Using Special Relativity (SR) as a starting point, then noting a few empirical 4-Vector facts, one can instead *derive* the Principles that are normally considered to be the Axioms of Quantum Mechanics (QM). Hence, [SR → QM]

Since many of the QM Axioms are rather obscure, this seems a far more logical and understandable paradigm than QM as a separate theory from SR, and sheds light on the origin and meaning of the QM Principles. For instance, the properties of SR <Events> can be “quantized by the Metric”, while SpaceTime & the Metric are not themselves “quantized”, in agreement with all known experiments and observations to-date.

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

or: Why General Relativity (GR) is *NOT* wrong
or: Don’t bet against Einstein ;)
or: QM, the easy way...

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com) version 2020-May-16 .3

And yes, I did the Math...
Ad Astra...Magnum Opus

Recommended viewing: via a .PDF Viewer/WebBrowser with Fit-To-Page & Page-Up/Down ex. Firefox Web Browser

SR 4-Tensor
(2,0)-Tensor $T_{\mu\nu}^{iv}$
(1,1)-Tensor $T_{\mu}^{v}$, or $T_{\mu}^{\nu}
(0,2)$-Tensor $T_{\mu \nu}$

SR 4-Vector
(1,0)-Tensor $V_{\mu} = V = (v_0, v)$
SR 4-CoVector: OneForm
(0,1)-Tensor $V_{\mu} = (v_0, -v)$
SR 4-Scalar
(0,0)-Tensor $S$ or $S_{\mu}$ Lorentz Scalar
4-Vectors = 4D (1,0)-Tensors are a fantastic language/tool for describing the physics of both relativistic and quantum phenomena. They easily show many interesting properties and relations of our Universe, and do so in a simple and concise mathematical way. Due to their tensorial nature, these 4-Vectors are automatically 4D coordinate-frame invariant, and can be used to generate *ALL* of the physical Lorentz Scalar (0,0)-Tensors and higher-rank Tensors of Special Relativity (SR). Let me repeat: You can mathematically build *ALL* of the SR Lorentz Scalars and larger SR Tensors from empirical SR 4-Vectors.

SR 4-Vectors are likewise easily shown to be related to the standard 3D vectors { 3-vectors = 3D (1,0)-tensors } that are used in Newtonian classical mechanics (CM), Maxwellian classical electromagnetism (EM), and standard quantum mechanics (QM). In addition, each SR 4-Vector also fundamentally connects a special relativistically-related temporal scalar to a spatial 3-vector:

- **Temporal** time \( t \) & **Spatial** 3-position \( (x, y, z) \) as SR 4-Position \( \mathbf{R} = (ct, \mathbf{r}) \)
- **Temporal** energy \( E \) & **Spatial** 3-momentum \( (p_x, p_y, p_z) \) as SR 4-Momentum \( \mathbf{P} = (E/c, \mathbf{p}) \)

**Why 4-Vectors and Tensors as opposed to some of the more abstract mathematical approaches to Quantum Mechanics?**

Because the components of 4-Vectors and 4-Tensors are physical properties that can actually be empirically measured. Experiment is the ultimate arbiter of which theories actually correspond to reality. If your quantum logics and string theories give no testable/measurable predictions, then they are basically useless for real, actual, empirical physics.

In this treatise, I will first extensively demonstrate how 4-Vectors are used in the context of Special Relativity (SR), and then show that their use in Relativistic Quantum Mechanics (RQM) is really not fundamentally different. Quantum Principles, without need of QM Axioms, emerge in a natural and elegant way. **SR is a theory of Measurement, even in QM.**

I also introduce the **SRQM Diagramming Method**: a highly instructive, graphical charting-method, which visually shows how the SRQM 4-Vectors, Lorentz 4-Scalars, and higher rank 4-Tensors are all related to each other. This symbolic representation clarifies a lot of physics and is a great tool for teaching and understanding.

**SRQM: A treatise of SR → QM by John B. Wilson (SciRealm@aol.com)**
SRQM

Some Physics: Mathematics

Abbreviations & Notation

Gravitational Physics: 4D = 4-Dimensional = \{0,1,2,3\}
QM = Quantum Mechanics
CM = Classical Mechanics
GR = General Relativity
SR = Special Relativity
EM = Electromagnetism/ElectroMagnetics
QED = Quantum ElectroDynamics
RQM = Relativistic Quantum Mechanics
NRQM = Non-Relativistic Quantum Mechanics
RPE = Partial Differential Equation
PDE = Partial Differential Equation

4-Vector SRQM Interpretation of QM

SRQM = The [SR→QM] Interpretation of Quantum Mechanics, by John B. Wilson
In full: [GR→SR→RQM→QM→(CM & EM)]
Some Physics: Mathematics 
Conventions & Notation

4-Tensor, 4-Vector, 4-Scalar Conventions:
4-Vectors (4D) in bold UPPERCASE: \( \mathbf{A} \)
3-vectors (3D) in bold lowercase: \( \mathbf{a} \)
Temporal scalars (1D) in non-bold, usually lowercase, \( a^0 \)
Individual scalar components in non-bold: ex. \( \mathbf{A} = (a^0, a^1, a^2, a^3) \)
Rest scalars in normal non-bold, denoted with naught: \( a_0 \)
Tensor-index-notation in normal non-bold: ex. \( \mathbf{A}^{\mu} = (a^\mu) = (a^0, a^1) \)
4D Tensors use Greek indices: ex. \{ \mu, \nu, \sigma, \rho \} 
3D tensors use Latin indices: ex. \{ i, j, k \} 
4-Vector: \( \mathbf{A} \) or \( \mathbf{A}^\mu \); ex. 4-UnitTemporal \( T \)
4-CoVector or OneForm: \( \mathbf{A} \) or \( \mathbf{A}_\mu \); ex. GradientOneForm \( \partial_\mu = \partial_\mu \)
Null 4-Vector \( \mathbf{N} \sim (|a|, a) \), which has Lorentz Scalar \( \mathbf{N} \cdot \mathbf{N} = 0 \)

SR: Metric Convention: Particle Physics,Time-0th-Positive (+,-,-,-)

RQM & QM are derivable from principles of SR

Let that sink in...

