etc.

(2b) There is a constant, (\( \hbar \)), which can be measured by classical experiment – eg. the Photoelectric Effect, the inverse Photoelectric Effect, LED's=Injection Electroluminescence, Duane-Hunt Law in Bremsstrahlung, the Watt/Kibble-Balance, etc. All known particles obey this constant.

(2c) The use of complex numbers (i) and differential operators \( \partial_t \) and \( \nabla = (\partial_x, \partial_y, \partial_z) \) in wave-type equations comes from pure mathematics: not necessary to assume any QM Axioms

These few things are enough to derive the RQM Klein-Gordon equation, the most basic of the relativistic wave equations. Taking the low-velocity limit \( |\mathbf{v}| << c \) (a standard SR technique) leads to the Schrödinger Equation.
SRQM Basic Idea (part 2)

Klein-Gordon RWE implies QM

If one has a Relativistic Wave Equation, such as the Klein-Gordon equation, then one has RQM, and thence QM via the low-velocity limit \(|v|<<c\).

The physical and mathematical properties of QM, usually regarded as axiomatic, are inherent in the Klein-Gordon RWE itself.

QM Principles emerge not from \(\{\text{QM Axioms} + \text{SR} \rightarrow \text{RQM}\}\), but from \(\{\text{SR} + \text{Empirical Facts} \rightarrow \text{RQM}\}\).

The result is a paradigm shift from the idea of \(\{\text{SR and QM as separate theories}\}\) to \(\{\text{QM derived from SR}\}\) – leading to a new interpretation of QM: 

*The SRQM or \([\text{SR} \rightarrow \text{QM}]\) Interpretation.*

\[
\begin{align*}
\text{GR} & \rightarrow \ (\text{low-mass limit} = \{\text{curvature } \sim 0\} \text{ limit}) \rightarrow \text{SR} \\
\text{SR} & \rightarrow \ (+ \text{ a few empirical facts}) \rightarrow \text{RQM} \\
\text{RQM} & \rightarrow \ (\text{low-velocity limit} \ {\{\text{v}|<<c\}) \rightarrow \text{QM}
\end{align*}
\]

The results of this analysis will be facilitated by the use of SR 4-Vectors.
## SRQM 4-Vector Path to QM

<table>
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<tr>
<th>SR 4-Vector</th>
<th>Definition Component Notation</th>
<th>Unites</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>( R = R^\mu = (ct, r) )</td>
<td>Time, Space -when &amp; where</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( U = U^\mu = \gamma(c, u) )</td>
<td>Lorentz Gamma * (c, Velocity) -nothing faster than c</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( P = P^\mu = (E/c, p) = (mc, p) )</td>
<td>Mass:Energy, Momentum -used in 4-Momenta Conservation ( \Sigma P_{\text{final}} = \Sigma P_{\text{initial}} )</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( K = K^\mu = (\omega/c, k) = (\omega/c, \omega \hat{n}/v_{\text{phase}}) )</td>
<td>Ang. Frequency, WaveNumber -used in Relativistic Doppler Shift ( \omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma(1 - \beta \cos[\theta])], k = \omega/c ) for photons</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial = \partial^\mu = (\partial/c, -\nabla) ) ( = (\partial/c, -\partial_x, -\partial_y, -\partial_z) )</td>
<td>Temporal Partial, Spatial Partial -used in SR Continuity Eqns., ProperTime -eg. ( \partial \cdot A = 0 ) means ( A ) is conserved</td>
</tr>
</tbody>
</table>

All of these are standard SR 4-Vectors, which can be found and used in a totally relativistic context, with no mention or need of QM. I want to emphasize that these objects are ALL relativistic in origin.
SRQM 4-Vector Invariants

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Lorentz Invariant</th>
<th>What it means in SR...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>( \mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct_o)^2 = (ct)^2 )</td>
<td>SR Invariant Interval</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( \mathbf{U} \cdot \mathbf{U} = \gamma^2 (c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2 )</td>
<td>Events move into future at magnitude c</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2 )</td>
<td>Einstein Mass:Energy Relation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_o/c)^2 )</td>
<td>Dispersion Invariance Relation</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial \cdot \partial = (\partial t/c)^2 - \nabla \cdot \nabla = (\partial t/c)^2 )</td>
<td>The d'Alembert Operator</td>
</tr>
</tbody>
</table>

All 4-Vectors have invariant magnitudes, found by taking the scalar product of the 4-Vector with itself. Quite often a simple expression can be found by examining the case when the spatial part is zero. This is usually found when the 3-velocity is zero. The temporal part is then specified by its “rest” value.

For example: \( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2 = (m_o c)^2 \)
\( E = \text{Sqrt}[(E_o)^2 + \mathbf{p} \cdot \mathbf{p} c^2] \), from above relation
\( E = \gamma E_o \), using \{\gamma = 1/\text{Sqrt}[1-\beta^2] = \text{Sqrt}[1+\gamma^2 \beta^2]\} and \{\beta = v/c\}
meaning the relativistic energy \( E \) is equal to the relative gamma factor \( \gamma \) * the rest energy \( E_o \).
SR + A few empirical facts: SRQM Overview

<table>
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<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>SI Dimensional Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position ( R = (ct,r) ); alt. ( X = (ct,x) )</td>
<td>( R = &lt;\text{Event}&gt; ); alt. ( X )</td>
<td>[m]</td>
</tr>
<tr>
<td>4-Velocity ( U = \gamma(c,u) )</td>
<td>( U = dR/d\tau )</td>
<td>[m/s]</td>
</tr>
<tr>
<td>4-Momentum ( P = (E/c,p) = (mc,p) )</td>
<td>( P = m_0U )</td>
<td>[kg·m/s]</td>
</tr>
<tr>
<td>4-WaveVector ( K = (\omega/c,k) )</td>
<td>( K = P/\hbar )</td>
<td>[{rad}/m]</td>
</tr>
<tr>
<td>4-Gradient ( \partial = (\partial_t/c,-\nabla) )</td>
<td>( \partial = -iK )</td>
<td>[1/m]</td>
</tr>
</tbody>
</table>

The Axioms of SR, which are actually GR limiting-cases, lead us to the use of Minkowski Space and Physical 4-Vectors, which are elements of Minkowski Space (4D SpaceTime).

Empirical Observation leads us to the transformation relations between the components of these SR 4-Vectors, and to the chain of relations between the 4-Vectors themselves.

These relations all turn out to be Lorentz Invariant Constants, whose values are measured empirically.

The combination of these SR objects and their relations is enough to derive RQM.
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR

\{c,\tau,m_o,\hbar,i\}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors: Related by these SR Lorentz Invariants

<table>
<thead>
<tr>
<th>4-Position</th>
<th>4-Velocity</th>
<th>4-Momentum</th>
<th>4-WaveVector</th>
<th>4-Gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>R = (ct,r)</td>
<td>U = γ(c,u)</td>
<td>P = (E/c,p)</td>
<td>K = (ω/c,k)</td>
<td>∂ = (∂t/c,-∇)</td>
</tr>
<tr>
<td>(\langle\text{Event}\rangle)</td>
<td>((\mathbf{U} \cdot \partial)\mathbf{R} = \left(\frac{d}{dt}\right)\mathbf{R} = d\mathbf{R}/d\tau)</td>
<td>(\mathbf{P} = m_o\mathbf{U})</td>
<td>(\mathbf{K} = \mathbf{P}/\hbar)</td>
<td>(\partial \cdot \partial = -(m_o c/\hbar)^2)</td>
</tr>
</tbody>
</table>

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit \(\{|v| \ll c\}\), giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM Diagram: RoadMap of SR (4-Vectors)

4-Velocity:
\[ U = \gamma(c, u) \]

4-Momentum:
\[ P = (mc, p) = (E/c, p) \]

4-WaveVector:
\[ K = (\omega/c, k) \]

4-Gradient:
\[ \partial = (\partial_t/c, -\nabla) \]

4-Position:
\[ R = (ct, r) = \langle \text{Event} \rangle \]

SR 4-Vector
(1,0)-Tensor \( V^\mu = V = (\vec{v}, v) \)
(0,2)-Tensor \( V_\mu = (v_0, \vec{v}) \)

SR 4-Scalar
(0,0)-Tensor \( S \), Lorentz Scalar

Trace[\( T^{\mu\nu} \)] = \( \eta_{\mu\nu}T^{\mu\nu} = T \)
\[ V\cdot V = V^\mu\eta_{\mu\nu}V^\nu = (v_0)^2 - \vec{v}\cdot\vec{v} = (v_0)^2 \]
= Lorentz Scalar

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson
SRQM Diagram: RoadMap of SR (Connections)

SR 4-Tensor
(2,0)-Tensor $T^{uv}_{\mu\nu}
(1,1)-Tensor \ T_u^v
(0,2)-Tensor \ T_{\mu
u}$

SR 4-Vector
(1,0)-Tensor $V^\mu$ = $V$ = $(\vec{v}, v)$
SR 4-CoVector
(0,1)-Tensor $V_\mu$ = $(v_\mu, \vec{v})$

SR 4-Scalar
(0,0)-Tensor $S$

Lorentz Scalar
$V \cdot V = V^\mu V_\mu = (\vec{v} \cdot \vec{v}) = (v_0^2 - \vec{v} \cdot \vec{v})$

Trace $[T^{uv}_{\mu\nu}] = \eta_{\mu\nu} T^{uv}_{\mu\nu} = T$

$\partial \cdot R = \delta (ct, \vec{r}) = <Event>$

4-Position $R = (ct, \vec{r})$

4-Gradient $\partial = (\partial/c, -\nabla)$

Minkowski Metric $\delta^{uv}[R^v] = \eta^{uv}$

Lorentz Transform $\delta_{\mu}[R^v] = \Lambda^v_\mu$

Derivative $\partial = d/d\tau = \gamma d/dt$

ProperTime $\mathbf{U} \cdot \partial = d/d\tau = \gamma d/dt$

Total SR Action $P \cdot R = S_{\text{action}}$

Total SR Phase $- \partial R = \Phi_{\text{phase,free}}$

SR 4-Vector $K = (\omega/c, \vec{k})$

SR 4-Momentum $P = (mc, p) = (E/c, p)$

SR 4-Position $R = (ct, \vec{r}) = <Event>$

Hamilton-Jacobi $P_\tau = -\partial[S_{\text{free}}]$

Plane-Waves $K_\tau = -\partial[\Phi_{\text{phase,free}}]$

SR 4-Scalar $Lorentz \ Scalar$
A Tensor Study of Physical 4-Vectors

SR 4-Tensor

(2,0)-Tensor $T^{μν}$
(1,1)-Tensor $T^μ_ν$, or $T^{μν}_ν$
(0,2)-Tensor $T_μ^{νν}$

SR 4-Vector

(1,0)-Tensor $V^μ = V = (V^0, V^ν)$

SR 4-CoVector

(0,1)-Tensor $V_ν = (V_0, -V^ν)$

SR 4-Scalar

(0,0)-Tensor $S$ Lorentz Scalar

Trace of a Tensor $T^{μη}_η = T^{μη}_η = T^{ηη}_η = T$

$V\cdot V = V^μ V^ν = (V^0)^2 - V^ν V_ν = (V^0)^2$

= Lorentz Scalar

---

SRQM Diagram:

RoadMap of SR (Free Particle)

- 4-Velocity $U = γ(c, u)$
- 4-Momentum $P = (mc, p) = (E/c, p)$
- 4-WaveVector $K = (ω/c, k)$
- 4-Position $R = (ct, r) = \langle\text{Event}\rangle$
- 4-Gradient $\partial = (∂/c, -\vec{V})$

Minkowski Metric $δ^{μη}[R^{νι}] = η^{μν}$

Lorentz Transform $δ[R^{νι}] = Λ^{νι}_μ$

Space Time Dim $δR = 4$

ProperTime $U \cdot \partial = d/dτ = γ d/dt$

Derivative $∂μ[R_ν] = Λ_ν_μ$

- $K \cdot R = Φ_{\text{phase,free}}$
- $-K \cdot R = Φ_{\text{phase}}$

SR Action $-P \cdot R = S_{\text{action,free}}$

SR Proper Time $\partial \cdot R = 4$ SpaceTime Dim

SR & QM Waves

SR Wave <Events> have 4-WaveVector = Substantiation oscillations proportional to mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum = Substantiation mass:energy & 3-momentum

*START HERE*: <Events> have 4-Position = Location in SR SpaceTime

ProperTime Derivative $γ d/dτ = d/dt$

Hamilton-Jacobi $P_τ = -∂[S]$ Wave Velocity $v_{\text{group}} * v_{\text{phase}} = c$ Einstein $E = mc^2 = γm_0 c^2$

Plane-Waves $K_τ = -∂[Φ]$
SRQM Diagram:
RoadMap of SR (Free Particle) with Magnitudes

SR 4-Tensor
(2,0)-Tensor $T^{uv}$
(1,1)-Tensor $T^{ur}$
(0,2)-Tensor $T_{uv}$

SR 4-Vector
(2,0)-Tensor $V^u = (\sqrt{v^0}, v)$
SR 4-CoVector
(1,0)-Tensor $V_u = (v^0, v)$
SR 4-Scalar
(0,0)-Tensor $S$ Lorentz Scalar

Trace $[T^{uv}] = n_{uv} T^{uv} = T^v_v = T$
$V \cdot V = V^u n_u V^v = (v^0)^2 - v \cdot v = (v^0)^2$

Lorentz Scalar
### SRQM: The Empirical 4-Vector Facts

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<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>Discoverer</th>
<th>Physics</th>
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<tr>
<td>4-Position</td>
<td>$\mathbf{R} = \langle \text{Event} \rangle$</td>
<td>Newton+Einstein</td>
<td>$[ t &amp; \mathbf{r}]$ Time &amp; Space Dimensions $[ \mathbf{R}=(ct,\mathbf{r}) ]$ SpaceTime</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>$\mathbf{U} = d\mathbf{R}/d\tau$</td>
<td>Newton</td>
<td>$[ v=dr/dt \ ]$ Calculus of motion $[ \mathbf{U}=\gamma(c,\mathbf{u})=d\mathbf{R}/d\tau ]$ Gamma &amp; Proper Time</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>$\mathbf{P} = m_0\mathbf{U}$</td>
<td>Newton</td>
<td>$[ p=mv \ ]$ Classical Mechanics $[ \mathbf{P}=(E/c,p)=m_0\mathbf{U} ]$ SR Mechanics</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>$\mathbf{K} = \mathbf{P}/\hbar$</td>
<td>Planck</td>
<td>$[ \hbar ]$ Thermal Distribution $[ E=\hbar\nu=\hbar\omega ]$ Photoelectric Effect ($\hbar=h/2\pi$) $[ p=\hbar\mathbf{k} ]$ Matter Waves</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>$\partial = -i\mathbf{K}$</td>
<td>Schrödinger</td>
<td>$[ \omega=i\partial_t, \mathbf{k}=-i\mathbf{\nabla} ]$ (SR) Wave Mechanics</td>
</tr>
</tbody>
</table>

(1) The SR 4-Vectors and their components are related to each other via constants
(2) We have not taken any 4-vector relation as axiomatic, the constants come from experiment.
(3) $c$, $\tau$, $m_0$, $\hbar$ come from physical experiments, $(-i)$ comes from the general mathematics of waves
The SRQM 4-Vector Relations Explained

<table>
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<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>What it means in SRQM...</th>
<th>Lorentz Invariant</th>
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<tbody>
<tr>
<td>4-Position $\mathbf{R} = (ct,\mathbf{r})$</td>
<td>$\mathbf{R} = \langle\text{Event}\rangle$</td>
<td>SpaceTime as Unified Concept</td>
<td>$c = \text{LightSpeed}$</td>
</tr>
<tr>
<td>4-Velocity $\mathbf{U} = \gamma(c,\mathbf{u})$</td>
<td>$\mathbf{U} = d\mathbf{R}/d\tau$</td>
<td>Velocity is ProperTime Derivative</td>
<td>$\tau = t_o = \text{ProperTime}$</td>
</tr>
<tr>
<td>4-Momentum $\mathbf{P} = (E/c,\mathbf{p})$</td>
<td>$\mathbf{P} = m_o\mathbf{U}$</td>
<td>Mass:Energy-Momentum Equivalence</td>
<td>$m_o = \text{RestMass}$</td>
</tr>
<tr>
<td>4-WaveVector $\mathbf{K} = (\omega/c,\mathbf{k})$</td>
<td>$\mathbf{K} = \mathbf{P}/\hbar$</td>
<td>Wave-Particle Duality</td>
<td>$\hbar = \text{UniversalAction}$</td>
</tr>
<tr>
<td>4-Gradient $\partial = (\partial/c,-\nabla)$</td>
<td>$\partial = -i\mathbf{K}$</td>
<td>Unitary Evolution, Operator Formalism</td>
<td>$i = \text{ComplexSpace}$</td>
</tr>
</tbody>
</table>

Three old-paradigm QM Axioms: Particle-Wave Duality $[(\mathbf{P})=\hbar(\mathbf{K})]$, Unitary Evolution $[\partial=(-i)\mathbf{K}]$, Operator Formalism $[(\partial)=-i\mathbf{K}]$ are actually just empirically-found constant relations between known SR 4-Vectors. Note that these constants are in fact all Lorentz Scalar Invariants.