Quantum Mechanics is derivable from Special Relativity

GR \( \rightarrow \) SR \( \rightarrow \) RQM \( \rightarrow \) QM \( \rightarrow \) {CM & EM}

Existing SR Rules

Quantum Principles

SR 4-Tensor
\( (2,0) \)-Tensor \( T^{\mu}_{\nu} \)
\( (1,1) \)-Tensor \( T^{\nu}_{\nu} \) or \( T_{\nu}^{\nu} \)
\( (0,2) \)-Tensor \( T_{\nu \mu} \)

SR 4-Vector
\( (1,0) \)-Tensor \( V^{\mu} = \mathbf{V} = (v^0, v^1) \)
\( (0,1) \)-Tensor \( V_\mu = (v_0, v_1) \)

SR 4-Scalar
\( (0,0) \)-Tensor \( S \) or \( S_\mu \)
Lorentz Scalar

3-Tensor 3D
\( (2,0) \)-Tensor \( T^k_{\nu} \)
\( (1,1) \)-Tensor \( T^\nu_{\mu} \) or \( T_{\nu}^{\mu} \)
\( (0,2) \)-Tensor \( T_{\nu \mu} \)

Classical (scalar)

3D Galilean Invariant

3-vector
not Lorentz Invariant

3-Scalar
3D (0,0)-Tensor
Special Relativity → Quantum Mechanics

The SRQM Interpretation: Links

See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document: most current ver. at SciRealm.org)
SRQM Study: Physical / Mathematical Tensors

4D Tensor Types: 4-Scalar, 4-Vector, 4-Tensor

Component Types: Temporal, Spatial, Mixed

Matrix Format

SR 4-Scalar S
a "number": magnitude

SR 4-Vector V^μ
an "arrow": magnitude and 1 direction

SR 4-Tensor T^μν
a "matrix or dyadic": magnitude and 2 directions

SR 4-Vector V = \vec{V}
(0,0)-Tensor S often as S, Lorentz Scalar

SR 4-Vector V = (V^0, V^1, V^2, V^3)
→ (V^t, V^x, V^y, V^z)

SR 4-Vector 4D (1,0)-Tensor V = \vec{V}
V^μ = (v^μ) = (v^0, v^1, v^2, v^3)

SR 4-Vector 4D (0,1)-Tensor \vec{V} = (V_0, V_1, V_2, V_3)
\vec{V} = (V^t, V^x, V^y, V^z)

SR 4-Tensor 4D (2,0)-Tensor T
T^μν = \begin{bmatrix} T^{00} & T^{01} \\ T^{10} & T^{11} \end{bmatrix}
\begin{bmatrix} T^{02} & T^{03} \\ T^{12} & T^{13} \end{bmatrix}

SR 4-Tensor 4D (1,1)-Tensor \Sigma
\Sigma^μν = \eta^μρ T^ρν

SR 4-Tensor 4D (1,1)-Tensor \Sigma = \eta^μρ T^ρν
\Sigma^μν = \delta^μ_ρ T^ρν

SR 4-Vector T^μν = T^\text{row:column}

SR 4-Scalar S
4-Position R^μ = (ct, r) = \langle \text{Event} \rangle

Mixed TimeSpace region: purple
The mnemonic being red and blue mixed make purple

Technically, all these objects are "SR 4-Tensors", but we usually reserve the name "4-Tensor" for objects with 2 (or more) indices, and use the "(m,n)-Tensor" notation to specify all the objects more precisely.

Rank = # of indices
0 = a Scalar
1 = a Vector
2 = a Dyadic:Matrix
c, etc.,
Dimension = # of values an index can use.
(4D:SR Tensors=4)

Trace[T^μν] = \eta^μ_ρ T^ρν = T^\rho_\mu
V \cdot V = V^μ \eta^\mu_ρ V^ρ = (V^ν)^2 - V^μ V^ν = (V^ν)^2 = Lorentz Scalar

SR:Minkowski Metric
a[\mathbf{R}] = \partial^2[\mathbf{R}] = \eta^μν \mathbf{V}^μ + \mathbf{H}^μν →
Diag[+1, -1, -1, -1] = \text{Diag}[1, -\delta^μ_ν] \{\text{in Cartesian form} \}
"Particle Physics" Convention
\{\eta^\mu_μ\} = 1/\eta^\mu_μ : \nu^μ = \delta^μ_ν \text{ Tr}[\eta^\mu_μ] = 4

4-Gradient \partial^μ
\partial = \partial^\mu \partial_\mu = (\partial/\partial c, \partial/\partial V)

4-Position R^μ = R^\mu = \langle \text{Event} \rangle

SpaceTime Dimensions

4D Tensor Octagon:
4-Tensors, 2 index = rank 2
4'2 = 8 corners in diagram
4'2 = (1+6+3) = 16 components

for 2-index tensor components:
6 Anti-Symmetric (Skew)
+10 Symmetric
16 General components

SR 4-CoVector = “Dual” 4-Vector
4D (0,1)-Tensor \Sigma = 4D One-Form
C^μ = \eta^μ_ρ \Sigma^ρ
C^μ = (c_0, c_1) → (c_0, c_x, c_y, c_z)
(c^0, -c) = (c_0, -c_x, -c_y, -c_z)

SR 4-Scalar S
SR 4-Vector V
SR 4-CoVector:OneForm
SR 4-Vector V = (V^0, V^1, V^2, V^3)

SR 4-CoVector V = \Sigma
V = (V^0, V^1, V^2, V^3)

SR 4-Tensor 4D (1,0)-Tensor V = \vec{V}
V^μ = (v^μ) = (v^0, v^1, v^2, v^3)

SR 4-Vector 4D (1,1)-Tensor \Sigma = \eta^μρ T^ρν
\Sigma^μν = \eta^μρ \eta_ρ^ν
\Sigma^μν = \delta^μ_ν \text{ Tr}[\eta^\mu_μ] = 4

Technically, all these objects are “SR 4-Tensors”, but we usually reserve the name “4-Tensor” for objects with 2 (or more) indices, and use the “(m,n)-Tensor” notation to specify all the objects more precisely.
In Classical Mechanics (CM), the magnitude of a 3-vector is computed using the Pythagorean theorem. Examine 3-position $\mathbf{r}$, which is a 3-position vector with the base at the origin $\mathbf{r}_0 \rightarrow 0$ = (0,0,0).

The 3D dot product of $\mathbf{r} \cdot \mathbf{r} = r^2 = (x^2 + y^2 + z^2)$ is the Pythagorean Theorem. It is equivalent to the 3D Kronecker Delta $\delta_{kk} = \text{Diag}[1,1,1] = I_3 = E_3$.

The 3D magnitude $r = \sqrt{r^2} = |r|$. 3D magnitudes are always positive.

$t$ & $|r|$ are scalar invariants only in Euclidean 3D space. However, our universe is locally Minkowski 4D.

The magnitude of an SR 4-Vector is very similar to the magnitude of a 3-vector, but there are some interesting differences. One uses the Lorentz Scalar Product, a 4D dot product, which includes time & space components, and is based on the SR/Minkowski Metric Tensor. Typically, the scalability convention of the Minkowski metric tensor $\eta_{\mu\nu} \rightarrow \text{Diag}[-1,-1,-1,1]$ (Cartesian form), with the other entries zero. Note the difference in signs.

It is equivalent to the 4D Kronecker Delta in mixed (1,1)-Tensor form. The 4D Kronecker Delta $\delta_{\mu\nu} = \text{Diag}[1,1,1,1] = I_4$.

Using Einstein Summation Convention which has upper/lower paired indices summed over.

$R = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct)^2 - (x^2 + y^2 + z^2) = (ct)^2 - \mathbf{r} \cdot \mathbf{r}$

for 4-Position $R = (ct,t)$

4D magnitude can be: negative(-), zero: null(0), positive(+)

The 4-Vector version has the Pythagorean element in the spatial components, the temporal component is of opposite sign. This gives a "causality condition", with SpaceTime intervals (in the [+1,-1,-1,1] Minkowski Metric) that can be:

$\Delta \mathbf{R} = [(c\Delta t)^2 - (\Delta r)^2]$

\text{Time-like:Temporal} $(\Delta r, \Delta t)$ = (0,0) (causal = 1D temporally-ordered, \textit{spatially relative})

\text{Light-like:Null} $(\Delta r, \Delta t)$ = (c,0) (causal & topological, maximum signal speed $|\Delta \mathbf{R}| = c$)

\text{Space-like:Space} $(\Delta r, \Delta t)$ = (0,c) (temporally relative, topological = 3D spatially-ordered)

3D Classical:Euclidean Metric
$\nabla[r] = \nabla[r^2] = \text{Kronecker Delta } \delta_{jk} = \delta_{jk} = \delta_{jk}$

Diag[1,1,1] = Diag[I_3] = Diag[E_3]

$\{\delta_{kk}\} = 1/\{\delta_{kk}\}

\text{Tr} [\delta_{kk}] = \delta_{+1} + \delta_{+2} + \delta_{+3} = 1 + 1 + 3 = 5$

3D Space
$\nabla \cdot \mathbf{r} = \nabla \cdot \delta_{jk} \mathbf{r}^k = \nabla \cdot \mathbf{r}^k = 3$

$(\partial / \partial x)[x] + (\partial / \partial y)[y] + (\partial / \partial z)[z] = (1+1+1) = 3$

4D SR:Minkowski Metric
$\delta[R] = \delta[R^2] = \eta_{\mu\nu} = \eta_{\mu\nu}^2 = V^\mu + H^\nu \rightarrow$

$\text{Diag}[+1,-1,-1,1] = \text{Diag}[1,1,1,1]$

$\{\eta_{\mu\nu}\} = 1/\{\delta_{\mu\nu}\} : \eta_{\mu\nu} = \delta_{\mu\nu} = \text{Diag}[1,1,1,1]$

$\text{Tr} [\eta_{\mu\nu}] = n_1 + n_2 + n_3 + 1 + 1 + 1 + 1 = 4$

4D SpaceTime
$\delta \mathbf{R} = \partial \eta_{\mu\nu} \mathbf{R}^\nu = \delta_0 \mathbf{R}^\nu = 4$

$\hspace{0.5cm} = (\partial / \partial x)[ct] - (\partial / \partial y)[x] - (\partial / \partial z)[y] - (\partial / \partial z)[z]$

$\hspace{0.5cm} = (1+1+1+1) = 4$

Dimension

$\text{Trace} [T^\nu] = \eta_{\mu\nu} T^{\nu} = T^\mu = T$

$\mathbf{V} \cdot \mathbf{V} = V^\mu V^\nu = [(V^\mu)^2 - \mathbf{V} \cdot \mathbf{V}] = (|V|^2)^2$

= Lorentz Invariant
SRQM Study: SR Minkowski SpaceTime

SR: Minkowski Metric \([ \eta ]\) Operations

Invariant Lorentz Scalar Product & Tensor Index Raising & Lowering

4-Vectors = tensorial entities of Minkowski SpaceTime which maintain covariance for inertial observers, meaning that they may have different \textit{relativistic} components for different observers, but describe the \textit{same physical object}. (like viewing a sculpture from different angles – snapshot pictures look different, but it's actually the same object)

There are also 4-CoVectors, aka. \{ One-Forms=4D \((0,1)\)-Tensors \} and dual to \{ 4-Vectors=4D \((1,0)\)-Tensors \}

Both GR and SR use a metric tensor \(g^{\mu\nu}\) to describe measurements in SpaceTime \((\text{TimeSpace})\). SR uses the “flat” Minkowski Metric \(g^{\mu\nu} \to \eta^{\mu\nu} \to \text{Diag}[1,1,1,1] = \text{Diag}[1,-1,-1,-1] \) \{Cartesian form\}, which is the \{curvature = 0 limit = low-mass limit\} of the GR metric \(g^{\mu\nu}\). SR is valid everywhere except extreme gravity, like near BH’s.

4-Vectors = 4D \((1,0)\)-Tensors

\[
\begin{align*}
A &= A^\mu = \eta^{\mu\nu} A^\nu = (a_0, a_1, a_2, a_3) = (a^0, a^1, a^2, a^3) = (a_0, a_1, a_2, a_3) \\
B &= B^\mu = \eta^{\mu\nu} B^\nu = (b_0, b_1, b_2, b_3) = (b^0, b^1, b^2, b^3) = (b_0, b_1, b_2, b_3) \\

\end{align*}
\]

Invariant Lorentz Scalar Product \((0,0)\)-Tensor

\[
\begin{align*}
A \cdot B &= A^\mu B_\mu = \eta^{\mu\nu} A^\mu B_\nu = \eta^{\mu\nu} A^\nu B^\mu \\
&= (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) = (a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3) = (a^0 b^0

\]

using the Einstein Summation Convention where upper-lower-paired indices are summed over

Proof of invariance \(\text{(using Tensor gymnastics and the properties of the Minkowski Metric } \eta \text{ & Lorentz Transforms } \Lambda)\):

\[
\begin{align*}
A' \cdot B' &= A'^\mu B'^\nu = \eta^{\mu\nu} A'^\nu B'^\mu \\
&= \Lambda^\mu_\nu A^\nu \eta^{\nu\lambda} \Lambda^\lambda^\beta B^\beta = (\eta^{\mu\nu} \Lambda^\nu_\lambda) A^\lambda B^\beta = (\eta^{\mu\nu} \Lambda^\nu_\lambda) A^\beta B^\beta = (\eta^{\mu\nu} \Lambda^\nu_\lambda) A_\lambda B^\beta = \Lambda^\mu_\nu A^\lambda B^\beta = \Lambda^\mu_\nu A^\nu B^\mu

\end{align*}
\]

Lorentz Scalar Product of 4-Vectors \(\leftrightarrow\) Lorentz Invariant Scalar = Same measured value for all inertial observers

Lorentz Invariant Scalars are also tensorial entities: \((0,0)\)-Tensors, which have the same value in all reference-frames.

Einstein & Lorentz “saw” the physics of SR, Minkowski & Poincaré “saw” the mathematics of SR.

We are indebted to all of them for the simplicity, beauty, and power of how SR and 4-vectors work...
**SRQM (Physics) Diagramming Method**

The SRQM Diagramming Method shows the properties and relationships of various physical objects/tensors in a graphical way. This “flowchart” method aids understanding.

**Representation:** 4-Scalars by ellipses, 4-Vectors by rectangles, 4-Tensors by octagons. Physical/mathematical equations and descriptions inside each shape/object. Sometimes there will be additional clarifying descriptions around a shape/object.