Minkowski Space and 4-Vectors also lead to idea of Lorentz Invariance. A Lorentz Invariant is a quantity that always has the same value, independent of the motion of inertial observers. Lorentz Invariants can typically be derived using the scalar product relation.

$\mathbf{U} \cdot \mathbf{U} = c^2$, $\mathbf{U} \cdot \partial = d/d\tau$, $\mathbf{P} \cdot \mathbf{U} = m_o c^2$, etc.

A very important Lorentz invariant is the Proper Time $\tau$, which is defined as the time displacement between two points on a worldline that is at rest wrt. an observer. It is used in the relations between 4-Position $\mathbf{R}$, 4-Velocity $\mathbf{U} = d\mathbf{R}/d\tau$, and 4-Acceleration $\mathbf{A} = d\mathbf{U}/d\tau$. 
SRQM: The SR Path to RQM
Follow the Invariants...

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<tr>
<td>4-Position</td>
<td>( \mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct)^2 )</td>
<td>SR Invariant Interval</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( \mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2 )</td>
<td>Events move into future at magnitude ( c )</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( \mathbf{P} \cdot \mathbf{P} = (m_0c)^2 )</td>
<td>Einstein Mass:Energy Relation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} \cdot \mathbf{K} = (m_0c/\hbar)^2 = (\omega_0/c)^2 )</td>
<td>Matter-Wave Dispersion Relation</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial \cdot \partial = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2 )</td>
<td>The Klein-Gordon Equation ( \rightarrow ) RQM!</td>
</tr>
</tbody>
</table>

\( \mathbf{U} = \frac{d\mathbf{R}}{d\tau} \)

Remember, everything after 4-Velocity was just a constant times the last 4-vector, and the Invariant Magnitude of the 4-Velocity is itself a constant

\( \mathbf{P} = m_0\mathbf{U}, \mathbf{K} = \mathbf{P}/\hbar, \partial = -i\mathbf{K} \), so e.g. \( \mathbf{P} \cdot \mathbf{P} = m_0\mathbf{U} \cdot m_0\mathbf{U} = m_0^2\mathbf{U} \cdot \mathbf{U} = (m_0c)^2 \)

The last equation is the Klein-Gordon RQM Equation, which we have just derived without invoking any QM axioms, only SR plus a few empirical facts.
SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

Treating motion like a particle
Moving particles have a 4-Velocity
4-Momentum is the negative 4-Gradient of the SR Action (S)

Treating motion like a wave
Moving waves have a 4-Velocity
4-WaveVector is the negative 4-Gradient of the SR Phase (Φ)


See SR Wave Definition for info on the Lorentz Scalar Invariant SR WavePhase.
SRQM: Wave-Particle Diffraction/Interference Types

A Tensor Study of Physical 4-Vectors

The 4-Vector Wave-Particle relation is inherent in all particle types: Einstein-de Broglie $P = (E/c, p) = \hbar K = \hbar(\omega/c, k)$.

All waves can diffract: Water waves, gravitational waves, photonic waves of all frequencies, etc. In all cases: experiments using single particles build the diffraction/interference pattern over the course many iterations.

Photons of any frequency encounter a "solid" object or grating. Most often encountered are diffraction gratings and the famous double-slit experiment.

Electrons, neutrons, atoms, small molecules, buckyballs (fullerenes), macromolecules, etc. have been shown to diffract through crystals.
Crystals may be solid single pieces or in powder form.

Kapitsa-Dirac Diffraction: Matter particles diffracted by photonic standing waves.
Electrons, atoms, super-sonic atom beams have been diffracted from resonant standing waves of light.

Photonic-Photonic Diffraction?: Delbruck scattering
Normally, photons do not interact, but at high enough relative energy, virtual particles can form which allow interaction.

4-Momentum $P=(mc, p) = (E/c, p)$

Kap. Diffraction $P = \hbar K$
Hold on, aren't you getting the “ℏ” from a QM Axiom?

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>SR Empirical Fact</th>
<th>What it means...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} = \left( \frac{\omega}{c}, \mathbf{k} \right) = \left( \frac{\omega}{c}, \omega \hat{n} / v \right) )</td>
<td>Wave-Particle Duality</td>
</tr>
</tbody>
</table>

ℏ is actually an empirically measurable quantity, just like e or c. It can be measured classically from the photoelectric effect, the inverse photoelectric effect, from LED’s (injection electroluminescence), from the Duane-Hunt Law in Bremsstrahlung, Electron Diffraction in crystals, the Watt/Kibble-Balance, etc.

For the LED experiment, one uses several different LED’s, each with its own characteristic wavelength. One then makes a chart of wavelength (λ) vs threshold voltage (V) needed to make each individual LED emit. One finds that: \( \lambda = h \times \frac{c}{eV} \), where \( e \) is Electron Charge and \( c \) is Light Speed. ℏ is found by measuring the slope.

Consider this as a blackbox where no assumption about QM is made. However, we know the SR relations \( \{E = eV\} \), and \( \{\lambda f = c\} \).

The data force one to conclude that \( \{E = hf = \hbar \omega\} \).

Due to manifest tensor invariance, this means that 4-Momentum \( \mathbf{P} = \left( \frac{E}{c}, \mathbf{p} \right) = h \mathbf{K} = h(\omega/c, \mathbf{k}) = h^* 4\text{-WaveVector} \mathbf{K} \).

The spatial component (due to De Broglie) follows naturally from the temporal component (due to Einstein) via the nature of 4-Vector mathematics.

This is also derivable from pure SR 4-Vector (Tensor) arguments: \( \mathbf{P} = m_o \mathbf{U} = (E_o/c^2) \mathbf{U} \) and \( \mathbf{K} = (\omega_o/c^2) \mathbf{U} \).

Since \( \mathbf{P} \) and \( \mathbf{K} \) are both Lorentz Scalar proportional to \( \mathbf{U} \), then by the rules of tensor mathematics, \( \mathbf{P} \) must also be Lorentz Scalar proportional to \( \mathbf{K} \).

i.e. Tensors obey certain mathematical structures:

Transitivity{if a~b and b~c, then a~c} & Euclideaness: {if a~c and b~c, then a~b} **Not to be confused with the Euclidean Metric**

This invariant proportional constant is empirically measured to be (ℏ) for each known particle type, massive (\( m_o > 0 \)) or massless (\( m_o = 0 \)):

\( \mathbf{P} = m_o \mathbf{U} = (E_o/c^2) \mathbf{U} = (E_o/c^2)(\omega_o/c^2) \mathbf{K} = (E_o/\omega_o) \mathbf{K} = (\gamma E_o/\gamma \omega_o) \mathbf{K} = (E/\omega) \mathbf{K} = (\hbar) \mathbf{K} \)
Hold on, aren't you getting the “K” from a QM Axiom?

<table>
<thead>
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<th>SR 4-Vector</th>
<th>SR Empirical Fact</th>
<th>What it means...</th>
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<tr>
<td>4-WaveVector</td>
<td>$k = (\omega/c, k) = (\omega/c, \omega \hat{n}/v_{phase}) = (\omega/c^2)U$</td>
<td>Wave-Particle Duality</td>
</tr>
</tbody>
</table>

$K$ is a standard SR 4-Vector, used in generating the SR formulae:

**Relativistic Doppler Effect:**

$$\omega_{obs} = \omega_{emit} / [\gamma(1 - \beta \cos[\theta])], \quad k = \omega/c \text{ for photons}$$

**Relativistic Aberration Effect:**

$$\cos[\theta_{obs}] = (\cos[\theta_{emit}] + |\beta|) / (1 + |\beta|\cos[\theta_{emit}])$$

The 4-WaveVector $K$ can be derived in terms of periodic motion, where families of surfaces move through space as time increases, or alternately, as families of hypersurfaces in SpaceTime, formed by all events passed by the wave surface. The 4-WaveVector is everywhere in the direction of propagation of the wave surfaces.

$$K = -\partial[\Phi_{phase}]$$

From this structure, one obtains relativistic/wave optics without ever mentioning QM.
Hold on, aren't you getting the “-i” from a QM Axiom?

SR 4-Vector | SR Empirical Fact | What it means...
--- | --- | ---
4-Gradient | \(\partial = (\partial_t/c, -\nabla) = -iK\) | Unitary Evolution of States Operator Formalism

\([\partial = -iK]\) gives the sub-equations \([\partial_t = -i\omega]\) and \([\nabla = iK]\), and is certainly the main equation that relates QM and SR by allowing Operator Formalism. But, this is a basic equation regarding the general mathematics of plane-waves; not just quantum-waves, but anything that can be mathematically described by plane-waves and superpositions of plane-waves… This includes purely SR waves, an example of which would be EM plane-waves (i.e. photons)...

\(\psi(t,r) = ae^{[i(k \cdot r - \omega t)]}\): Standard mathematical plane-wave equation

\(\partial[\psi(t,r)] = \partial[ae^{[i(k \cdot r - \omega t)]}] = (-i\omega)[ae^{[i(k \cdot r - \omega t)]}] = (-i\omega)\psi(t,r), \) or \([\partial_t = -i\omega]\)
\(\nabla[\psi(t,r)] = \nabla[ae^{[i(k \cdot r - \omega t)]}] = (iK)[ae^{[i(k \cdot r - \omega t)]}] = (iK)\psi(t,r), \) or \([\nabla = iK]\)

In the more economical SR notation:
\(\partial[\psi(R)] = \partial[ae^{(-iK \cdot R)]} = (-iK)[ae^{(-iK \cdot R)]} = (-iK)\psi(R), \) or \([\partial = -iK]\)

This one is more of a mathematical empirical fact, but regardless, it is not axiomatic. It can describe purely SR waves, again without any mention of QM.
Hold on, aren't you getting the “∂” from a QM Axiom?

| SR 4-Vector | SR Empirical Fact | What it means...
<table>
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<tr>
<td>4-Gradient</td>
<td>∂ = (∂/c, -∇) = -iK</td>
<td>4D Gradient Operator</td>
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</table>

[∂ = (∂/c, -∇)] is the SR 4-Vector Gradient Operator. It occurs in a purely relativistic context without ever mentioning QM.

\[ \partial \cdot \mathbf{X} = (\partial/ct, -\nabla) \cdot (ct, \mathbf{x}) = (\partial/ct - (-\nabla \cdot \mathbf{x})) = (\partial[t] + \nabla \cdot \mathbf{x}) (1)+(3) = 4 \]

The 4-Divergence of the 4-Position \((\partial \cdot \mathbf{X} = \partial^\mu \eta_{\mu\nu} X^\nu)\) gives the dimensionality of SpaceTime.

\[ \partial[\mathbf{X}] = (\partial/ct, -\nabla)(ct, \mathbf{x}) = (\partial/ct, -\nabla[\mathbf{x}]) = \text{Diag}[1, -1] = \eta^{\mu\nu} \]

The 4-Gradient acting on the 4-Position \((\partial[\mathbf{X}] = \partial^\mu[X^\nu])\) gives the Minkowski Metric Tensor.

\[ \partial \cdot \mathbf{J} = (\partial/ct, -\nabla) \cdot (pc, \mathbf{j}) = (\partial/ct[pc] - (-\nabla \cdot \mathbf{j})) = (\partial[p] + \nabla \cdot \mathbf{j}) = 0 \]

The 4-Divergence of the 4-CurrentDensity is equal to 0 for a conserved current. It can be rewritten as \((\partial[p] = - \nabla \cdot \mathbf{j})\), which means that the time change of ChargeDensity is balanced by the space change or divergence of CurrentDensity. It is a Continuity Equation, giving local conservation of ChargeDensity. It is related to Noether’s Theorem.
Hold on, doesn’t using “∂” in an Equation of Motion presume a QM Axiom?

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<tr>
<td>4-(Position)Gradient</td>
<td>∂ₚ = ∂ = (∂/c,-∇) = -iK</td>
<td>4D Gradient Operator</td>
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</table>

Klein-Gordon Relativistic Quantum Wave Equation
∂⋅∂[Ψ] = -(m₀/c²)[Ψ] = -(ω₀/c²)[Ψ]

Relativistic Euler-Lagrange Equations
∂ₚ[L] = (d/dτ)∂u[L]: {particle format}
∂[Φ][L] = (∂ₚ) ∂[∂ₚ(Φ)][L]: {density format}

[∂ = (∂/c,-∇)] is the SR 4-Vector (Position)Gradient Operator.
It occurs in a purely relativistic context without ever mentioning QM.
There is a long history of using the gradient operator on classical physics functions, in this case the Lagrangian. And, in fact, it is another area where the same mathematics is used in both classical and quantum contexts.
The QM Schrödinger Relation

\[ P = i\hbar \partial \]

This is derived from the combination of:

The Einstein-de Broglie Relation

\[ P = \hbar K \]

Complex Plane-Waves

\[ K = i\partial \]

These are the standard QM Schrödinger Relations. It is this Lorentz Scalar Invariant relation \((i\hbar)\) which connects the 4-Momentum to the 4-Gradient, making it into a QM operator.

Note that these 4-Vectors are already connected in multiple ways in standard SR.