**Relationships:** Lorentz Scalar Products or tensor compositions of different 4-Vectors are on lines with arrows(→) between related 4-Vectors. Lorentz Scalar Products of a single 4-Vector or Lorentz Scalar Products or tensor compositions of different 4-Vectors are on lines with ellipses. Lorentz 4-Scalars by ellipses, 4-Vectors by rectangles, 4-Tensors by octagons.

**Flow:** Objects that are some function of a Lorentz 4-Scalar with another 4-Vector or 4-Tensor are on lines with arrows(→) indicating the direction of flow. (ex. multiplication)

**Properties:** Some objects will also have a symbol representing its properties nearby, and sometimes there will be color highlighting within the object to emphasize temporal-spatial properties. I will use blue=Temporal & red=Spatial → purple=mixed TimeSpace.

Alternate ways of writing 4-Vector expressions in physics: 

\[(\mathbf{A} \cdot \mathbf{B})\] is a 4-Vector style, which uses vector-notation (ex. inner product "dot=·" or exterior product "wedge=\wedge"), and is typically more compact, always using **bold** UPPERCASE to represent the 4-Vector, ex. \((\mathbf{A} \cdot \mathbf{B}) = (A^\mu \eta_{\mu \nu} B^\nu)\), and **bold** lowercase to represent 3-vectors, ex. \((\mathbf{a} \cdot \mathbf{b}) = (a^i \delta_{ij} b^j)\). Most 3-vector rules have analogues in 4-Vector mathematics.

\((\mathbf{A}^\mu \eta_{\mu \nu} B^\nu)\) is a Ricci Calculus style, which uses tensor-index-notation and is useful for more complicated expressions, especially to clarify those expressions involving tensors with more than one index, such as the Faraday EM Tensor \(F^{\mu \nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)\) = (\(\partial^\mu A^\nu\)).
Special Relativity → Quantum Mechanics

SRQM Tensor Invariants

Inherent 4D SpaceTime Properties

One of the extremely important properties of Tensor Mathematics is the fact that there are numerous ways to generate Tensor Invariants. These Invariants lead to Physical Properties that are fundamental in our Universe. They are totally independent of the coordinate systems used to measure them. Thus, they represent symmetry properties that are inherent in the fabric of SpaceTime (TimeSpace). See the Cayley-Hamilton Theorem, esp. for the Anti-Symmetric Tensor Products.

SRQM (Physics) Diagramming Method

4-Vector SRQM Interpretation of Relativity

4-Gradient \( \partial^\mu = (\partial / c, \vec{\nabla}) \)

4-Vector 4D (1.0)-Tensor

4-Tensor 4D (2.0)-Tensor

4-Displacement \( \Delta R = (c \Delta t, \Delta r) \)

4-Position \( \vec{R} = (ct, \vec{r}) = \langle \text{Event} \rangle \)

U: \( \gamma d\mu / dt \) → Lorentz Tensor Invariant

Derivative

\( \partial / c \)

\( \vec{\nabla} \)

4-Velocity \( \vec{u} \)

\( \gamma u \)

\( c \)

\( \beta = u/c \)

Relativistic Gamma \( \gamma = 1/\sqrt{1 - \beta^2} \), \( \beta = u/c \)

These invariants are not all totally independent, some invariants are functions of other invariants.
SRQM Study: Physical/Mathematical Tensors

**Tensor Types: 4-Scalar, 4-Vector, 4-Tensor**

**Physical Examples – Venn Diagram**

---

**0 index-count Tensors:**
- SR 4-Scalar (0,0)-Tensors: Lorentz Scalar $S$
- Rest Mass $(m_0)$
- Speed of Light $(c=\sqrt{\frac{1}{1-U^2}})$
- Planck's Constant $(h)$
- Proper Time $\Delta t = d\tau/dt$
- Derivative $\frac{d}{dx}$
- Space-Time Dimension $4D$
- Spatial Projection $V\cdot V$
- Proper Time $\Delta t$

**1 index-count Tensors:**
- SR 4-Vector (1,0)-Tensors: Lorentz 4-Vector $V$
- Momentum $p = m_0 c \vec{v}$
- Position $x = x^\mu$
- Lorentz scalar $\Lambda_{\mu\nu}$
- Lorentz Transform $\Lambda_{\mu\nu}$
- Temporal Projection $V_\nu$
- Spatial Projection $V^{\mu'}$

**2 index-count Tensors:**
- SR Mixed 4-Tensor (1,1)-Tensors: Lorentz Tensor $T_{\mu\nu}$
- Riemann Curvature Tensor $R_{\mu\nu\rho\sigma}$
- Ricci Decomposition $R_{\mu\nu} = S_{\mu\nu} + E_{\mu\nu} + C_{\mu\nu}$
- Weyl (Conformal) Curvature Tensor $C_{\mu\nu}$

**Higher index-count Tensors:**
- SR & GR 4-Tensors $T^{\mu\nu}$
- Ricci Tensor $R_{\mu\nu}$
- Weyl Tensor $C_{\mu\nu}$
- Riemann Curvature Tensor $R_{\mu\nu\rho\sigma}$
- Minkowski Metric $\eta_{\mu\nu}$

---

**SRQM Study of Physical 4-Vectors**

**SR 4-Vector:**
- $\mathbf{V} = (V^1, V^2, V^3, V^4)$
- Covariant Components $V_\mu$
- Contravariant Components $V^\mu$

**SR 4-Scalar:**
- $\mathbf{S} = (S^0, S^1, S^2, S^3, S^4)$
- Covariant Components $S_\mu$
- Contravariant Components $S^\mu$

**SR 4-Covector:**
- $\mathbf{V'} = (V'_0, V'_1, V'_2, V'_3)$
- Covariant Components $V'_\mu$
- Contravariant Components $V'^\mu$

---

**SR 4-Mixed Tensor:**
- $\mathbf{T} = (T^\mu_\nu)_{\mu\nu}$
- Covariant Components $T^{\mu\nu}$
- Contravariant Components $T_{\mu\nu}$
**SRQM Study:**

**SRQM 4-Vectors = 4D (1,0)-Tensors**

**SRQM 4-Tensors = 4D (2,0)-Tensors**

{or higher index #}

*Made from 4-Vector relations*

---

4-Vector = 4D (1,0)-Tensor

- 4-Position: \( R = c t, r = x^i \) (alt notation)
- 4-Velocity: \( U = \gamma c u = (\gamma u/c, \gamma u) \)
- 4-Momentum: \( P = \gamma mc = (mc/c, p) \)
- 4-Acceleration: \( \mathbf{a} = \gamma (\gamma u/c, u) \)
- 4-UnitSpatial: \( \mathbf{e} = \gamma (1, \mathbf{u}) \) (\( u^2 = 1 \))
- 4-UnitTemporal: \( \mathbf{E} = \gamma (\gamma u/c, u) \) (\( u^2 = 1 \))

4-Tensors of 2 or more indices. Use \((m,n)\)-Tensor notation to specify types more precisely.