A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson
Review of SR 4-Vector Mathematics

4-Gradient \( \partial = (\partial /c, -\nabla) \)

4-Position \( X = (ct, x) \)

4-Velocity \( U = \gamma (c, u) \)

4-Momentum \( P = (E/c, p) = (E_o/c^2)U \)

4-WaveVector \( K = (\omega/c, k) = (\omega_o/c^2)U \)

\( \partial \cdot X = (\partial /c, -\nabla)(ct, x) = (\partial /c[ct] - (-\nabla \cdot x)) = 1 - (-3) = 4: \) Dimensionality of SpaceTime

\( U \cdot \partial = \gamma (c, u) \cdot (\partial /c, -\nabla) = \gamma (\partial + u \cdot \nabla) = \gamma (d/dt) = d/d\tau: \) Derivative wrt. ProperTime is Lorentz Scalar

\( \partial[X] = (\partial /c, -\nabla)((ct, x)) = (\partial /c[ct], -\nabla[x]) = \text{Diag}[1, -1] = \eta^{ij}: \) The Minkowski Metric

\( \partial[K] = (\partial /c, -\nabla)(\omega/c, k) = (\partial /c[\omega/c], -\nabla[k]) = [0] \)

\( K \cdot X = (\omega/c, k) \cdot (ct, x) = (\omega t - k \cdot x) = \phi: \) Phase of SR Wave

\( \partial[K \cdot X] = \partial[K] \cdot X + K \cdot \partial[X] = K = -\partial[\phi]: \) Neg 4-Gradient of Phase gives 4-WaveVector

\( (\partial \cdot \partial)[K \cdot X] = ((\partial /c)^2 - \nabla \cdot \nabla)(\omega t - k \cdot x) = 0: \) Wave Continuity Equation, No sources or sinks

\( (\partial \cdot \partial)[K \cdot X] = \partial \cdot (\partial[K \cdot X]) = \partial \cdot K = 0: \)

let \( f = ae^b(K \cdot X): \)
then \( \partial[f] = (-iK)ae^{b-i(K \cdot X)} = (-iK)f: \) \( \partial = -iK: \)
and \( \partial \cdot \partial[f] = (-i)^2(K \cdot K)f = -(\omega_o/c)^2 f: \)
\( (\partial \cdot \partial) = (\partial /c)^2 - \nabla \cdot \nabla = -(\omega_o/c)^2: \) Standard mathematical plane-waves if \( b = -i \)

Unitary Evolution, Operator Formalism

The Klein-Gordon Equation \( \rightarrow \) RQM

Note that no QM Axioms are assumed: This is all just pure SR 4-vector (tensor) manipulation
Review of SR 4-Vector Mathematics

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = -(\omega_0 / c)^2 = -(1/\lambda c)^2 \)

Let \( X_T = (ct+c\Delta t, \mathbf{x}) \), then \( \partial [X_T] = (\partial / c, -\nabla)(ct+c\Delta t, \mathbf{x}) = \text{Diag}[1, -I(3)] = \partial[X] = \eta^{\mu \nu} \)
so \( \partial [X_T] = \partial[X] \) and \( \partial[K] = [[0]] \)

Let \( f = ae^{-i(K \cdot X_T)} \), the time translated version

\[
\begin{align*}
(\partial \cdot \partial)[f] \\
\partial \cdot (\partial[f]) \\
\partial \cdot (\partial[e^{-i(K \cdot X_T)}]) \\
\partial \cdot (e^{-i(K \cdot X_T)} \partial[-i(K \cdot X_T)]) \\
-i\partial \cdot (f \partial[K \cdot X_T]) \\
-i\partial[f] \partial[K \cdot X_T] + \Psi(\partial \cdot \partial)[K \cdot X_T]) \\
(-i)^2 f(\partial[K \cdot X_T])^2 + 0 \\
(-i)^2 f(\partial[K] \cdot X_T + K \cdot \partial[X_T])^2 \\
(-i)^2 f(0 + K \cdot \partial[X])^2 \\
(-i)^2 f(K)^2 \\
-(K \cdot K)f \\
-(\omega_0 / c)^2 f
\end{align*}
\]
What does the Klein-Gordon Equation give us?... **A lot of RQM!**

Relativistic Quantum Wave Equation: \[ \partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla = -(m_o c/\hbar)^2 = (im_o c/\hbar)^2 = -(\omega_o/c)^2 \]

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles (Scalars)
Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (Spinors)
Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Taking the low-velocity-limit of the KG leads to the standard QM non-relativistic Schrödinger Eqn, for spin=0
Taking the low-velocity-limit of the Dirac leads to the standard QM non-relativistic Pauli Eqn, for spin=1/2

Setting RestMass \( m_o \rightarrow 0 \) leads to the RQM Free Wave, Weyl, and Free Maxwell Eqns

In all of these cases, the equations can be modified to work with various potentials by using more SR 4-Vectors, and more empirically found relations between them, e.g. the Minimal Coupling Relations:
4-TotalMomentum \( \mathbf{P}_{\text{tot}} = \mathbf{P} + q\mathbf{A} \), where \( \mathbf{P} \) is the particle 4-Momentum, \( q \) is a charge, and \( \mathbf{A} \) is a 4-VectorPotential, typically the 4-EMVectorPotential.

Also note that generating QM from RQM (via a low-energy limit) is much more natural than attempting to “relativize or generalize” a given NRQM equation. Facts assumed from a non-relativistic equation may or may not be applicable to a relativistic one, whereas the relativistic facts are still true in the low-velocity limiting-cases. This leads to the idea that QM is an approximation only of a more general RQM, just as SR is an approximation only of GR.
# Relativistic Quantum Wave Eqns.

| Spin-(Statistics) | Relativistic Light-like | Relativistic Matter-like | Non-Relativistic Limit (|v|<<c) | Field Representation |
|-------------------|-------------------------|--------------------------|-------------------------------|----------------------|
| Bose-Einstein=n   | Mass = 0                | Mass > 0                 | Mass >0                       | Scalar (
| Fermi-Dirac=n/2  |                         |                          |                          | (0-Tensor) |

| Spin-(Statistics) | Relativistic Light-like | Relativistic Matter-like | Non-Relativistic Limit (|v|<<c) | Field Representation |
|-------------------|-------------------------|--------------------------|-------------------------------|----------------------|
| Bose-Einstein=n   | Mass = 0                | Mass > 0                 | Mass >0                       | Scalar (0-Tensor) |
| Fermi-Dirac=n/2  |                         |                          |                               | $\Psi = \Psi[0]$ |

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<th>Klein-Gordon</th>
<th>Schrödinger</th>
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<td>N-G Bosons</td>
<td>((\partial \cdot \partial))(\Psi = 0)</td>
<td>Higgs Bosons, maybe Axions</td>
<td>Common NRQM Systems</td>
</tr>
<tr>
<td></td>
<td></td>
<td>((\partial \partial + (m_\text{c}/c)^2))(\Psi = 0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>with minimal coupling</td>
<td>with minimal coupling</td>
</tr>
<tr>
<td></td>
<td></td>
<td>((i(\hbar\partial - \gamma \gamma \gamma))(\Psi = 0)</td>
<td>(i(\hbar\partial - \gamma \gamma \gamma))(\Psi = 0)</td>
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<tr>
<td></td>
<td></td>
<td>?Axions? are KG with EM invariant src term</td>
<td></td>
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<tr>
<td></td>
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<td>((\gamma \partial - m_\text{c}/c))(\Psi = 0)</td>
<td>(i(\hbar\partial - \gamma \gamma \gamma))(\Psi = 0)</td>
<td></td>
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<tr>
<td>factored to Right &amp; Left Spinors</td>
<td>with minimal coupling</td>
<td>with minimal coupling</td>
<td></td>
</tr>
<tr>
<td>((\sigma \partial))(\Psi = 0)</td>
<td>((\gamma \partial + im_\text{c}/c))(\Psi = 0)</td>
<td>(i(\hbar\partial - \gamma \gamma \gamma))(\Psi = 0)</td>
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<tr>
<td>L = i(\Psi \partial \partial \partial \partial \psi)</td>
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<td>1-Tensor</td>
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<tr>
<td>((\partial \partial))(\mathbf{A} = 0)</td>
<td>((\partial \partial))(\mathbf{A} = 0)</td>
<td>A = A'</td>
<td></td>
</tr>
<tr>
<td>((\partial \partial))(\mathbf{A} = \mu_\text{J}) w current src where ((\partial \partial))(\mathbf{A} = 0)</td>
<td>where ((\partial \partial))(\mathbf{A} = 0)</td>
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</tr>
<tr>
<td>((\partial \partial))(\mathbf{A} = \mu_\text{e}\mathbf{E}\mathbf{A}) QED</td>
<td>((\partial \partial))(\mathbf{A} = 0)</td>
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John B. Wilson
Klein-Gordon Equation: $\partial^2 - \nabla^2 = -(m_0 c^2 \hbar)^2$

Since the 4-vectors are related by constants, we can go back to the 4-Momentum description:

$$(\partial^2 - \nabla^2) = -(m_0 c^2 \hbar)^2$$

Factoring: $[ E - c \boldsymbol{\alpha} \cdot \boldsymbol{p} - \beta(m_0 c^2) ] [ E + c \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta(m_0 c^2) ] = 0$

$E$ & $p$ are quantum operators, $\alpha$ & $\beta$ are matrices which must obey $\alpha \beta = -\beta \alpha, \alpha_i \alpha_j = -\alpha_j \alpha_i, \alpha_i^2 = \beta^2 = I$

The left hand term can be set to 0 by itself, giving...

$[ E - c \boldsymbol{\alpha} \cdot \boldsymbol{p} - \beta(m_0 c^2) ] = 0$, which is one form of the Dirac equation

Remember: $P^\mu = (p^0, \boldsymbol{p}) = (E/c, \boldsymbol{p})$ and $\alpha^\mu = (\alpha^0, \alpha)$ where $\alpha^0 = I_2$

$$[ E - c \boldsymbol{\alpha} \cdot \boldsymbol{p} - \beta(m_0 c^2) ] = [ c\alpha^0 p^0 - c \boldsymbol{\alpha} \cdot \boldsymbol{p} - \beta(m_0 c^2) ] = [ c\alpha^0 P_\mu - \beta(m_0 c^2) ] = 0$$

$\alpha^\mu \partial_\mu = -\beta(Im_0 c/\hbar)$

Transforming from Pauli Spinor (2 component) to Dirac Spinor (4 component) form:

Dirac Equation: $(\gamma^\mu \partial_\mu)[\psi] = -(Im_0 c/\hbar)\psi$

Thus, the Dirac Eqn is guaranteed by construction to be one solution of the KG Eqn

The KG Equation is at the heart of all the various relativistic wave equations, which differ based on mass and spin values, but all of them respect $E^2 - c^2 \boldsymbol{p} \cdot \boldsymbol{p} - (m_0 c^2)^2 = 0$
Relativistic Quantum Wave Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = - (m_o c / \hbar)^2 = (im_o c / \hbar)^2 = -(\omega_o / c)^2 \)

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles \{Higgs\} (4-Scalars)
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Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Setting RestMass \( m_o \rightarrow 0 \) leads to the:
RQM Free Wave (4-Scalar massless)
RQM Weyl (4-Spinor massless)
Free Maxwell Eqns (4-Vector massless)

So, the same Relativistic Quantum Wave Equation is simply applied to different SR Tensorial Quantum Fields
See Mathematical_formulation_of_the_Standard_Model at Wikipedia:

4-Scalar (massive) Higgs Field \( \varphi \) \[ \partial \cdot \partial = - (m_o c / \hbar)^2 \varphi \] Free Field Eqn→Klein-Gordon Eqn \( \partial \cdot \partial \varphi = -(m_o c / \hbar)^2 \varphi \)

4-Vector (massive) Weak Field \( Z^\mu, W^\mu \) \[ \partial \cdot \partial = - (m_o c / \hbar)^2 Z^\mu \] Free Field Eqn→Proca Eqn \( \partial \cdot \partial Z^\mu = -(m_o c / \hbar)^2 Z^\mu \)

4-Vector (massless \( m_o = 0 \)) Photon Field \( A^\mu \) \[ \partial \cdot A^\mu = 0 \] Free Field Eqn→EM Wave Eqn \( \partial \cdot A^\mu = 0 \)

4-Spinor (massive) Fermion Field \( \psi \) \[ \gamma^\mu \cdot \partial = -i m_o c / \hbar \psi \] Free Field Eqn→Dirac Eqn \( \gamma^\mu \cdot \partial \psi = -(i m_o c / \hbar) \psi \)

*The Fermion field is a special case, the Dirac Gamma Matrices \( \gamma^\mu \) and 4-Spinor field \( \psi \) work together to preserve Lorentz Invariance.
In relativistic quantum mechanics and quantum field theory, the Bargmann–Wigner equations describe free particles of arbitrary spin $j$, an integer for bosons ($j = 1, 2, 3 \ldots$) or half-integer for fermions ($j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields.

Bargmann–Wigner equations: $\left(-\gamma^\mu P_\mu + mc_\alpha^\alpha' r^\alpha_1 \ldots^\alpha_r |^\alpha_1 \ldots^\alpha_r = 0\right)$

In relativistic quantum mechanics and quantum field theory, the Joos–Weinberg equation is a relativistic wave equations applicable to free particles of arbitrary spin $j$, an integer for bosons ($j = 1, 2, 3 \ldots$) or half-integer for fermions ($j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields. The spin quantum number is usually denoted by $s$ in quantum mechanics, however in this context $j$ is more typical in the literature.

Joos–Weinberg equation: $\left[\gamma^{\mu_1 \mu_2 \ldots \mu_2 j} P_{\mu_1} P_{\mu_2} \ldots P_{\mu_2 j} + (mc)^2\right] \Psi = 0$

The primary difference appears to be the expansion in either the wavefunctions for (BW) or the Dirac Gamma’s for (JW)

For both of these: A state or quantum field in such a representation would satisfy no field equation except the Klein-Gordon equation.

Yet another form is the Duffin-Kemmer-Petiau Equation vs Dirac Equation

DKP Eqn (spin 0 or 1): $(i\hbar \beta^a \delta_a - m_c)\Psi = 0$, with $\beta^a$ as the DKP matrices

Dirac Eqn (spin $\frac{1}{2}$): $(i\hbar \gamma^a \delta_a - m_c)\Psi = 0$, with $\gamma^a$ as the Dirac Gamma matrices
A few more SR 4-Vectors

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<th>SR 4-Vector</th>
<th>Definition</th>
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<td>4-Position</td>
<td>$\mathbf{R} = (ct,r)$; alt. $\mathbf{X} = (ct,x)$</td>
<td>Time, Space</td>
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<tr>
<td>4-Velocity</td>
<td>$\mathbf{U} = \gamma(\mathbf{c},\mathbf{u})$</td>
<td>Gamma, Velocity</td>
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<td>4-Momentum</td>
<td>$\mathbf{P} = (E/c,\mathbf{p}) = (mc,\mathbf{p})$</td>
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<td>4-VectorPotential</td>
<td>$\mathbf{A} = (\varphi/c,\mathbf{a})$</td>
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<td>4-TotalMomentum</td>
<td>$\mathbf{P}_{\text{tot}} = (E/c+q\varphi/c,\mathbf{p}+qa)$</td>
<td>Energy-Momentum inc. EM fields</td>
</tr>
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<td>$\mathbf{K}_{\text{tot}} = (\omega/c+(q/\hbar)\varphi/c,\mathbf{k}+(q/\hbar)a)$</td>
<td>Freq-WaveNum inc. EM fields</td>
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<td>4-CurrentDensity</td>
<td>$\mathbf{J} = (c\rho,\mathbf{j}) = q\mathbf{J}_{\text{prob}}$</td>
<td>Charge Density, Current Density</td>
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<td>4-ProbabilityCurrentDensity</td>
<td>$\mathbf{J}<em>{\text{prob}} = (c\rho</em>{\text{prob}},\mathbf{j}_{\text{prob}})$</td>
<td>QM Probability (Density, Current Density)</td>
</tr>
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- \(\varphi\) is the scalar potential
- \(\mathbf{n}\) is the unit vector in the direction of propagation
- \(\rho\) is the charge density
- \(\mathbf{j}\) is the current density
- \(\rho_{\text{prob}}\) and \(\mathbf{j}_{\text{prob}}\) are the probability density and current density in QM
# More SR 4-Vectors Explained

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<td>( \mathbf{R} = (ct, r) )</td>
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<td>( \mathbf{U} = d\mathbf{R}/d\tau )</td>
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<td>4-Momentum</td>
<td>( \mathbf{P} = m_o \mathbf{U} = (E_o/c^2) \mathbf{U} )</td>
<td>Mass-Energy-Momentum Equivalence</td>
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</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial = -i\mathbf{K} )</td>
<td>Unitary Evolution of States</td>
</tr>
<tr>
<td>Operator Formalism, Complex Waves</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4-VectorPotential</td>
<td>( \mathbf{A} = (\phi/c, a) = (\phi_o/c^2) \mathbf{U} )</td>
<td>Potential Fields...</td>
</tr>
<tr>
<td>4-TotalMomentum</td>
<td>( \mathbf{P}_{\text{tot}} = \mathbf{P} + q\mathbf{A} )</td>
<td>Energy-Momentum inc. Potential Fields</td>
</tr>
<tr>
<td>4-TotalWaveVector</td>
<td>( \mathbf{K}_{\text{tot}} = \mathbf{K} + (q/\hbar)\mathbf{A} )</td>
<td>Freq-WaveNum inc. Potential Fields</td>
</tr>
<tr>
<td>4-CurrentDensity</td>
<td>( \mathbf{J} = \rho_o \mathbf{U} = q\mathbf{J}_{\text{prob}} )</td>
<td>ChargeDensity-CurrentDensity Equivalence</td>
</tr>
<tr>
<td></td>
<td>( \partial \cdot \mathbf{J} = 0 )</td>
<td>CurrentDensity is conserved</td>
</tr>
<tr>
<td>4-Probability</td>
<td>( \mathbf{J}<em>{\text{prob}} = (c\rho</em>{\text{prob}}, j_{\text{prob}}) )</td>
<td>QM Probability from SR</td>
</tr>
<tr>
<td>CurrentDensity</td>
<td>( \partial \cdot \mathbf{J}_{\text{prob}} = 0 )</td>
<td>Probability Worldlines are conserved</td>
</tr>
</tbody>
</table>
Minimal Coupling = Potential Interaction