---

**SI Dimensional Units**

- [m]
- [m/s]
- [m/s²]
- [m²/s³]

---

**Temporal : Spatial components**

- [Time (t)] (Space/length extent (r))
- [Temporal “velocity” factor (γ), Spatial “velocity” factor (γu), Spatial 3-velocity (u)]
- [Temporal “velocity” factor (γ), Spatial normalized “velocity” factor (γβ), Spatial 3-beta (β)]
- [Temporal “velocity” factor (γβ · n), Spatial normalized “velocity” factor (γβ · n), Spatial 3-beta (β · n)]
- [mass (m), energy (E), 3-momentum (p)] (with \( E = mc^2 = \gamma m c^2 = \gamma E_0 \))

---

**Temporal-Temporal : Temporal-Spatial : Spatial-Spatial components**

- [Charge-density (ρ) : 3-current-density = 3-charge-flux (J)]
- [Scalar-potential = voltage (φ) : 3-vector-potential (A), typically the EM versions (φEM) : (AEM)]
- [Potential-energy (V = \( q \phi \)) : 3-potential-momentum (q = A), EM ver (VEM = \( q \phiEM \)) : (qAEM)]
- [Temporal differential (\( \delta \)) : Spatial 3-gradient = spatial differentials (\( \delta = \partial_x + \partial_y + \partial_z \))]

---

**SR 4-Tensors**

- 4-Tensor of Physical 4-Vectors

---

4-Tensors can be constructed from the Tensor Products of 4-Vectors. Technically, 4-Tensors refer to all SR objects (4-Scalars, 4-Vectors, etc), but typically reserve the name 4-Tensor for SR Tensors of 2 or more indices. Use \((m,n)\)-Tensor notation to specify types more precisely.
SRQM Study:

4-Scalars = 4D (0,0)-Tensors = Lorentz Scalars = 4D SR Invariants ↔ Physical Constants

*Made from 4-Vector relations*

### SI Dimensional Units

- Time (s)
- Length (m)
- Mass (kg)
- Electric Charge (C)
- Velocity (m/s)
- Acceleration (m/s²)
- Current (A)
- Magnetic Flux (Wb)
- Electric Field (N/C)
- Magnetic Field (A/m)
- Temperature (K)

### 4-Scalar = 4D (0,0)-Tensor (generally composed of 4-Vector combinations with LSP)

\[
(t) = [R·U]/[U·U] = [R·R]/[R·U] \quad \text{**Time as measured in the at-rest frame**}
\]

\[
(\tau) = [dR·U]/[U·U] \quad \text{**Differential Time as measured in the at-rest frame**}
\]

\[
(\rho) = [U·U] = \gamma[d/dt] \quad \text{**Note that the 4-Gradient operator is to the right of 4-Velocity**}
\]

\[
(\phi) = [U·U] = \gamma[1, \beta] \quad \text{with 4-UnitTemporal} \quad T = \gamma[1, \beta] \quad \text{and} \quad [T·T] = +1 \quad \text{“Unit”}
\]

\[
(m_e) = [P·U]/[U·U] = [P·R]/[R·U] 
\quad \text{as Electron RestMass}
\]

\[
(E_e) = [P·U] 
\quad \text{as Electron Energy}
\]

\[
(\omega_e) = [K·U] 
\quad \text{as Electron Frequency}
\]

\[
(P_e) = [J·U]/[U·U] = (q)[N·U]/[U·U] = (q)(n_e) 
\quad \text{as Electron Momentum}
\]

\[
(\varphi_e) = [A·U] 
\quad \text{as the EM version RestScalarPotential}
\]

\[
(n_e) = [N·U]/[U·U] 
\quad \text{as Electron NumberDensity}
\]

\[
(F_{\text{phase,free}}) = [-K·R] = [K(r - \omega t)] \quad \text{**Units [Angle] = [WaveVec.] [Length] = [Freq.] [Time]**}
\]

\[
(S_{\text{action}}) = [-P·R] = [P(r - E t)] \quad \text{**Units [Action] = [Momentum] [Length] = [Energy] [Time]**}
\]

\[
(h) = [P·U]/[K_{\text{cyc}}·U] = [P·R]/[K_{\text{cyc}}] 
\quad \text{K_{cyc} = K/(2}\pi) 
\]

\[
(S) = [P·U]/[K_{\text{cyc}}·U] = [P·R]/[K_{\text{cyc}}] = K = (2\pi)K_{\text{cyc}} 
\]

\[
(\delta) = [T_{\text{dij}}] = \Lambda_{\alpha\beta}A^{\alpha\beta}_{\text{SR}} 
\quad \text{SR Dim = 4, InnerProduct} |\text{any Lorentz Transf.(cont., discrete)}| = 4
\]

\[
\delta F_{\text{SR}} = (\mu_e)J = (1/\epsilon_0 C^2)J 
\quad \text{Maxwell EM Egn. w/ source} \quad \mu_0\epsilon_0 = 1/\epsilon^2
\]

\[
\delta T_{\text{SR}} = (\mu_e)J = (1/\epsilon_0 C^2)J 
\quad \text{Maxwell EM Egn. w/ source} \quad \mu_0\epsilon_0 = 1/\epsilon^2
\]

\[
U_{\text{SR}} = (1/2)F 
\quad \text{Lorentz Force Egn.} \quad (q\rightarrow-\epsilon) \text{ as Electron Charge}
\]

\[
(Q) = \int[pdx dydz] = \int[p\gamma d^2x] = \int[(p\gamma)(d\gamma)(\gamma dy)] 
\quad \text{Integration of volume charge density}
\]

\[
(N) = \int[n dx dydz] = \int[n\gamma d^2x] = \int[(n\gamma)(d\gamma)(\gamma dy)] 
\quad \text{Integration of volume number density}
\]

\[
(V_{\text{SR}}) = (1/2)\int[d\gamma d\gamma] = \int[dA] = \int[(dA)(dy)] 
\quad \text{Integration of volume elements} \ (\text{Riemannian Volume Form})
\]

\[
(p_{\text{dij}}) = V_{\text{SR}} T_{\text{dij}}^\text{Temporal} = \text{“Vertical” Projection of PerfectFluid Stress-Energy Tensor}
\]

\[
(p_{\text{dij}}) = (1/3)H_{\text{SR}} T_{\text{dij}}^\text{Spatial} = \text{“Horizontal” Projection of PerfectFluid Stress-Energy Tensor}
\]

\[
2(b·b·e·c^2) = |P·F_{SR}| = F_{SR}^T_{SR} 
\quad \text{(e·b/c)^2 = Det}[F_{SR}^T_{SR}] \quad \text{since F_{SR}^T is 2n x 2n square anti-symmetric}
\]

Lorentz Scalars = (0,0)-Tensors can be constructed from the Lorentz Scalar Products (LSP) of 4-Vectors: (A·B) = Lorentz Scalar
SRQM Study: Primary/Primitive/Elemental 4-Vectors: 4-UnitTemporal \( T \) & 4-UnitSpatial \( S \)

4-UnitTemporal, Dimensionless
- Magnitude\(^2 = +1\)
  - "Magnitude" = \((\pm 1)\)
  - \(|\text{Magnitude}| = (1)\)

\[ T \cdot T = \gamma(1, \beta) \cdot \gamma(1, \beta) = \gamma^2(1 + \beta \cdot \beta) = +1 \]

4-UnitSpatial, Dimensionless
- Magnitude\(^2 = 0\)
  - "Magnitude" = \((0)\)
  - \(|\text{Magnitude}| = (0)\)

\[ T \cdot S = \gamma(1, \beta) \gamma_{\beta \alpha} (\beta \cdot n, \hat{n}) = \gamma(1, \beta) (1 + \beta \cdot n) \]

4-UnitSpatial, Dimensionless
- Magnitude\(^2 = -1\)
  - "Magnitude" = \((\pm i)\)
  - \(|\text{Magnitude}| = (1)\)

\[ S \cdot S = \gamma_{\beta \alpha}(\beta \cdot n, \hat{n}) \cdot \gamma_{\beta \alpha}(\beta \cdot n, \hat{n}) = \gamma_{\beta \alpha}^2((\beta \cdot n) \cdot (\beta \cdot n)) = \gamma_{\beta \alpha}^2(1 - (\beta \cdot \hat{n})^2) = -1 \]

SR LightCone

**Relativistic Gamma**
\[ \gamma = 1/\sqrt{1 - \beta \cdot \beta} = 1/\sqrt{1 - |\beta|^2} \]

**SR Light Cone**
\[ \gamma_{\beta \alpha} = 1/\sqrt{1 - |\beta|^2} \]

The 4-UnitTemporal \( T \) and 4-UnitSpatial \( S \) are both dimensionless, which allows them to make dimensional 4-Vectors via multiplication by a 4-Scalar, as shown here.

**4-Velocity**
- Magnitude\(^2 = +c^2\)
  - "Magnitude" = \((c)\)
  - \(|\text{Magnitude}| = (c)\)

\[ U = U^\mu = \gamma(c, u) = c\gamma(1, \beta) = dR/d\tau = cT \]

**4-Spin**
- Magnitude\(^2 = -(s_o)^2\)
  - "Magnitude" = \(-(s_o)\)
  - \(|\text{Magnitude}| = (s_o)\)

\[ S_{\text{spin}} = S_{\text{spin}}^\mu = s_o S \]

\[ S_{\text{spin}} \cdot S_{\text{spin}} = (s_o^2, s) \cdot (s_o^2, s) = (s_o^2, s) \cdot (s_o^2, s) = -(s_o)^2 \]

**Trace**
\[ T^{\alpha \beta} = \eta_{\alpha \beta} T^{\alpha \beta} = T^{\alpha \beta} \]

\[ V \cdot V = V^{\alpha \mu} V^{\alpha \nu} \]

\[ (V^{\alpha \mu})^2 = (V^{\alpha \mu})^2 = (V^{\alpha \mu})^2 \]

\[ \text{Lorentz Scalar} \]

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http://scirealm.org/SRQM.pdf
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Differential, 4-Gradient

SR SpaceTime Calculus & Invariants

---

4-Position

\[ R^\text{v} = (ct, r) = \langle \text{Event} \rangle \rightarrow (ct, x, y, z) \]

4-Velocity

\[ U = U^\nu = \gamma(c, u) \]

\[ d^\nu / d^\tau = (U \cdot \partial) R \]

4-Differential

\[ dR = dR^\nu = (cdt, dr) \text{ [infinitesimal]} \]

\[ \Delta R = \Delta R^\nu = (c\Delta t, \Delta r) = R_2 - R_1 \text{ [finite]} \]

4-Gradient

\[ \partial = \partial_R = \partial_x = \partial^\nu = (\partial / c, -\nabla) \]

\[ = (\partial / c) / \partial \gamma + (\partial / \partial y) + (\partial / \partial z) \]

Total Derivative Chain Rule

\[ dR^\nu \eta_{\mu\nu} (\partial^\nu) = dR^\nu (\partial_R) = dR^\nu (\partial_R R^\nu) = (dt \partial_t + dx / \partial x + dy / \partial y + dz / \partial z) \]

Relativistic Gamma

\[ \gamma = 1 / \sqrt{1 - \beta \cdot \beta} \], \( \beta = u / c \)

---

The 4-Velocity is interesting in that it sort of bootstraps itself into existence:

\[ U = \gamma(c, u) \]

\[ = (d^\nu / d^\tau) \]

\[ = (U \cdot \partial) R \]

The bootstrap is because d/d\tau = (U \cdot \partial)
SRQM Study:  
Physical 4-Vectors 
Some 4-Acceleration Relations 

4-Derivative 

\[ \partial = \partial_\gamma = \partial_0 = \partial_x = \partial_y = \partial_z = \partial/c, \rightarrow \partial/\partial R \rightarrow \partial/\partial x, \partial/\partial y, \partial/\partial z \] 

4-Position 

\[ R = R^\mu = (c,t) \rightarrow (c,t,x,y,z) \] 

4-Velocity 

\[ U = U^\mu = \gamma(u,c) = \gamma(c,u) - \frac{\partial}{\partial \gamma} \frac{\partial}{\partial t} = \frac{\partial}{\partial \gamma} \frac{\partial}{\partial t} + cT \] 

Invariance LightSpeed 

\[ U \cdot U = c^2 \] 

4-Acceleration 

\[ A = A^\mu = \gamma(c', \gamma' u + \gamma a) = \gamma(\gamma^2 (u^a) / c^2 + a) = dU / dt = dR / dt \] 

\[ = -\gamma^2 (u^a) / c^2 - \gamma^4 (a^a) \rightarrow -\gamma^2 (a^a) : (u || a) \] 

The tedious algebra...
SRQM Study: Physical 4-Vectors
Some 4-Gradient Relations
These are relations for the 4-(Position)Gradient, one can have 4-Gradients wrt. other 4-Vector variables as well... ex. $\partial_k$
SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