Klein-Gordon Eqn $\rightarrow$ Schrödinger Eqn

$P_T = P + Q = P + qA$

Minimal Coupling: Total = Dynamic + Charge_Coupled to 4-(EM)VectorPotential

$K = i\partial$

Complex Plane-Waves

$P = \hbar K$

Einstein-de Broglie QM Relations

$P = i\hbar \partial$

Schrödinger Relations

$P = (E/c, \mathbf{p}) = (E_T - q\varphi/c, \mathbf{p}_T - qa)$

$\delta = (\partial/c, \nabla) = \delta_T + (i\hbar/\hbar)\mathbf{A} = (\partial_T/c + (i\hbar/\hbar)\varphi/c, -\nabla_T + (i\hbar/\hbar)a)$

The Klein-Gordon RQM Wave Equation (relativistic QM)

$\delta \cdot \delta = (\partial/c)^2 - \nabla^2 = -(m_o c/\hbar)^2$

Einstein Mass:Energy:Momentum Equivalence

$P \cdot P = (E/c)^2 - \mathbf{p}^2 = (m_o c)^2$

Relativistic

$E \sim [ (m_o c)^2 + \mathbf{p}^2/2m_o ]$

Low velocity limit \{ |v| \ll c \} from \((1+x)^n \sim [1 + nx + O(x^2)]\) for \(|x| \ll 1\)

$(E_T - q\varphi)^2 = (m_o c)^2 + c^2(p_{t-q}a)^2$

Relativistic with Minimal Coupling

Low velocity with Minimal Coupling

$(i\hbar\partial_T - q\varphi)^2 = (m_o c)^2 + c^2(i\hbar\nabla_T - qa)^2$

Relativistic with Minimal Coupling

Low velocity with Minimal Coupling

$(i\hbar\partial_T - q\varphi) \sim [ (m_o c)^2 + c^2(i\hbar\nabla_T - qa)^2/2m_o ]$

Low velocity with Minimal Coupling

Low velocity with Minimal Coupling

$(i\hbar\partial_T) \sim [ \mathbf{V} + (i\hbar\nabla_T + qa)^2/2m_o ]$

Typically the 3-vector_potential $\mathbf{a} \sim 0$ in many situations

$(i\hbar\partial_T)|\Psi> \sim [ \mathbf{V} - (i\hbar\nabla_T)^2/2m_o ]|\Psi> :$

The Schrödinger NRQM Wave Equation (non-relativistic QM)
Once one has a **Relativistic Wave Eqn...**

Klein-Gordon Equation: \( \partial \cdot \partial = \left( \frac{\partial}{c} \right)^2 - \nabla \cdot \nabla = \frac{-i m_0 c}{\hbar} \) \( \equiv \frac{-m_0 c}{\hbar} \)

Once we have derived a RWE, what does it imply?

The KG Eqn. was derived from the physics of SR plus a few empirical facts. It is a 2\(^{nd}\) order, linear, wave PDE that pertains to physical objects of reality from SR.

Just being a linear wave PDE implies all the mathematical techniques that have been discovered to solve such equations generally: Hilbert Space, Superpositions, \(<\text{Bra}|,|\text{Ket}>\) notation, wavevectors, wavefunctions, etc. These things are from mathematics in general, not only and specifically from an Axiom of QM.

Therefore, if one has a physical RWE, it implies the mathematics of waves, the formalism of the mathematics, and thus the mathematical Principles and Formalism of QM. Again, QM Axioms are not required – they emerge from the physics and math...
Once one has a Relativistic Wave Eqn... Examine **Photon Polarization**

From the Wikipedia page on [Photon Polarization]

Photon polarization is the quantum mechanical description of the classical polarized sinusoidal plane electromagnetic wave. An individual photon can be described as having right or left circular polarization, or a superposition of the two. Equivalently, a photon can be described as having horizontal or vertical linear polarization, or a superposition of the two.

The description of photon polarization contains many of the physical concepts and much of the mathematical machinery of more involved quantum descriptions and forms a fundamental basis for an understanding of more complicated quantum phenomena. Much of the mathematical machinery of quantum mechanics, such as state vectors, probability amplitudes, unitary operators, and Hermitian operators, emerge naturally from the classical Maxwell's equations in the description. The quantum polarization state vector for the photon, for instance, is identical with the Jones vector, usually used to describe the polarization of a classical wave. Unitary operators emerge from the classical requirement of the conservation of energy of a classical wave propagating through lossless media that alter the polarization state of the wave. Hermitian operators then follow for infinitesimal transformations of a classical polarization state.

Many of the implications of the mathematical machinery are easily verified experimentally. In fact, many of the experiments can be performed with two pairs (or one broken pair) of polaroid sunglasses.

The connection with quantum mechanics is made through the identification of a minimum packet size, called a photon, for energy in the electromagnetic field. The identification is based on the theories of Planck and the interpretation of those theories by Einstein. The correspondence principle then allows the identification of momentum and angular momentum (called spin), as well as energy, with the photon.
Principle of Superposition: From the mathematics of waves

Klein-Gordon Equation: $\partial^2 = (\partial^2/c)^2 - \nabla^2 = -(m_o c / \hbar)^2 = -(\omega_o/c)^2$

The Extended Superposition Principle for Linear Equations

Suppose that the non-homogeneous equation, where $L$ is linear, is solved by some particular $u_p$

Suppose that the associated homogeneous problem is solved by a sequence of $u_i$.

$L(u_p) = C; \ L(u_0) = 0, \ L(u_1) = 0, \ L(u_2) = 0 ...$

Then $u_p$ plus any linear combination of the $u_n$ satisfies the original non-homogeneous equation:

$L(u_p + \sum a_n u_n) = C,$

where $a_n$ is a sequence of (possibly complex) constants and the sum is arbitrary.

Note that there is no mention of partial differentiation. Indeed, it's true for any linear equation, algebraic or integro-partial differential-whatever.

QM superposition is not axiomatic, it emerges from the mathematics of the Linear PDE.
Klein-Gordon Equation: \( \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_o c/\hbar)^2 = -(\omega_o/c)^2 \)

\( \mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_o/c)^2 \): The particular solution (w rest mass)

\( \mathbf{K}_n \cdot \mathbf{K}_n = (\omega_n/c)^2 - \mathbf{k}_n \cdot \mathbf{k}_n = 0 \): The homogenous solution for a (virtual photon?) microstate n

Note that \( \mathbf{K}_n \cdot \mathbf{K}_n = 0 \) is a null 4-vector (photonic)

Let \( \Psi_p = Ae^{-i(\mathbf{K} \cdot \mathbf{X})} \), then \( \partial \cdot \partial[\Psi_p] = (-i)^2(\mathbf{K} \cdot \mathbf{K})\Psi_p = -(\omega_o/c)^2\Psi_p \)

which is the Klein-Gordon Equation, the particular solution...

Let \( \Psi_n = A_n e^{-i(\mathbf{K}_n \cdot \mathbf{X})} \), then \( \partial \cdot \partial[\Psi_n] = (-i)^2(\mathbf{K}_n \cdot \mathbf{K}_n)\Psi_n = (0)\Psi_n \)

which is the Klein-Gordon Equation homogeneous solution for a microstate n

We may take \( \Psi = \Psi_p + \sum_n \Psi_n \)

Hence, the Principle of Superposition is not required as an QM Axiom, it follows from SR and our empirical facts which lead to the Klein-Gordon Equation. The Klein-Gordon equation is a linear wave PDE, which has overall solutions which can be the complex linear sums of individual solutions – i.e. it obeys the Principle of Superposition. This is not an axiom – it is a general mathematical property of linear PDE's. This property continues over as well to the limiting case { \(|\mathbf{v}|<<c\} of the Schrödinger Equation.
QM Hilbert Space: From the mathematics of waves

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_\sigma c^2/\hbar)^2 \)

Hilbert Space (HS) representation:
if \( |\Psi> \in HS \), then \( c|\Psi> \in HS \), where \( c \) is complex number
if \( |\Psi_1> \) and \( |\Psi_2> \in HS \), then \( |\Psi_1> + |\Psi_2> \in HS \)
if \( |\Psi> = c_1|\Psi_1> + c_2|\Psi_2> \), then \( <\Phi|\Psi> = c_1<\Phi|\Psi_1> + c_2<\Phi|\Psi_2> \) and \( <\Psi| = c_1^*<\Psi_1| + c_2^*<\Psi_2| \)

\(<\Phi|\Psi> = <\Psi|\Phi>\)
\(<\Psi|\Psi> \geq 0\)
if \( <\Psi|\Psi> = 0 \), then \( |\Psi> = 0 \)

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. They are indispensable tools in the theories of partial differential equations, Fourier analysis, signal processing, heat transfer, ergodic theory, and Quantum Mechanics.

The QM Hilbert Space emerges from the fact that the KG Equation is a linear wave PDE – Hilbert spaces as solutions to PDE’s are a purely mathematical phenomenon – no QM Axiom is required.

Likewise, this introduces the \(<\text{bra}|,|\text{ket}>\) notation, wavevectors, wavefunctions, etc.

Note:

One can use Hilbert Space descriptions of Classical Mechanics using the Koopman-von Neumann formulation. One can not use Hilbert Space descriptions of Quantum Mechanics by using the Phase Space formulation of QM.
Canonical Commutation Relation: Viewed from standard QM

Standard QM Canonical Commutation Relation: $[x^j, p^k] = i\hbar \delta^{jk}$

The Standard QM Canonical Commutation Relation is simply an axiom in standard QM. It is just given, with no explanation. You just had to accept it.

I always found that unsatisfactory.

There are at least 4 parts to it:

Where does the commutation ($[ , ]$) come from?
Where does the imaginary constant ($i$) come from?
Where does the Planck constant ($\hbar$) come from?
Where does the Kronecker Delta ($\delta^{jk}$) come from?

See the next page for SR enlightenment...
Let $(f)$ be an arbitrary SR function $X[f] = Xf$, $\partial f = \partial f$.
$X$, function or not, has no effect on $(f)$
$\partial = \partial f$ is definitely an SR function:operator

$X[a[f]] = Xa[f]$
$\partial[a[f]] - Xa[f]$
$\partial[a[f]] - Xa[f] = \partial [a[f]]$

Recognize this as a commutation relation
$[\partial, X]f = \partial Xf$

$\partial \cdot X = \partial \cdot X^\prime = 4$
$\partial \cdot X = \partial \cdot X^\prime = 4$

$\partial \cdot [X^\prime] = \partial \cdot X^\prime = \Lambda^\prime$

$\partial \cdot [X^\prime] = \partial \cdot X^\prime = \Lambda^\prime$

$\partial = (\partial/c, -\nabla)$
$\partial = (\partial/c, -\nabla)$

$\Lambda = \text{Diag}[1,-1,-1,-1]$
$\Lambda = \text{Diag}[1,-1,-1,-1]$

$\Lambda = \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-\delta^k]$

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$\Lambda = \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-\delta^k]$
SRQM Study:
4-Position and 4-Gradient

SR: Minkowski Metric
\[ \delta[R] = \delta^\mu \delta^\nu = \eta^{\mu\nu} \]
\[ \rightarrow \text{Diag}[1,-1,-1,-1] \]
\[ = \text{Diag}[1,-1,-1] \]
\[ = \text{Diag}[1,-\delta^k_k] \]
\[ \text{(in Cartesian form)} \]
\[ \{\eta_{\mu\nu}\} = 1/\{\eta^{\mu\nu}\} \]
\[ \text{Tr}[\eta^{\mu\nu}] = 4 \]
\[ \eta^\mu_\nu = \delta^\mu_\nu \]

Rotations
Boosts
CPT

SR: Lorentz Transform
\[ \partial_[[R]] = \partial R^\mu/\partial R^\nu = \Lambda^\mu_\nu \]
\[ \Lambda^\mu_\alpha \Lambda^\alpha_\beta = \eta^\mu_\beta \]
\[ \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta} \]
\[ \text{(Det}[\Lambda]\text{)}^2 = 1 \]
\[ \text{Det}[\Lambda] = \pm 1 \]
\[ \Lambda^\mu_\nu = (\Lambda^{-1})^\nu_\mu \]
\[ \Lambda_{\mu\nu} \Lambda^\nu_\mu = 4 \]

SpaceTime
\[ \partial\cdot R = \partial_\mu R^\mu = 4 \]
Dimension

SRQM: Tensor Zero Exterior Product
\[ \partial^\mu R = \partial^\mu R^\nu - \partial^\nu R^\mu = \eta^{\mu\nu} - \eta^{\nu\mu} = 0^{\mu\nu} \]

SRQM: Non-Zero Commutation
\[ [\partial, R] = [\partial^\mu, R^\nu] \]
\[ = \partial^\mu R^\nu - \partial^\nu R^\mu \]
\[ = \eta^{\mu\nu} \]

Invariant Interval
\[ R\cdot R = (ct)^2 - r\cdot r = (ct)^2 \]

Invariant d’Alembertian
Wave Equation
\[ \partial\cdot\partial = (\partial/c, -\nabla) \]
\[ = (\partial/c, -\partial_x, -\partial_y, -\partial_z) \]

4-Gradient
\[ \partial = (\partial/c, -\nabla) \]
\[ = (\partial/c, -\partial_0, -\partial_x, -\partial_y, -\partial_z) \]

4-Position
\[ R = (ct, r) \]

SR 4-Tensor
\[ (2,0)-\text{Tensor } T^\mu_\nu \]
\[ (1,1)-\text{Tensor } T_\mu^\nu, \text{ or } T^\nu_\mu \]
\[ (0,2)-\text{Tensor } T_{\mu\nu} \]

SR 4-Vector
\[ (1,0)-\text{Tensor } V^\nu = V = (\vec{v}, v) \]
\[ (0,1)-\text{Tensor } V_\nu = (v_0, \vec{v}) \]

SR 4-Scalars
\[ (0,0)-\text{Tensor } S \]
Lorentz Scalar
\[ \text{Trace}[T^\mu_\nu] = \eta_{\mu\nu} T^\mu_\nu = T^\nu_\nu = T \]
\[ V\cdot V = V^\nu \eta_{\nu\nu} V^\nu = (v^0)^2 - \vec{v}\cdot\vec{v} = (v^0)^2 \]
\[ = \text{Lorentz Scalar} \]
Heisenberg Uncertainty Principle: Viewed from SRQM

Heisenberg Uncertainty \( \{ \sigma^2_x, \sigma^2_y \} \geq (1/2)|\langle [A,B]\rangle| \)
arises from the non-commuting nature of certain operators.

The commutator is \([A,B] = AB-BA\), where \(A\) & \(B\) are functional “measurement” operators. The Operator Formalism arose naturally from our SR \(\rightarrow\) QM path: \(\partial = -iK\).

The Generalized Uncertainty Relation: \(\sigma^2_x \sigma^2_y \geq (\Delta F) \cdot (\Delta G) \geq (1/2)|\langle i[A,B]\rangle|\)

The uncertainty relation is a very general mathematical property, which applies to both classical or quantum systems. From Wikipedia: Photon Polarization: "This is a purely mathematical result. No reference to a physical quantity or principle is required."