- Lorentz Identity Transform: \( \Lambda^{\mu \nu} \rightarrow \eta^{\mu \nu} = \delta^{\mu \nu} = I_4 \)
- SR Lorentz Transforms
  - Lorentz x-Boost Transform: \( \Lambda^{\mu \nu} \rightarrow B^{\mu \nu} = \gamma \delta^{\mu \nu} \)
  - Lorentz z-Rotation Transform: \( \Lambda^{\mu \nu} \rightarrow R^{\mu \nu} = \gamma \delta^{\mu \nu} \)
  - General Time-Space Boost: \( \gamma = \frac{1}{\sqrt{1-v^2}} \)
  - General Space-Space Rotation: \( \Lambda_\alpha \Lambda_\beta = 4 = \Lambda_{\alpha'} \Lambda_{\beta'} \)
- SR Minkowski Metric: \( \delta[R] = \delta[R^{\mu \nu}] = \eta^{\mu \nu} = V^{\mu \nu} + H^{\mu \nu} \)
- Faraday EM: \( F^{\alpha \beta} = \delta[R]^{A_\alpha} - \delta[R]^{A_\beta} = \delta^{\alpha A} \)
- 4-Angular Momentum: \( M^{\alpha \beta} = \gamma X P^\alpha - \gamma X P^\beta = X^\alpha P^\beta = X \cdot P \)

Note that all the Lorentz Transforms and the Minkowski Metric are unit dimensionless. The Perfect Fluid has units of [energy density = pressure = J/m^4 = N/m^2 = kg/m^3].
SRQM Study: Physical 4-Tensors

Some SR 4-Tensors and Symbols

- **4-Unit Temporal Tensor**: $T^{iv} = (1, \beta) = (1, 1/c) = U/c$
- **Temporal Projection (2,0)-Tensor**: $P^{iv} \to \delta^{i}_{\mu} T^{x\nu} = \delta^{i}_{\mu} T^{x\nu}$
- **SR Minkowski Metric**: $\delta[R] = \delta^{x}_{\mu} \gamma^{x}_{\nu} = \gamma^{x}_{\nu} = H^{iv} + T^{iv}$
- **SR: Minkowski Metric Tensor**: $\delta[R] = \delta^{x}_{\mu} \gamma^{x}_{\nu} = \gamma^{x}_{\nu} = H^{iv} + T^{iv}$
- **Faraday EM Tensor**: $T^{iv} = \delta^{2} R^{i} = \delta^{2} R^{i} - \delta^{i}_{\mu} \alpha^{j}$
- **Maxwell EM Stress-Energy Tensor**: $T^{iv} = (1/\mu_{0}) [F_{iv} F_{x}^{j} - (1/4) \eta^{iv} F_{x}^{j} F_{x}^{k}]$ (No RestFrame, Light-Like, Null)

- **SRM 4-Vector**
  - $(1,0)$-Tensor $V^{iv} = (V^{iv}, v^{iv})$
  - $(0,1)$-Tensor $V^{iv} = (0, V^{iv})$

- **SRM 4-Scalar**
  - Scalar $S_{0}$
  - Lorentz Scalar $V^{iv} = (v^{iv}, 0)$

**Equation of State**

$\text{EoS}[T^{iv}] = w = p_{e}/\rho_{e}$

**Temporal “(V)vertical” Projection (2,0)-Tensor**

- **Projection (2,0)-Tensor**: $P^{iv} \to \delta^{i}_{\mu} T^{x\nu} = \delta^{i}_{\mu} T^{x\nu}$
- **SR 4-Tensor Symmetric, Spatial Isotropic**: $T^{iv} = \text{Diag}[1, -1, 1, -1]$
- **SRM Stress-Energy 4-Tensor**

**Perfect Fluid Stress-Energy Tensor**

$T^{iv} = (\rho_{e}) [H^{iv} + (-p_{e}) \delta^{i}_{\mu}]$ (MCRF)

**Cold Matter-Dust**

$T^{iv} = P^{i} N^{v} = \text{Diag}[p_{e}, 0, 0, 0]$ (MCRF)

**Stress-Energy 4-Tensor**

- **SRM Stress-Energy 4-Tensor**
  - Symmetric, Spatial Isotropic
  - Pressureless

**Null-Dust=Photon Gas**

$T^{iv} = (\rho_{e}) [H^{iv} + (-p_{e}/3) \delta^{i}_{\mu}]$ (MCRF)

**Lambda Vacuum**

$T^{iv} = (\rho_{e}) [H^{iv} + (\Lambda) \delta^{i}_{\mu}]$ (MCRF)

**Dark Energy?**

$T^{iv} = (\rho_{e}) [H^{iv} - (1/3) \delta^{i}_{\mu}]$ (MCRF)

**Zero:Nothing Vacuum**

$T^{iv} = (0, 0, 0)$

**SR 4-Tensor**

- $(2,0)$-Tensor $T^{iv}$
- $(1,1)$-Tensor $T^{iv}$, or $T_{x}^{iv}$
- $(0,2)$-Tensor $T^{iv}$

**SR 4-Vector**

- $(1,0)$-Tensor $V^{iv}$
- $(0,1)$-Tensor $V^{iv}$

**SR 4-Scalar**

- $S_{0}$
- Lorentz Scalar $V^{iv} = (v^{iv}, 0)$

**Note**: The projection tensors & the Minkowski Metric are unit independent. [1] Energy Density (temporal) & Pressure (spatial) have the same dimension measurement units. [J/m³ = kg/m³ = m²/kg s⁻¹]

**Trace Tensor**

$T^{iv} = \eta^{iv} T^{iv} = T^{iv} = T^{iv}$

**4-Vector SRQM Interpretation**

- of QM

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SRQM Study: Physical 4-Tensors

Projection 4-Tensors \( \{ P^{\mu \nu} : P^{\mu \nu} : P^{\mu \nu} \} \)

The projection tensors can work on 4-Vectors to give a new 4-Vector, or on 4-Tensors to give either a 4-Scalar component or a new 4-Tensor.

4-Unit Temporal \( T^T = \gamma(1, \beta) \)
4-Generic \( A^T = (a^0, a^1, a^2, a^3) = (a^0, a^1, a^2, a^3) \)
4-Unit Spatial \( S^T = \gamma(0, \hat{n}) \)

\( V^\mu A^\mu = (1-a^0 a^0 + a^1 a^1 + a^2 a^2 + a^3 a^3) \)
\( H^\nu A^\nu = (a^0 a^0 + a^1 a^1 + a^2 a^2 + a^3 a^3) \)

Spatial projection \( (0, a^i, a^i, a^i) = (0, a^i, a^i, a^i) \)

Temporal projection \( (0, \hat{n}, \hat{n}, \hat{n}) \)

Minkowski Metric

\( V^\mu T^\mu = V^\mu ((p^0)^2 V^\mu + (p^i)^2 H^i)^{-1} = (p^0)^2 V^\mu + (p^i)^2 H^i \)
\( H^\mu T^\mu = H^\mu ((p^0)^2 V^\mu + (p^i)^2 H^i)^{-1} = (p^0)^2 V^\mu + (p^i)^2 H^i \)

Note that the projection tensors are unit dimensionless; the object projected retains its own dimensional measurement units. Also note that the (2,0) & (0,2) Spatial Projectors have opposite signs from the (1,1) - Spatial due to the (t, r, s, i) metric signature convention
SRQM: The [SR→QM] Interpretation of Quantum Mechanics

Special Relativity (SR) Axioms: Invariant Interval + LightSpeed (c) as Universal Physical Constant lead to SR, although technically SR is itself the Minkowski-SpaceTime low-curvature: "flat" limiting-case of GR.