The Cauchy–Schwarz inequality asserts that (for all vectors \(f\) and \(g\) of an inner product space, with either real or complex numbers):

\[\sigma^2_x \sigma^2_y = \langle |f| f \rangle \cdot \langle |g| g \rangle \geq \langle |f| g \rangle^2\]

But first, let's back up a bit; Using standard complex number math, we have:

\[z = a + ib\]
\[z^* = a - ib\]
\[|z|^2 = a^2 + b^2 = [\text{Re}(z)]^2 + [\text{Im}(z)]^2 = [(z + z^*)/2]^2 + [(z - z^*)/(2i)]^2\]

or
\[|z|^2 = [(z + z^*)/(2i)]^2 + [(z - z^*)/(2i)]^2\]

Now, generically, based on the rules of a complex inner product space we can arbitrarily assign:

\[z = \langle f | g \rangle, \quad z^* = \langle g | f \rangle\]

Which allows us to write:

\[
\langle |f| g \rangle^2 = \langle \langle f | g \rangle + \langle g | f \rangle \rangle^2 \geq \langle |f| g \rangle \cdot \langle |g| f \rangle \cdot (2i)\]

We can also note that:

\[
\langle |f\rangle = \langle F | \Psi \rangle \quad \text{and} \quad \langle |g\rangle = \langle G | \Psi \rangle
\]

Thus,

\[
\langle |f \rangle |g\rangle^2 = \langle \langle \Psi | [F^* G | \Psi \rangle + \langle \Psi | [G^* F | \Psi \rangle \rangle^2 + \langle \langle \Psi | [F^* G | \Psi \rangle - \langle \Psi | [G^* F | \Psi \rangle \rangle^2 (2i)\]

For Hermetian Operators…

\[F^* = +F, \quad G^* = +G\]

For Anti-Hermetian (Skew-Hermetian) Operators…

\[F^* = -F, \quad G^* = -G\]

Assuming that \(F\) and \(G\) are either both Hermetian, or both anti-Hermetian…

\[
\langle |f \rangle |g\rangle^2 = \langle \langle \Psi | [\pm F G | \Psi \rangle + \langle \Psi | [\pm G F | \Psi \rangle \rangle^2 + \langle \langle \Psi | [\pm F G | \Psi \rangle - \langle \Psi | [\pm G F | \Psi \rangle \rangle^2 (2i)\]

We can write this in commutator and anti-commutator notation…

\[
\langle |f \rangle |g\rangle^2 = \langle [\pm F, G] \rangle \cdot \langle \Psi \rangle \rangle^2 + \langle [\pm G, F] \rangle \cdot \langle \Psi \rangle \rangle^2 (2i)\]

Due to the squares, the \((\pm)\)'s go away, and we can also multiply the commutator by an \((i^2)\)

\[
\langle |f \rangle |g\rangle^2 = \langle [F, G] \rangle \cdot \langle \Psi \rangle \rangle^2 + \langle [iF, G] \rangle \cdot \langle \Psi \rangle \rangle^2 (2i)\]

The Cauchy–Schwarz inequality again…

\[
\sigma^2_x \sigma^2_y = \langle |f| g \rangle \cdot \langle |g| f \rangle \rangle^2 = \langle |f \rangle |g\rangle^2 \geq \langle [F, G] \rangle \cdot \langle \Psi \rangle \rangle^2 + \langle [iF, G] \rangle \cdot \langle \Psi \rangle \rangle^2 (2i)\]

Taking the root:

\[
\sigma^2_x \geq (1/2)\langle |f| G \rangle \]

Which is what we had for the generalized Uncertainty Relation.

*Note* This is not a QM axiom - This is just pure math. At this stage we already see the hints of commutation and anti-commutation.

It is true generally, whether applying to a physical or purely mathematical situation.
Heisenberg Uncertainty Principle: Simultaneous vs Sequential

Heisenberg Uncertainty \( \{ \sigma_A^2 \sigma_B^2 \geq (1/2)|<[A,B]>| \} \) arises from the non-commuting nature of certain operators.

\[
[\partial^\mu, X^\nu] = \partial[X] = \eta^{\mu\nu} = \text{Minkowski Metric}
\]

\[
[P^\mu, X^\nu] = [i\hbar \partial^\mu, X^\nu] = i\hbar[\partial^\mu, X^\nu] = i\hbar \eta^{\mu\nu}
\]

Consider the following:
Operator A acts on System |\(\Psi>\) at SR Event A: \(A|\Psi> \rightarrow|\Psi'\rangle\)
Operator B acts on System |\(\Psi'>\rangle\) at SR Event B: \(B|\Psi'> \rightarrow|\Psi''\rangle\)
or \(BA|\Psi> = B|\Psi'> = |\Psi''\rangle\)

If measurement Events A & B are space-like separated, then there are observers who can see \{ A before B, A simultaneous with B, A after B \}, which of course does not match the quantum description of how Operators act on Kets.

If Events A & B are time-like separated, then all observers will always see A before B. This does match how the operators act on Kets, and also matches how |\(\Psi>\) would be evolving along its worldline, starting out as |\(\Psi>\), getting hit with operator A at Event A to become |\(\Psi'>\rangle\), then getting hit with operator B at Event B to become |\(\Psi''\rangle\).

The Uncertainty Relation here does NOT refer to simultaneous (space-like separated) measurements, it refers to sequential (time-like separated) measurements. This removes the need for ideas about the particles not having simultaneous properties. There are simply no “simultaneous measurements” of non-zero commuting properties on an individual system, a single worldline – they are sequential, and the first measurement places the system in such a state that the outcome of the second measurement will be altered wrt. if the order of the operations had been reversed.
The Pauli Exclusion Principle is a result of the empirical fact that nature uses identical particles, and this combined with the Spin-Statistics theorem from SR, leads to an exclusion principle for fermions (anti-symmetric, Fermi-Dirac statistics) and an aggregation principle for bosons (symmetric, Bose-Einstein statistics). The Spin-Statistics Theorem is related as well to the CPT Theorem.

For large numbers and/or mixed states these both tend to the Maxwell-Boltzmann statistics. In the \( \{kT>>(\varepsilon_i-\mu)\} \) limit, Bose-Einstein reduces to Rayleigh-Jeans. The commutation relations here are based on space-like separation particle exchanges, unlike the time-like separation for measurement operator exchanges in the Uncertainty Principle.

<table>
<thead>
<tr>
<th>Spin</th>
<th>Particle Type</th>
<th>Quantum Statistics</th>
<th>Classical ( {kT&gt;&gt;(\varepsilon_i-\mu)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>spin:(0,1,...,N)</td>
<td>Indistinguishable, Commutation relation (( ab = ba ))</td>
<td>Bose-Einstein: ( n_i = g_i / [ e^{(\varepsilon_i-\mu)/kT} -1 ] ) aggregation principle</td>
<td>Rayleigh-Jeans: from ( e^x \sim (1 + x +...) ) ( n_i = g_i / [ (\varepsilon_i-\mu)/kT ] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \downarrow ) Limit as ( e^{(\varepsilon_i-\mu)/kT} &gt;&gt;1 ) ( \downarrow )</td>
<td></td>
</tr>
<tr>
<td>Multi-particle Mixed</td>
<td>Distinguishable, or high temp, or low density</td>
<td>Maxwell-Boltzmann: ( n_i = g_i / [ e^{(\varepsilon_i-\mu)/kT} +0 ] )</td>
<td>Maxwell-Boltzmann: ( n_i = g_i / [ e^{(\varepsilon_i-\mu)/kT} ] )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \uparrow ) Limit as ( e^{(\varepsilon_i-\mu)/kT} &gt;&gt;1 ) ( \uparrow )</td>
<td></td>
</tr>
<tr>
<td>spin:(1/2,3/2,...,N/2)</td>
<td>Indistinguishable, Anti-commutation relation (( ab = -ba ))</td>
<td>Fermi-Dirac: ( n_i = g_i / [ e^{(\varepsilon_i-\mu)/kT} +1 ] ) exclusion principle</td>
<td></td>
</tr>
</tbody>
</table>

SR → QM

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

SciRealm.org

John B. Wilson
Complex 4-vectors are simply 4-Vectors where the components may be complex-valued

\[ A = A^\mu = (a^0, a) = (a^0, a^1, a^2, a^3) \rightarrow (a^t, a^x, a^y, a^z) \]

\[ B = B^\mu = (b^0, b) = (b^0, b^1, b^2, b^3) \rightarrow (b^t, b^x, b^y, b^z) \]

Examples of 4-Vectors with complex components are the 4-Polarization and the 4-ProbabilityCurrentDensity

Minkowski Metric \( g^{\mu\nu} \rightarrow \eta^{\mu\nu} = \eta_{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-I(3)] \),
which is the \{curvature\~0 limit = low-mass limit\} of the GR metric \( g^{\mu\nu} \).

Applying the Metric to raise or lower an index also applies a complex-conjugation *

Scalar Product = Lorentz Invariant \( \rightarrow \) Same value for all inertial observers

\[ A \cdot B = \eta_{\mu\nu} A^{\mu} B^{\nu} = A^{\nu} B^{\nu} = A^{\mu} B^{\ast \mu} = (a^0 b^0 - a^* \cdot b) \] using the Einstein summation convention

This reverts to the usual rules for real components

However, it does imply that \( A \cdot B = B \cdot A \)
SRQM: CPT Theorem
Phase Connection, Lorentz Invariance

A Tensor Study of Physical 4-Vectors

The Phase is a Lorentz Scalar Invariant – all observers must agree on its value.

\[ K \cdot X = (\omega/c)(c \cdot X) = (\omega/c^2) = -\Phi \]  
Phase of SR Wave

We take the point of view of an observer operating on a particle at 4-Position \( X \), which has an initial 4-Momentum \( K \). The 4-Position \( X \) of the particle, the operation’s event, will not change: we are applying the various operations only to the particle’s 4-Momentum \( K \).

Note that for matter particles, the operation’s event, will not change: we are applying the various operations only to the particle’s 4-Momentum \( K \).

\[ \text{Note that for matter particles, } (\omega/c)N, \text{ where } N \text{ is the “Unit”-Null 4-Vector } N = (1,n) \text{ and } n \text{ is a unit-spatial 3-vector.} \]

All operations listed below work similarly on the Null 4-Vector.

Do a Time Reversal Operation: \( T \)
The particle’s temporal direction is reversed & complex-conjugated: \( T = -T^* = (1,-\beta)^* \)

Do a Parity Operation (Space Reflection): \( P \)
Only the spatial directions are reversed:
\[ T_P = (\gamma(1,-\beta)^*) \]

Do a Charge Conjugation Operation: \( C \)
Charge Conjugation actually changes all internal quantum #s:
charge, lepton #, etc.

Feynman showed this is the equivalent of a world-line reversal & complex-conjugation:
\[ T_C = (\gamma(1,-\beta)^*)^* \]

Pairwise combinations:
\[ T_{TP} = T_{PT} = T_C = (\gamma(1,-\beta)^*)^* \]
\[ T_{TC} = T_{CT} = T_c = (\gamma(1,-\beta)^*) \]
\[ T_{PC} = T_{CP} = T_T = (\gamma(1,-\beta)^*) \]
\[ T_{CC} = T = (\gamma(1,\beta)) \]

A CP event is mathematically the same as a \( T \) event
\[ T_{CP} = T = (\gamma(1,\beta)) \]

They all remain temporal 4-vectors
\[ T_{CC} = T = (\gamma(1,\beta)) \]

\[ T_{PPT} = T = (\gamma(1,\beta)) \]

Matter-like
\[ T = (\gamma(1,\beta)) \]
\[ T_T = (\gamma(1,\beta))^* \gamma(1,\beta) = \gamma^2(1,\beta) = 1: \text{It's a temporal 4-vector} \]

Light-like/Photonic
\[ N = (1,n) \]
\[ N^* = (1,n)^*(-1) = (1^2 - n^2)(-n) = (1-1) = 0: \text{It's a null 4-vector} \]

\[ N \cdot N = (1,n)^*(-1) = (1^2 - n^2)(-n) = (1-1) = 0 \]

\[ N \cdot N = (1,n)^*(-1) = (1^2 - n^2)(-n) = (1-1) = 0 \]

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\[ N \cdot N = (1,n)^*(-1) = (1^2 - n^2)(-n) = (1-1) = 0 \]

Trace\[ T_{CC} = T = (\gamma(1,\beta)) \]
\[ V \cdot V = (\gamma(1,\beta))^* \gamma(1,\beta) = 1 \]

Lorentz Scalar
SRQM: CPT Theorem
(Charge) vs (Parity) vs (Time)

Identity and Space-Parity are Unitary.
Time-Reversal and Charge-Conjugation are Anti-Unitary.

Trace[T^μν] = n^μ_ν T^μν = 1
\[ V^\ast V = V^\ast n^\nu V = (V^\nu)^2 - V^\mu V = (V^\nu)^2 \]
= Lorentz Scalar

Tr[(CPT)] = (CTP)
= (PCT)
= (TCP)
= (TPC)

SR 4-Tensor
(2,0)-Tensor T^μν
(1,1)-Tensor T^μν or T^νμ
(0,2)-Tensor T^μν

SR 4-Vector
(1,0)-Tensor V^μ = V = (V^0, V^ν)

SR 4-CoVector
(0,1)-Tensor V_μ = (V^ν_μ)

SR 4-Scalar
(0,0)-Tensor S = Lorentz Scalar

4-Vector SRQM Interpretation of QM
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SRQM Transforms: Venn Diagram

### Poincaré = Lorentz + Translations

<table>
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<tr>
<th>Lorentz Transform $\Lambda_{\mu}^{\nu}$</th>
<th>Translation Transform $\Delta X_{\mu}$</th>
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<td>Time-reversal $\Lambda_{\mu}^{\nu} \rightarrow T_{\mu}^{\nu}$</td>
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<td>Parity-Inversion $\Lambda_{\mu}^{\nu} \rightarrow P_{\mu}^{\nu}$</td>
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<td>r $\rightarrow$ -r</td>
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<tr>
<td>Charge-Conjugation $\Lambda_{\mu}^{\nu} \rightarrow C_{\mu}^{\nu}$</td>
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<td>R $\rightarrow$ -R, q $\rightarrow$ -q</td>
<td>4-Linear $\Lambda_{\mu}^{\nu} \rightarrow (0, \Delta x)$</td>
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<tr>
<td>CPT Symmetry {Charge}</td>
<td>Homogeneity</td>
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<tr>
<td>{Charge}</td>
<td>{same all points}</td>
</tr>
<tr>
<td>{Time}</td>
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**SR: Lorentz Transform**

$$\Lambda_{\mu}^{\nu} = (\Lambda^{-1})_{\mu}^{\nu} : \Lambda_{\alpha}^{\mu} = \eta_{\alpha\beta}\Lambda_{\nu}^{\beta} = \eta_{\nu\alpha}$$

Rotations $J_i = -\varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} 2$, Boosts $K_i = M_{i0}$

Amusingly, Inhomogeneous Lorentz adds homogeneity.

---

**A Tensor Study of Physical 4-Vectors**

John B. Wilson

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---

**Charges-Conjugation**

- **Charge**
- **Anti-Charge**

**Discrete**

- **Time-reversal**
- **Parity-inversion**
- **Charge-conjugation**

**Continuous**

- **Rotation**
- **Boost**

**Space (3)**

- **Identity** $I_{\mu\nu} = \eta_{\mu\nu}$
- **No mixing**

**Time (1)**

- **Temporal** $\Delta t$
- **Spatial** $\Delta x$

**4-Vectors**

- **Lorentz Matrices**
- **Translations**

**Discrete**

- **Time-reversal** $T$
- **Parity-inversion** $P$
- **Charge-conjugation** $C$

**Continuous**

- **Rotation** $R$
- **Boost** $B$

**Lorentz Group**

- **Inhomogeneous Lorentz**
- **Homogeneity**

**SR: Lorentz Transform**

$$\Lambda_{\mu}^{\nu} = (\Lambda^{-1})_{\mu}^{\nu} : \Lambda_{\alpha}^{\mu} = \eta_{\alpha\beta}\Lambda_{\nu}^{\beta} = \eta_{\nu\alpha}$$

Rotations $J_i = -\varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} 2$, Boosts $K_i = M_{i0}$

Amusingly, Inhomogeneous Lorentz adds homogeneity.

---

**SRQM Transforms: Venn Diagram**

- **Poincaré = Lorentz + Translations**

**Transformations**

- **# of independent parameters = # continuous symmetries = # Lie Dimensions**

**Poincaré Transformation Group** aka. Inhomogeneous Lorentz Transformation

Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)

General Linear, Affine Transform $X_{\mu} = \Lambda_{\mu}^{\nu} X_{\nu} + \Delta X_{\mu}$ with $\text{Det} [\Lambda_{\mu}^{\nu}] = \pm 1$

$(6 + 4 = 10)$

**SRQM Transforms**

- **Charge Conjugation**
- **Parity Inversion**
- **Time Reversal**
- **Charge Parity**
- **Space Parity**
- **Time Parity**

**SRQM Transforms**

- **Charge**
- **Time**
- **Space**

**SRQM Transforms**

- **Identity** $I_{\mu\nu} = \eta_{\mu\nu}$
- **No mixing**

**SRQM Transforms**

- **Temporal** $\Delta t$
- **Spatial** $\Delta x$
The Hermitian Generators that lead to translations and rotations via unitary operators in QM...

These all ultimately come from the Poincaré Invariance → Lorentz Invariance that is at the heart of SR and Minkowski Space.

Infintesimal Unitary Transformation
\[ \hat{U}_t(\hat{G}) = I + i\epsilon \hat{G} \]

Finite Unitary Transformation
\[ \hat{U}_\alpha(\hat{G}) = e^{i\alpha \hat{G}} \]

let \( \hat{G} = \frac{P}{\hbar} = K \)
let \( \alpha = \Delta x \)

\[ \hat{U}_{\Delta x}(\frac{P}{\hbar})\Psi(X) = e^{i(\Delta x \cdot P/\hbar)}\Psi(X) = e^{i(-\Delta x \cdot \partial)}\Psi(X) = \Psi(X - \Delta x) \]

Time component: \( \hat{U}_{\Delta t}(\frac{P}{\hbar})\Psi(ct) = e^{i(\Delta tE/\hbar)}\Psi(ct) = e^{i(-\Delta t \partial_t)}\Psi(ct) = \Psi(ct - c\Delta t) = c\Psi(t - \Delta t) \)
Space component: \( \hat{U}_{\Delta x}(\frac{P}{\hbar})\Psi(x) = e^{i(\Delta x \cdot p/\hbar)}\Psi(x) = e^{i(\Delta x \cdot \nabla)}\Psi(x) = \Psi(x + \Delta x) \)

By Noether's Theorem, this leads to \( \partial \cdot K = 0 \)

We had already calculated
\[ (\partial \cdot \partial)[K \cdot X] = ((\partial/c)^2 - \nabla \cdot \nabla)(\omega t - k \cdot x) = 0 \]
\[ (\partial \cdot \partial)[K \cdot X] = \partial \cdot (\partial[K \cdot X]) = \partial \cdot K = 0 \]

Poincaré Invariance also gives the Casimir invariants of mass and spin, and ultimately leads to the spin-statistics theorem of RQM.
QM Correspondence Principle:
Analogous to the GR and SR limits

Basically, the old school QM Correspondence Principle says that QM should give the same results as classical physics in the realm of large quantum systems, i.e. where macroscopic behavior overwhelms quantum effects. Perhaps a better way to state it is when the change of system by a single quantum has a negligible effect on the overall state.

There is a way to derive this limit, by using Hamilton-Jacobi Theory:

\[(i\hbar \partial_t)|\Psi\rangle \sim [V - (\hbar \nabla T)^2/2m_o]|\Psi\rangle : \text{The Schrödinger NRQM Equation for a point particle (non-relativistic QM)}\]

Examine solutions of form \( \Psi = \Psi_o e^{i\Phi} = \Psi_o e^{iS/\hbar} \), where \( S \) is the QM Action

\[\partial_t[\Psi] = (i/\hbar)\Psi \partial_t[S] \text{ and } \partial_x[\Psi] = (i/\hbar)\Psi \partial_x[S] \text{ and } \nabla^2[\Psi] = (i/\hbar)\Psi \nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2\]

\[(i)(i\hbar)\Psi \partial_t[S] = V\Psi - (\hbar^2/2m_o)((i/\hbar)\Psi \nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2)\]

\[(i)(i\hbar)\Psi \partial_x[S] = V\Psi - ((i\hbar/2m_o)\Psi \nabla^2[S] - (\Psi/2m_o)(\nabla[S])^2)\]

\[\partial_t[S] = -V + (i\hbar/2m_o)\nabla^2[S] - (1/2m_o)(\nabla[S])^2\]

\[\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2] = (i\hbar/2m_o)\nabla^2[S] : \text{Quantum Single Particle Hamilton-Jacobi}\]

\[\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2] = 0 : \text{Classical Single Particle Hamilton-Jacobi}\]

Thus, the classical limiting case is:

\[\nabla^2[\Phi] \ll (\nabla[\Phi])^2\]

\[\hbar \nabla^2[S] \ll (\nabla[S])^2\]

\[\hbar \nabla \cdot \mathbf{p} \ll (\mathbf{p} \cdot \mathbf{p})\]

\[(\mathbf{p} \lambda) \nabla \cdot \mathbf{p} \ll (\mathbf{p} \cdot \mathbf{p})\]
QM Correspondence Principle: Analogous to the GR and SR limits

\[ \partial_t[S] + \left[V + \frac{1}{2m_0}\nabla^2[S]\right] = \frac{\text{i\hbar}}{2m_0} \nabla^2[S] : \text{Quantum Single Particle Hamilton-Jacobi} \]

\[ \partial_t[S] + \left[V + \frac{1}{2m_0}\nabla^2[S]\right] = 0 : \text{Classical Single Particle Hamilton-Jacobi} \]

Thus, the quantum→classical limiting-case is:

\[ \hbar \nabla^2[S]_{\text{action}} << (\nabla[S]_{\text{action}})^2 \]
\[ \nabla^2[\Phi]_{\text{phase}} << (\nabla[\Phi]_{\text{phase}})^2 \]
\[ \hbar \nabla \cdot p << (p \cdot p) \]
\[ (p \lambda) \nabla \cdot p << (p \cdot p) \]

with

\( P = (E/c, p) = -(\partial[S]_{\text{action}}) = -(-\partial/c, \nabla)[S]_{\text{action}} = (-\partial/c, \nabla)[S]_{\text{action}} \]
\( K = (\omega/c, k) = -(\partial[\Phi]_{\text{phase}}) = -(-\partial/c, \nabla)[\Phi]_{\text{phase}} = (-\partial/c, \nabla)[\Phi]_{\text{phase}} \)

It is analogous to GR → SR in limit of low curvature (low mass), or SR → CM in limit of low velocity \{ |v| << c \}. It still applies, but is now understood as the same type of limiting-case as these others.

*Note* The commonly seen form of \((c \rightarrow \infty, \hbar \rightarrow 0)\) as limits are incorrect!

\( c \) and \( \hbar \) are universal constants – they never change.

If \( c \rightarrow \infty \), then photons (light-waves) would have infinite energy \{ E = pc \}. This is not true classically.

If \( \hbar \rightarrow 0 \), then photons (light-waves) would have zero energy \{ E = \hbar \omega \}. This is not true classically.

Always better to write the SR Classical limit as \{ |v| << c \}, the QM Classical limit as \{ \nabla^2[\Phi]_{\text{phase}} << (\nabla[\Phi]_{\text{phase}})^2 \}.

Again, it is more natural to find a limiting-case of a more general system than to try to unite two separate theories which may or may not ultimately be compatible. From logic, there is always the possibility to have a paradox result from combination of arbitrary axioms, whereas deductions from a single true axiom will always give true results.
SRQM: 4-Vector Quantum Probability

Conservation of Probability Density

Consider the following purely mathematical argument (based on Green’s Vector Identity):
\[ \partial \cdot (f \partial g - \partial f g) = f \partial \cdot \partial g - \partial \cdot \partial f g \]
with \((f)\) and \((g)\) as SR Lorentz Scalar functions

Proof:
\[ \partial \cdot (f \partial g - \partial f g) = \partial \cdot (f \partial g) - \partial \cdot \partial f g \]
\[ = f \partial \cdot \partial g - \partial \cdot \partial f g \]

We can also multiply this by a Lorentz Invariant Scalar Constant \(s\)
\[ s (f \partial \cdot \partial g - \partial \cdot \partial f g) = \partial \cdot s (f \partial g - \partial f g) \]

Ok, so we have the math that we need…

Now, on to the physics… Start with the Klein-Gordon Eqn.
\[ \partial \cdot \partial = (-i\hbar c/m_o)^2 = -(m_o c/\hbar)^2 \]
\[ \partial \cdot \partial + (m_o c/\hbar)^2 = 0 \]

Let it act on SR Lorentz Invariant function \(g\)
\[ \partial \cdot \partial g + (m_o c/\hbar)^2 g = 0 \]
Then pre-multiply by \(f\)
\[ [f] \partial \cdot \partial g + [f] (m_o c/\hbar)^2 g = 0 \]
\[ [f] \partial \cdot \partial g + (m_o c/\hbar)^2 [f] g = 0 \]

Now, subtract the two equations
\[ \{ [f] \partial \cdot \partial g + (m_o c/\hbar)^2 [f] g = 0 \} - \{ \partial \cdot \partial [f] g + (m_o c/\hbar)^2 [f] g = 0 \} \]
\[ [f] \partial \cdot \partial g + (m_o c/\hbar)^2 [f] g - \partial \cdot \partial [f] g + (m_o c/\hbar)^2 [f] g = 0 \]
\[ \partial \cdot \partial [f] g - \partial \cdot \partial [f] g = 0 \]

And as we noted from the mathematical Green’s Vector identity at the start…
\[ [f] \partial \cdot \partial [g] - \partial \cdot \partial [f] g = \partial \cdot (f \partial g - \partial f g) = 0 \]

Therefore,
\[ s \partial \cdot (f \partial g - \partial f g) = 0 \]
\[ \partial \cdot s (f \partial g - \partial f g) = 0 \]

Thus, there is a conserved current 4-Vector, \(J_{prob} = s (f \partial g - \partial f g)\), for which \(\partial \cdot J_{prob} = 0\), and which also solves the Klein-Gordon equation.

Let’s choose as before \((\partial = -iK)\) with a plane wave function \(f = ae^{-i(K \cdot X)} = \psi\), and choose \(g = f^* = ae^{i(K \cdot X)} = \psi^*\) as its complex conjugate.

At this point, I am going to choose \(s = (i\hbar/2m_o)\), which is Lorentz Scalar Invariant, in order to make the probability have dimensionless units and be normalized to unity in the rest case.
4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn

A Tensor Study of Physical 4-Vectors

John B. Wilson
SciRealm.org

4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux

\[ J_{\text{prob}} = (\rho_{\text{prob}}, j_{\text{prob}}) = (\hbar/2m, j) \]

\[ \rho = (\rho_{\text{prob}}) \] (c, u) = (\gamma p_{\text{prob}})(c, u) = (\rho_{\text{prob}})(c, u)

with 4-Divergence of Probability \{ \partial \cdot J_{\text{prob}} = 0 \} by construction via Green’s Vector Identity and the Klein-Gordon RQM Eqn.

The reason for \( s = (\hbar/2m) \)

\[ \text{becomes more clear by examining our diagram:} \]

\[ \text{Start at the 4-Gradient and follow the arrows toward the 4-ProbabilityFlux} \]

You immediately see where the \( (\hbar/m_c) \) factor comes from.

The \( \rho_{\text{prob}, o} \) is then a function of the \( \psi \)'s divided by 2.

\[ \partial (f \cdot g) - \partial(f) \cdot g = f \partial \cdot (g) - \partial(g) \cdot f \]

Green’s Vector Identity

Examine the temporal component, the Relativistic Probability Density

\[ \rho_{\text{prob}} = (\hbar/2m_c^2)(\psi^* \cdot \partial\psi - \partial(\psi^*) \psi) \]

Assume wave solution in following general form:

\[ \psi = A f [k] e^{(+i\omega t)} \]

\[ \psi^* = A^* f [k]^* e^{(-i\omega t)} \]

then

\[ \partial \psi = (\pm \omega) A f [k] e^{(\mp i\omega t)} \]

\[ \partial \psi^* = (\mp \omega) A^* f [k]^* e^{(+i\omega t)} \]

Finally, multiply by charge \( q \) to get standard SR EM

\[ 4\text{-CurrentDensity} = 4\text{-ChargeFlux} = J = (c p, j) = q \rho_{\text{prob}} = q(c p_{\text{prob}, j}) \]

4-Gradient

\[ \delta \cdot (\partial / c - \nabla) \]

Klein-Gordon

4-Vector SRQM Interpretation of QM
4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn with Minimal Coupling

A Tensor Study of Physical 4-Vectors

4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux
\[ J_{\text{prob}} = (c \rho_{\text{prob}} \mathbf{i}_{\text{prob}}) = (\pm i/2m_{\gamma} (\psi^* \sigma_{\gamma} \psi - \sigma_{\gamma} \psi^* \psi) = (\rho_{\text{prob}} \gamma) (c, u) = (\gamma p_{\text{prob}})(c, u) = (\rho_{\text{prob}})(c, u) \]
with 4-Divergence of Probability \( \partial J_{\text{prob}} = 0 \) by construction via Green’s Vector Identity and the Klein-Gordon RQM Eqn.

If we include minimal coupling:
\[ J_{\text{prob}} = (c \rho_{\text{prob}} \mathbf{i}_{\text{prob}}) = (\pm i/2m_{\gamma} (\psi^* \sigma_{\gamma} \psi - \sigma_{\gamma} \psi^* \psi) + (q/m_{\gamma})(\psi^* \psi)A \]

Start at \( A \) on the chart
Follow past \( (q) \) factor to get to \( Q = qA \)
Minimal Coupling allows passage back to \( P \) with no factors
Follow back past \( (1/m_{\gamma}) \) to get to \( U \)
Follow past Born Rule \( (\psi^* \psi) \)
Now have the additional factor:
\[ + (q/m_{\gamma})(\psi^* \psi)A \]

4-Vector SRQM Interpretation of QM

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4-Vector Quantum Probability

Newtonian Limit

Examine the temporal component:

\[ \rho_{\text{prob}} = (i\hbar/2m_c)(\psi^* \partial_t \psi) + (q/m_o)(\psi^* \psi) (\phi/c^2) \]

\[ \rho_{\text{prob}} \rightarrow (\gamma)(\psi^* \psi) + (\gamma)(q\phi_o/m_o c^2)(\psi^* \psi) = (\gamma)[1 + q\phi_o/E_o](\psi^* \psi) \]

Typically, the particle EM potential energy \((q\phi_o)\) is much less than the particle rest energy \((E_o)\), else it could generate new particles. So, take \((q\phi_o << E_o)\), which gives the EM factor \((q\phi_o/E_o) \sim 0\).

Now, taking the low-velocity limit \((\gamma \rightarrow 1)\), \(\rho_{\text{prob}} \rightarrow \gamma[1 + \sim 0](\psi^* \psi)\), \(\rho_{\text{prob}} \rightarrow (\psi^* \psi) = (\rho_{\text{prob}^o})\) for \(|v|<<c\).

The Standard Born Probability Interpretation, \((\psi^* \psi) = (\rho_{\text{prob}^o})\), only applies in the low-potential-energy & low-velocity limit.

This is why the {non-positive-definite} probabilities and \(|\text{probabilities}| > 1\) in the RQM Klein-Gordon equation gave physicists fits, and is the reason why one must regard the probabilities as charge conservation instead.

The original definition from SR is Continuity of Worldlines, \(\partial \cdot \mathbf{J}_{\text{prob}} = 0\), for which all is good and well in the RQM version.

The definition says there are no external sources or sinks of probability = conservation of probability.

The Born idea that \((\rho_{\text{prob}}) \rightarrow \text{Sum}[(\psi^* \psi)] = 1\) is just the Low-Velocity QM limit.

Only the non-EM rest version \((\rho_{\text{prob}^o}) = \text{Sum}[(\psi^* \psi)] = 1\) is true.

It is not a fundamental axiom, it is an emergent property which is valid only in the NRQM limit.

We now multiply by charge \((q)\) to instead get a

4-"Charge"CurrentDensity \(\mathbf{J} = (c\rho, j) = q\mathbf{J}_{\text{prob}} = q(c\rho_{\text{prob}}, j_{\text{prob}})\), which is the standard SR EM 4-CurrentDensity.
Compton Scattering Derivation: Compton Effect

\[ P - P' = (m_e c)^2 \text{ generally } \rightarrow 0 \text{ for photons (} m_e = 0 \text{)} \]

\[ P_{\text{phot}} - P_{\text{mass}} = h \cdot K_{T} \text{, where } K_{T} = (h \cdot \omega/c^2, 0, 0, 0) \]

\[ P_{\text{phot}}^2 = (P_{\text{mass}} + P_{\text{phot}})^2 = (P_{\text{mass}}^2 + P_{\text{phot}}^2) \]

\[ (0 + 2P_{\text{mass}} \cdot P^*_{\text{phot}} - 2P_{\text{mass}} \cdot P^*_{\text{phot}}^* + P_{\text{mass}}^* \cdot P^*_{\text{phot}} + P_{\text{mass}}^* \cdot P^*_{\text{phot}}^*) = (P_{\text{mass}}^2)^2 \]

\[ \lambda_{C} = (h/m_e c)(1 - \cos[\theta]) \]

The Compton Effect: Compton Scattering

Calculated wavelength shift of a photon scattering from an electron (ignoring spin)

Proves that light does not have a "wave-only" description, photon 4-Momentum required

\[ E/\omega = \gamma E/\gamma \omega = E/\omega = h \]

\[ K_{\text{photon}} = (h/\omega c^2)(1, \hat{n}, 0) = \text{null} \]

\[ \omega_{\lambda} = \nu \lambda = c \] for photons
The EM 4-Vector Potential gives the Aharonov-Bohm Effect.

\[ \Phi_{pot} = -\frac{q}{\hbar}A \cdot dX \]

or taking the differential...

\[ d\Phi_{pot} = -\frac{q}{\hbar}A \cdot dX \]

over a path...

\[ \Delta \Phi_{pot} = \int_{\text{path}} d\Phi_{pot} \]

\[ \Delta \Phi_{pot} = -\frac{q}{\hbar} \int_{\text{path}} A \cdot dX \]

\[ \Delta \Phi_{pot} = \int_{\text{path}} \left[ \left( \frac{q}{\hbar} \right) \left( cdt \right) - a \cdot dX \right] \]

\[ \Delta \Phi_{pot} = \int_{\text{path}} (\varphi dt - a \cdot dX) \]

Aharonov-Bohm Effect

Note that both the Electric and Magnetic effects come out by using the 4-Vector notation.

Electric AB effect: \[ \Delta \Phi_{pot,\text{Elec}} = -\left( \frac{q}{\hbar} \right) \int_{\text{path}} (\varphi dt) \]

Magnetic AB effect: \[ \Delta \Phi_{pot,\text{Mag}} = +\left( \frac{q}{\hbar} \right) \int_{\text{path}} (a \cdot dX) \]

Proves that the 4-VectorPotential \( A \) is more fundamental than \( e \) and \( b \) fields, which are just components of the Faraday EM Tensor.
SRQM 4-Vector Study: The QM Josephson Junction Effect = SuperCurrent

EM 4-VectorPotential \( \mathbf{A} = -(\hbar/q) \partial \Delta \Phi_{pot} \)

Josephson Effect

The EM 4-VectorPotential gives the Aharonov-Bohm Effect.
Phase \( \Phi_{pot} = -(q/h) \mathbf{A} \cdot \mathbf{X} = -\mathbf{K}_{pot} \cdot \mathbf{X} \)

Rearrange the equation a bit:
\[
-(\hbar/q) \Delta \Phi_{pot} = \mathbf{A} \cdot \Delta \mathbf{X}
\]
\[
d/dt[\mathbf{A} \cdot \Delta \mathbf{X}] = d/dt[-(\hbar/q) \Delta \Phi_{pot}] = d/dt[\mathbf{A}] \cdot \Delta \mathbf{X} + \mathbf{A} \cdot d/dt[\Delta \mathbf{X}] = d/dt[\mathbf{A}] \cdot \Delta \mathbf{X} + \mathbf{A} \cdot \mathbf{U}
\]

Assume that ( \( d/dt[\mathbf{A}] \cdot \Delta \mathbf{X} \sim 0 \) :)
\[
\mathbf{A} = -(\hbar/q) \partial \Delta \mathbf{Phi}_{pot} \quad (\partial_c-a,=-\hbar/q)(\partial_c,-\nabla)[\Delta \Phi_{pot}]
\]

Which explains Josephson Effect criteria:
\[
\Delta \mathbf{X} \sim 0: \text{small gap}
\]
\[
d/dt[\mathbf{A}] \sim 0: \text{"critical current" & no voltage}
\]
\[
d/dt[\mathbf{A}] \cdot \Delta \mathbf{X} \sim \text{orthogonal: ??}
\]

Take the temporal part:
EM ScalarPotential \( \varphi = -(\hbar/q)(\partial_c)[\Delta \Phi_{pot}] \); \( \varphi = (q/h)\varphi \)

If the charge \( (q) \) is a Cooper-electron-pair: \( q = -2e \)

Voltage \( \mathbf{V}(t) = \varphi(t) = (\hbar/2e)(\partial_c)(\partial_c)[\Delta \Phi_{pot}] \); AngFreq \( \omega = -2eV/\hbar \)

This is the superconducting phase evolution equation of the Josephson Effect
\( (h/2e) \) is defined to be the Magnetic Flux Quantum \( \Phi_o \)

SR 4-Tensor (2,0)-Tensor \( T^\mu_\nu \), or \( T^\nu_\mu \)

SR 4-Vector (1,0)-Tensor \( V^\nu = (\mathbf{V} \cdot \mathbf{v})_\nu \)

SR 4-CoVector (0,1)-Tensor \( V_\nu = (\mathbf{V} \cdot \mathbf{v})^{\nu} \)

SR 4-Scalar (0,0)-Tensor \( S \)

Lorentz Scalar

4-Vector SRQM Interpretation of QM

4-Gradient \( \partial = (\partial/c,-V) \)

4-Velocity \( \mathbf{U} = (c,u) \)

4-VectorPotential \( \mathbf{A} = (c/p,a) \)

4-Momentum \( \mathbf{P} = (m_c,p) \)

4-PotentialMomentum \( \mathbf{Q} = (U,c,q)VqA \)

Minimal Coupling \( P + Q \)

Einstein de Broglie \( E = mc^2 \)

Wave Velocity \( v_{group} = c^2 \)

Rest Scalar Potential \( \varphi/c^2 \)

Rest Energy:Mass \( E = mc^2 \)

4-Displacement \( \Delta \mathbf{X} = c\Delta t,\Delta \mathbf{x} \)

\( \mathbf{d}\mathbf{X} = (cdt, dx) \)

4-Position \( \mathbf{X} = (ct, x) \)

4-WaveVector \( \mathbf{K} = (\omega/c,k) = (\omega/c,\omega\phi/v_{phase}) \)

4-WaveVectorIncField \( \mathbf{K}_I = (\omega/c,k) = K_{dyn}\mathbf{X} + K_{pot}\mathbf{X} \)

4-MomentumIncField \( \mathbf{P}_I = (E/c,p) \)

Quantum Principles

SRM → Quantum Principles
**SRQM Symmetries:**

**Hamilton-Jacobi vs Relativistic Action**

**Josephson vs Aharonov-Bohm**

**Differential (4-Vector) vs Integral (4-Scalar)**

**Differential Formats : 4-Vectors**

SR Hamilton-Jacobi Equation

\[
P_T = -\partial[\Delta S_{\text{action}}] \]
\[
P + qA = -\partial[\text{h}\Delta \Phi_{\text{phase}}] \]
\[
P + qA = -\partial[\text{h}\Delta \Phi_{\text{phase,dyn}} + (h)\Delta \Phi_{\text{phase, pot}}] \]

4-TotMom Conservation

\[
P_T = (P+Q) = (P+qA) \]

Minimal Coupling

\[
P = (P,-qA) = (P,-Q) \]

**4-Momentum**

\[
P = -\partial[\Delta S_{\text{act,dyn}}] \]
\[-\partial[\text{h}\Delta \Phi_{\text{phase,dyn}}] \]

Josephson Junction Relation

\[
A = -(h/q)\partial[\Delta \Phi_{\text{potential}}] \]
\[= -(1/q)\partial[\Delta S_{\text{act,pot}}] \]
\[= Q/q \]

**Integral Formats : 4-Scalars**

SR Action Equation

\[
\Delta S_{\text{action}} = \int_{\text{path}} P_T \cdot dX \]
\[
\Delta S_{\text{action}} = \int_{\text{path}} (P+qA) \cdot dX \]
\[
h\Delta \Phi_{\text{phase}} = \int_{\text{path}} (P+qA) \cdot dX \]
\[
h\Delta \Phi_{\text{phase,dyn}} + h\Delta \Phi_{\text{phase,pot}} = \int_{\text{path}} (P+qA) \cdot dX \]

4-TotMom Conservation

\[
P_T = (P+Q) = (P+qA) \]

Minimal Coupling

\[
P = (P,-qA) = (P,-Q) \]

**Action Equation**

\[
\Delta S_{\text{act,dyn}} = h\Delta \Phi_{\text{phase,dynamic}} \]
\[= \int_{\text{path}} (P) \cdot dX \]

**Action Equation**

\[
\Delta S_{\text{act,pot}} = h\Delta \Phi_{\text{phase,potential}} \]
\[= \int_{\text{path}} (qA) \cdot dX = \int_{\text{path}} (Q) \cdot dX \]

**Aharonov-Bohm Relation**

\[
\Delta \Phi_{\text{potential}} = -(q/h)\int_{\text{path}} A \cdot dX \]
\[= -(1/h)\int_{\text{path}} Q \cdot dX \]
\[= \Delta S_{\text{act,pot}}/h \]

Technically, the standard Josephson Junction uses just the temporal part \{ A = (\varphi/c,a) \} & Cooper-pair-electrons \{ q = -2e \}

giving \( V(t) = \varphi = (h/2e)\partial/\partial t[\Delta \Phi_{\text{pot}}] \).

There should be a spatial part as well.

**4-totMom Conservation**

\[
P_T = (P+Q) = (P+qA) \]

Minimal Coupling

\[
P = (P,-qA) = (P,-Q) \]

**Minimal Coupling**

\[
P_T = (P+Q) = (P+qA) \]

Minimal Coupling

\[
P = (P,-qA) = (P,-Q) \]

**Dynamic Part**

**Potential Part**

**Potential Part**

**Dynamic Part**

Existing SR Rules

4-Vector SRQM Interpretation of QM

SciRealm.org

John B. Wilson
The $\hbar$ Connection

$\mathbf{P} = \hbar \mathbf{K}$: Basic Einstein-de Broglie

$\mathbf{P} + \mathbf{Q} = \mathbf{P} + \mathbf{Q}$

$\mathbf{P} + \mathbf{Q} = \hbar \mathbf{K}_{\text{dyn}} + \hbar \mathbf{K}_{\text{pot}}$

$\mathbf{P} + \mathbf{Q} = \hbar (\mathbf{K}_{\text{dyn}} + \mathbf{K}_{\text{pot}})$

Sum over $n$ particles: $\mathbf{P}_r = \sum_n (\mathbf{P} + \mathbf{Q}), \mathbf{K}_r = \sum_n (\mathbf{K}_{\text{dyn}} + \mathbf{K}_{\text{pot}})$

$\mathbf{P}_r = \hbar \mathbf{K}_r$

{SR Hamilton-Jacobi} $= \hbar$QM Complex Plane-Waves

The SR Hamilton-Jacobi Equation, and the QM idea of Complex Plane-Waves, are related by a simple constant ($\hbar$) relation.
A Tensor Study of Physical 4-Vectors

### Dimensionless Physical Objects

There are a number of dimensionless physical objects in SR that can be constructed from Physical 4-Vectors. Most are 4-Scalars, but there are few 4-Vector and 4-Tensors.

- **Space Dimension**
  - $\Phi_{[X]} = \eta^{\mu \nu} \Phi_{\mu \nu}$: The SR Minkowski Metric
  - $T \cdot T = 1$: Lorentz Scalar “Magnitude” of the 4-UnitTemporal
  - $T \cdot S = 0$: Lorentz Scalar of 4-UnitTemporal with 4-UnitSpatial

- **4-Position**
  - $X = (ct, x)$

- **4-Momentum**
  - $K = (\omega/c, k)$

- **4-Velocity**
  - $U = \gamma(c, u)$

- **4-Current Density**
  - $J = (\rho, \mathbf{J})$

- **4-Planck’s Constant**
  - $\hbar$

- **4-Vector Potential**
  - $A = (\phi/c, a)$

- **4-Phase**
  - $\Phi_{phase\_dyn} = \Phi_{phase\_dyn}$

- **4-Wave Vector**
  - $K = (\omega/c, k, \omega \gamma/v)$

- **4-Displacement**
  - $\Delta X = (c \Delta t, \Delta x)$

- **4-Gradient**
  - $\partial = (\partial/c, \nabla)$

- **4-Position In Field**
  - $\mathbf{P} = (E/c, p)$

- **4-Momentum In Field**
  - $\mathbf{P} = (E/c, p + q)$

- **Temperature**
  - $\Theta = (\mu/c, \mathbf{t})$

- **4-Inverse Temporal Momentum**
  - $\mathbf{T} = (\mu/c, \mathbf{q})$

- **4-Volume**
  - $V^\nu = (\mathbf{v}^\nu)$

- **4-Spinner**
  - $\mathbf{S}$

- **Trace**
  - $\eta^{\mu \nu} T_{\mu \nu} = \mathbf{T}^\nu = \mathbf{T}$

- **4-Natural Units**
  - $\mathbf{V} = V^\nu (\nabla \mathbf{V})$

- **4-Dirac’s Gamma Matrix**
  - $\gamma^\mu$

- **4-Unit Spinor**
  - $\psi$
SRQM: QM Axioms Unnecessary

QM Principles emerge from SR

QM is derivable from SR plus a few empirical facts – the “QM Axioms” aren't necessary.
These properties are either empirically measured or are emergent from SR properties...

3 “QM Axioms” are really just empirical constant relations between purely SR 4-Vectors:
   Particle-Wave Duality [(P) = h(K)]
   Unitary Evolution [∂(t) = (-i)K]
   Operator Formalism [(∂) = -iK]

2 “QM Axioms” are just the result of the Klein-Gordon Equation being a linear wave PDE:
   Hilbert Space Representation (<bra|,|ket>, wavefunctions, etc.) & The Principle of Superposition

3 “QM Axioms” are a property of the Minkowski Metric and the empirical fact of Operator Formalism
   The Canonical Commutation Relation
   The Heisenberg Uncertainty Principle (time-like-separated measurement exchange)
   The Pauli Exclusion Principle (space-like-separated particle exchange)

1 “QM Axiom” only holds in the NRQM case
   The Born QM Probability Interpretation – Not applicable to RQM, use Conservation of Worldlines instead

1 “QM Axiom” is really just another level of limiting cases, just like SR → CM in limit of low velocity
   The QM Correspondence Principle (QM → CM in limit of {∇²[ϕ] << (∇[ϕ])²})

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM Interpretation: Relational QM & EPR

The SRQM interpretation fits fairly well with Carlo Rovelli’s Relational QM interpretation:

Relational QM treats the state of a quantum system as being observer-dependent, that is, the QM State is the relation between the observer and the system. This is inspired by the key idea behind Special Relativity, that the details of an observation depend on the reference frame of the observer.

All systems are quantum systems: no artificial Copenhagen dichotomy between classical/macroscopic/conscious objects and quantum objects.

The QM States reflect the observers’ information about a quantum system. Wave function “collapse” is informational – not physical. A particle always knows it’s complete properties. An observer has at best only partial information about the particle’s properties.

No Spooky Action at a Distance. When a measurement is done locally on an entangled system, it is only the partial information about the distant entangled state that “changes/becomes-available-instantaneously”. There is no superluminal signal. Measuring/physically-changing the local particle does not physically change the distant particle.

ex. Place two identical-except-for-color marbles into a box, close lid, and shake. Without looking, pick one marble at random and place it into another box. Send that box very far away. After receiving signal of the far box arrival at a distant point, open the near box and look at the marble. You now instantaneously know the far marble’s color as well. The information did not come by signal. You already had the possibilities (partial knowledge). Looking at the near marble color simply reduced the partial knowledge of both marble’s color to complete knowledge of both marbles’ color. No signal was required, superluminal or otherwise.

ex. The quantum version of the same experiment uses the spin of entangled particles. When measured on the same axis, one will always be spin-up, the other will be spin-down. It is conceptually analogous. Entanglement is only about correlations of system that interacted in the past and are determined by conservation laws.
Einstein and Bohr can both be “right” about EPR:
Per Einstein: The QM State measured is not a “complete” description, just one observer’s point-of-view.
Per Bohr: The QM State measured is a “complete” description, it’s all that a single observer can get.

The point is that many observers can all see the “same” system, but see different facets of it. But a single measurement is the maximal information that a single observer can get without re-interacting with the system, which of course changes the system in general. Remember, the Heisenberg Uncertainty comes from non-zero commutation properties which *require separate measurement arrangements*. The properties of a particle are always there. Properties define particles. We as observers simply have only partial information about them.

Relativistic QM, being derived from SR, should be local – The low-velocity limit to QM may give unexpected anomalous results if taken out of context, or out of the applicable validity range, such as with velocity addition \( v_{12} = v_1 + v_2 \), where the correct formula should be the relativistic velocity composition \( v_{12} = (v_1 + v_2)/[1 + v_1 v_2/c^2] \).

These ideas lead to the conclusion that the wavefunction is just one observer’s state of information about a physical system, not the state of the physical system itself. The “collapse” of the wavefunction is simply the change in an observer’s information about a system brought about by a measurement or, in the case of EPR, an inference about the physical state.

EPR doesn’t break Heisenberg because measurements are made on different particles. The happy fact is that those particles interacted and became correlated in the causal past. The EPR-Bell experiments prove that it is possible to maintain those correlations over long distances. It does not prove superluminal signaling.
We should not be surprised by the “quantum” probabilities being correct instead of “classical” in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Examples

* The limit of \( \hbar \to 0 \) \text{(Fallacy)}: \n\hbar \text{ is a Lorentz Scalar Invariant and Fundamental Physical Constant. It never becomes 0.} \text{(Fact)}

* The classical commutator being zero \([p^i,x^j] = 0\) \text{(Fallacy)}:
\[ [P^i,X^j] = i\hbar \eta^{ij}; [p^i,x^j] = -i\hbar \delta^{ij}; [p^0,x^0] = [E/c,ct] = [E,t] = i\hbar; \text{Again, it never becomes 0} \text{(Fact)} \]

* Using Maxwell-Boltzmann (distinguishable) statistics for counting probabilities of (indistinguishable) quantum states \text{(Fallacy)}: Must use Fermi-Dirac statistics for Fermions: Spin=(n+1/2); Bose-Einstein statistics for Bosons: Spin=(n) \text{(Fact)}

* Using sums of classical probabilities on quantum states \text{(Fallacy)}: Must use sums of quantum probability-amplitudes \text{(Fact)}

* Ignoring phase cross-terms and interference effects in calculations \text{(Fallacy)}: Quantum systems and entanglement require phase cross-terms \text{(Fact)}

* Assuming that one can simultaneously “measure” non-commuting properties at a single spacetime event \text{(Fallacy)}: Particle properties always exist. However, non-commuting ones require \text{separate} measurement arrangements to get information about the properties. The required measurement arrangements on a single particle/worldline are at best sequential events, where the temporal order plays a role; \text{(Fact)} However, EPR allows one to “infer (not measure)” the other property of a particle by the \text{separate} measurement of an entangled partner. \text{(Fact)} This does not break Heisenberg Uncertainty, which is about the order of operations (measurement events) on a single worldline. \text{(Fact)} In the entangled case, both/all of the entangled partners share common past-causal entanglement events, typically due to a conservation law. \text{(Fact)} Information is not transmitted at FTL. The particles simply carried their normal respective “correlated” properties (no hidden variables) with them. \text{(Fact)}

* Assuming that QM is a generalization of CM, or that classical probabilities apply to QM \text{(Fallacy)}: CM is a limiting-case of QM for when changes in a system by a few quanta have a negligible effect on the whole/overall system. \text{(Fact)}
SRQM Interpretation:
Quantum Information

We should not be surprised by the "quantum" probabilities being correct instead of "classical" in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

{from Wikipedia}

No-Communication Theorem/No-Signaling:
A no-go theorem from quantum information theory which states that, during measurement of an entangled quantum state, it is not possible for one observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. The theorem shows that quantum correlations do not lead to what could be referred to as "spooky communication at a distance". **SRQM: There is no FTL signaling.**

No-Teleportation Theorem:
The no-teleportation theorem stems from the Heisenberg uncertainty principle and the EPR paradox: although a qubit $|\psi\rangle$ can be imagined to be a specific direction on the Bloch sphere, that direction cannot be measured precisely, for the general case $|\psi\rangle$. The no-teleportation theorem is implied by the no-cloning theorem. **SRQM: Ket states are informational, not physical.**

No-Cloning Theorem:
In physics, the no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state. This no-go theorem of quantum mechanics proves the impossibility of a simple perfect non-disturbing measurement scheme. The no-cloning theorem is normally stated and proven for pure states; the no-broadcast theorem generalizes this result to mixed states. **SRQM: Measurements are arrangements of particles that interact with a subject particle.**

No-Broadcast Theorem:
Since quantum states cannot be copied in general, they cannot be broadcast. Here, the word "broadcast" is used in the sense of conveying the state to two or more recipients. For multiple recipients to each receive the state, there must be, in some sense, a way of duplicating the state. The no-broadcast theorem generalizes the no-cloning theorem for mixed states. The no-cloning theorem says that it is impossible to create two copies of an unknown state given a single copy of the state. **SRQM: Conservation of worldlines.**

No-Deleting Theorem:
In physics, the no-deleting theorem of quantum information theory is a no-go theorem which states that, in general, given two copies of some arbitrary quantum state, it is impossible to delete one of the copies. It is a time-reversed dual to the no-cloning theorem, which states that arbitrary states cannot be copied. **SRQM: Conservation of worldlines.**

No-Hiding Theorem:
The no-hiding theorem is the ultimate proof of the conservation of quantum information. The importance of the no-hiding theorem is that it proves the conservation of wave function in quantum theory. **SRQM: Conservation of worldlines. RQM wavefunctions are Lorentz Scalars (spin=0), Spinors (spin=1/2), 4-Vectors (spin=1), all of which are Lorentz Invariant.**
SRQM Interpretation: Quantum Information

We should not be surprised by the "quantum" probabilities being correct instead of "classical" probabilities in the EPR/Bell-Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Quantum information (qubits) differs strongly from classical information, epitomized by the bit, in many striking and unfamiliar ways. Among these are the following:

A unit of quantum information is the qubit. Unlike classical digital states (which are discrete), a qubit is continuous-valued, describable by a direction on the Bloch sphere. Despite being continuously valued in this way, a qubit is the smallest possible unit of quantum information, as despite the qubit state being continuously-valued, it is impossible to measure the value precisely.

A qubit cannot be (wholly) converted into classical bits; that is, it cannot be "read". This is the no-teleportation theorem.

Despite the awkwardly-named no-teleportation theorem, qubits can be moved from one physical particle to another, by means of quantum teleportation. That is, qubits can be transported, independently of the underlying physical particle. SRQM: Ket states are informational, not physical.

An arbitrary qubit can neither be copied, nor destroyed. This is the content of the no-cloning theorem and the no-deleting theorem. SRQM: Conservation of worldlines.

Although a single qubit can be transported from place to place (e.g. via quantum teleportation), it cannot be delivered to multiple recipients; this is the no-broadcast theorem, and is essentially implied by the no-cloning theorem. SRQM: Conservation of worldlines.

Qubits can be changed, by applying linear transformations or quantum gates to them, to alter their state. While classical gates correspond to the familiar operations of Boolean logic, quantum gates are physical unitary operators that in the case of qubits correspond to rotations of the Bloch sphere.

Due to the volatility of quantum systems and the impossibility of copying states, the storing of quantum information is much more difficult than storing classical information. Nevertheless, with the use of quantum error correction quantum information can still be reliably stored in principle. The existence of quantum error correcting codes has also led to the possibility of fault tolerant quantum computation.

Classical bits can be encoded into and subsequently retrieved from configurations of qubits, through the use of quantum gates. By itself, a single qubit can convey no more than one bit of accessible classical information about its preparation. This is Holevo's theorem. However, in superdense coding a sender, by acting on one of two entangled qubits, can convey two bits of accessible information about their joint state to a receiver.

Quantum information can be moved about, in a quantum channel, analogous to the concept of a classical communications channel. Quantum messages have a finite size, measured in qubits; quantum channels have a finite channel capacity, measured in qubits per second.
Minkowski still applies in local GR

QM is a local phenomenon

The QM Schrodinger Equation is not fundamental. It is just the low-energy limiting-case of the RQM Klein-Gordon Equation. All of the standard QM Axioms are shown to be empirically measured constants or emergent properties of SR. It is a bad approach to start with NRQM as an axiomatic starting point and try to generalize it to RQM, in the same way that one cannot start with CM and derive SR. Since QM *can* be derived from SR, this partially explains the difficulty of uniting QM with GR: QM is not a “separate formalism” outside of SR that can be used to “quantize” just anything...

Strictly speaking, the use of the Minkowski space to describe physical systems over finite distances applies only in the SR limit of systems without significant gravitation. In the case of significant gravitation, SpaceTime becomes curved and one must abandon SR in favor of the full theory of GR.

Nevertheless, even in such cases, based on the GR Equivalence Principle, Minkowski space is still a good description in a local region surrounding any point (barring gravitational singularities). More abstractly, we say that in the presence of gravity, SpaceTime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski Space. Thus, the structure of Minkowski Space is still essential in the description of GR.

So, even in GR, at the local level things are considered to be Minkowskian: i.e. SR → QM “lives inside the surface” of this local SpaceTime, GR curves the surface.
SRQM Interpretation: Main Result

QM is derivable from SR!

Hopefully, this interpretation will shed light on why Quantum Gravity has been so elusive. Basically, QM rules of "quantization" don't apply to GR. They are a manifestation-of/derivation-from SR. Relativity *is* the "Theory of Measurement" that QM has been looking for.

This would explain why no one has been able to produce a successful theory of Quantum Gravity, and why there have been no violations of Lorentz Invariance nor of the Equivalence Principle.

If quantum effects "live" in Minkowski SpaceTime with SR, then GR curvature effects are at a level above the RQM description, and two levels above standard QM. SR+QM are "in" SpaceTime, GR is the "shape" of SpaceTime...

Thus, this treatise explains the following:

- Why GR works so well in it's realm of applicability {large scale systems}.
- Why QM works so well in it's realm of applicability {micro scale systems and certain macroscopic systems}. i.e. The tangent space to any point in GR curvature is locally Minkowskian, and thus QM is typically found in small local volumes...
- Why RQM explains more stuff than QM without SR {because QM is just the low-velocity limiting-case of RQM}.
- Why all attempts to "quantize gravity" have failed {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}.
- Why all attempts to modify GR keep conflicting with experimental data {because GR is apparently fundamental}.
- Why QM works perfectly well with SR as RQM but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}.
- How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, and give the SRQM Interpretation of Quantum Mechanics.
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

SR Axioms: Invariant Interval + (c) as Physical Constant lead to SR, although technically SR is itself the low-curvature limiting-case of GR.

\{c,τ,m_0,ℏ,\iota\}: All Empirically Measured SR Lorentz Invariants and/or Mathematical Constants

Standard SR 4-Vectors:

<table>
<thead>
<tr>
<th>4-Position</th>
<th>( R = (ct,r) )</th>
<th>Related by these SR Lorentz Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Velocity</td>
<td>( U = γ(c,u) )</td>
<td>( (R·R) = (ct)^2 )</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( P = (E/c,p) )</td>
<td>( (U·U) = (c)^2 )</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( K = (ω/c,k) )</td>
<td>( (P·P) = (m_0c)^2 )</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial = (∂/c, -∇) )</td>
<td>( (K·K) = (m_0c/ℏ)^2 )</td>
</tr>
</tbody>
</table>

\( |\partial| \ll c \) leads to QM via the low-velocity limit, giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Eqn, and thence to QM via the low-velocity limit \( |\partial| \ll c \), giving the Schrödinger Eqn. The relation also leads to the Dirac, Maxwell, Pauli, Proca, Weyl, & Scalar Wave QM Eqns.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SR → QM

SR Action → 4-Momentum
SR SpaceTime Dimension=4
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

SR → RQM Klein-Gordon
Relativistic Quantum Mechanics
SR → RQM Klein-Gordon

http://scirealm.org/SciRealm@ao
SciRealm.org
John B. Wilson
4-Vector SRQM Interpretation of QM
SRQM Diagram: SRQM 4-Vectors and Lorentz Scalars / Physical Constants

Heart of SR
4-Displacement
\[ \Delta R = (c \Delta t, \Delta r) \]
4-Position
\[ R = (ct, r) = \langle \text{Event} \rangle \]
4-UnitTemporal
\[ T = \gamma(1, \beta) \]
Speed of Light
\[ c \]
4-UnitSpatial
\[ S = \gamma_{\text{sp}}(\hat{n}, \text{r}) \]
Rest Number Density
\[ n_0 \]
4-NumberFlux
\[ N = (n, c, n) = (n, c, u) \]
Probability Rule
\[ P_{\text{prob}} = \chi_{\psi} = |\psi|^2 \]
4-ProbCurrDensity
\[ J_{\text{prob}} = (\rho_{\text{pro}}, J_{\text{prob}}) \]
4-ProbabilityFlux
\[ J = (pc, j) = \rho(c, u) \]
4-CurrentDensity

ProperTime Derivative
\[ U \cdot \partial \cdots \gamma d/dt \cdots \]
4-Velocity
\[ U = \gamma(c, u) = dR/d\tau \]
4-Acceleration
\[ A = \gamma(\epsilon, \nu) + \gamma a = \frac{\partial U}{\partial \tau} \]
4-Polarization
\[ E = (\epsilon^0, \epsilon) = (\epsilon, \beta, \epsilon) \]
4-Spin
\[ S = (s_0, s) = (s, \beta, s) \]
4-Force
\[ F = \gamma(E/c, f) = dP/d\tau \]
4-Momentum
\[ P = (mc, p) = (E/c, p) \]
4-WaveVector
\[ K = (\omega/c, k) = (\omega, \nu/\gamma_{\text{phase}}) \]
4-TotalWaveVector
\[ K_t = (\omega/c, k_t) = -\partial[\Phi_{\text{phase}}] \]
4-TotalMomentum
\[ P_t = (E/c, p_t) = (H/c, p_t) = -\partial[S_{\text{action}}] \]

ProperTime Derivative
\[ U \cdot \partial \cdots \gamma d/dt \cdots \]
4-Position Derivative
\[ U \cdot E = 0 \]
4-UnitSpace
\[ U \cdot S = 0 \]

Invariant Interval
\[ R = (ct, r) = (ct)^2 - r^2 = (ct)^2 \]

Scalar / Physical Constants of Light
\[ c \]

Lorentz Scalars / Physical Constants
\[ \lambda_{\nu \nu} \]

Lorentz Metric
\[ \eta_{\mu \nu} = \Lambda_{\mu \nu} \]

Minkowski SpaceTime
\[ \partial^2 \text{r} \cdot \text{r} = 0 \]

EM Potential
\[ E = mc^2 \]
EM Charge
\[ q \]

EM Charge
\[ q \]

Minimal Coupling
\[ P = Q \]

Trace[T_{\nu\nu}] = \eta_{\nu \nu} T_{\nu \nu} = T_{\nu \nu} = T

\[ V \cdot V = V^2 + \nu V \cdot \nu = [(\nu^0)^2 - \nu V] = (\nu^0)^2 \]

Lorentz Scalar

SRQM Diagram
Special Relativity → Quantum Mechanics

The SRQM Interpretation: Links

See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document)
The 4-Vector SRQM Interpretation

QM is derivable from SR!

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

quantum relativity

SRQM = SciRealm QM?
A happy coincidence… :)