\{c, \tau, m_0, \hbar, i\} = \{c: \text{SpeedOfLight}, \tau: \text{ProperTime}, m_0: \text{RestMass}, \hbar: \text{Dirac/Planck Reduced Constant (h=h/2\pi)}, i: \text{Imaginary Number} \sqrt{-1}\}: are all Empirically Measured SR Lorentz Invariant Physical Constants and/or Mathematical Constants

Standard SR 4-Vectors:

4-Position \( R = (ct, r) \) = \langle \text{Event} \rangle
4-Velocity \( U = \gamma (c, u) \) = \( (U \cdot \partial) R = (\frac{d}{d\tau}) R = \frac{dR}{d\tau} \)
4-Momentum \( P = (Ec/c, p) \) = \( m_0 U \)
4-WaveVector \( K = (\omega/c, k) \) = \( P/\hbar \)
4-Gradient \( \partial = (\partial_t/c, \nabla) \) = \(-iK\)

Related by these SR Lorentz Invariants:

\( (R \cdot R) = (ct)^2 \)
\( (U \cdot U) = (c)^2 \)
\( (P \cdot P) = (m_0 c)^2 \)
\( (K \cdot K) = (m_0 c/\hbar)^2 \)

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Quantum Eqn, and thence to QM via the low-velocity limit \( \{ |v| << c \} \), giving the Schrödinger Eqn. This fundamental KG Relation also leads to the other Quantum Wave Equations:

\begin{align*}
\text{spin}=0 & \quad \text{boson field} = 4\text{-Scalar}:
\text{RQM (massless)} & \quad \text{RQM (with non-zero mass)} & \quad \text{QM} \\
\text{spin}=1/2 & \quad \text{fermion field} = 4\text{-Spinor}:
\text{Free Scalar Wave (Higgs)} & \quad \text{Klein-Gordon} & \quad \text{Schrödinger (regular QM)} \\
\text{spin}=1 & \quad \text{boson field} = 4\text{-Vector}:
\text{Maxwell (EM photonic)} & \quad \text{Dirac (w/ EM charge)} & \quad \text{Pauli (w/ EM charge)}
\end{align*}

SRQM Chart:

Special Relativity → Quantum Mechanics
SR→QM Interpretation Simplified

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM 4-Vector Topic Index

Mostly SR Stuff
4-Vector Basics, SR 4-Vectors = Physical 4D (1,0)-Tensors
Paradigm Assumptions: Right & Wrong
Minkowski:SR SpaceTime, TimeSpace, <Events>, WorldLines, 4D Minkowski Metric
SR (4-Scalars, 4-Vectors, 4-Tensors) & Tensor Invariants, Cayley-Hamilton Theorem
SR Lorentz Transforms, CPT Symmetry, Trace Identification, Antimatter, Feynman-Stueckelberg
Fundamental Physical Constants = Lorentz Scalar Invariants = SR 4-Scalars = (0,0)-Tensors
Projection Tensors: Temporal "(V)vertical" & Spatial "(H)orizontal"; (V),(H) refer to Light-Cone
Stress-Energy Tensors, Perfect Fluids, Special Cases (Dust, Radiation, EM, DarkEnergy, etc)
Invariant Intervals, Measurement
SpaceTime Kinematics & Dynamics, ProperTime Derivative
Einstein's E = mc² = γmc² = γE₀, Rest Mass:Rest Energy, Invariants
SpaceTime Orthogonality: Time-like 4-Velocity, Space-like 4-Acceleration
Relativity of Simultaneity:Stationary; Invariance/Absolutes of Causality:Topology
Relativity: Time Dilation (--→ clock moving →→), Length Contraction (→→ ruler moving ←←)
Invariants: Proper Time (| clock at rest |), Proper Length ( | ruler at rest |)
Temporal Ordering: (Time-like) Causality is Absolute; (Space-like) Simultaneity is Relative
Spatial Ordering: (Time-like) Stationary is Relative; (Space-like) Topology is Absolute
SR Motion * Lorentz Scalar = Interesting Physical 4-Vector
SR Conservation Laws & Local Continuity Equations, Symmetries
Relativistic Doppler Effect, Relativistic Aberration Effect
SR Wave-Particle Relation, Invarianτ of Alembertian Wave Eqn, SR Waves, 4-WaveVector
SpaceTime is 4D = (1+3)D: t=ct=ct, x=ct, y=ct, z=ct, t²=x²+y²+z²=ct² = x²+y²+z²=ct² = (a²,a¹,a²,a³) = 4 comps
Minimal Coupling = Interaction with a (Vector)Potential
Conservation of 4-TotalMomentum (TotalEnergy=Hamiltonian & 3-total-momentum)
SR Hamiltonian: Lagrangian Connection
Lagrangian, Lagrangian Density
Hamilton-Jacobi Equation (differential), Relativistic Action (integral)
Euler-Lagrange Equations
Noether's Theorem, Continuous Symmetries, Conservation Laws, Continuity Equations
Relativistic Equations of Motion, Lorentz Force Equation
c² Invariant Relations, The Speed-of-Light (c)
Thermodynamic 4-Vectors, Unruh-Hawking Radiation, Particle Distributions

Mostly QM & SRQM Stuff
Advanced SRQM 4-Vectors
Where is Quantum Gravity?
Relativistic Quantum Wave Equations
Klein-Gordon Equation/ Fundamental Quantum Relation
RoadMap from SR to QM: SR → QM, SRQM 4-Vector Connections
QM Schrödinger Relation
QM Axioms? - No, (QM Principles derived from SR) = SRQM
Relativistic Wave Equations: based on mass & spin & relative velocity:energy
RWE's: Klein-Gordon, Dirac, Proca, Maxwell, Weyl, Pauli, Schrödinger, etc.
Classical Limits: SR's ( |ω| << c ) ; QM's ( |h| |p| << ( p,p )
Photon Polarization
Linear PDE's→{(Principle of Superposition, Hilbert Space, <Bra|,|Ket> Notation)
Canonical QM Commutation Relations ↔ derived from SR
Heisenberg Uncertainty Principle (due to non-zero commutation)
Pauli Exclusion Principle (Fermion), Bose Aggregation Principle (Boson)
Complex 4-Vectors, Quantum Probability, Imaginary values
CPT Theorem, Lorentz Invariance, Poincaré Invariance, Isometry
Hermitian Generators, Unitarity:Anti-Unitarity
QM → Classical Correspondence Principle, similar to SR → Classical Low Velocity
The Compton Effect = Photon:Electron Interaction (neglecting Spin Effects)
Photon Diffraction, Crystal-Electron Diffraction, The Kapitza-Dirac Effect
The (h) Relation, Einstein-de Broglie, Planck: Dirac, Wave-Particle
The Aharonov-Bohm Effect ( integral ) , The Josephson Junction Effect ( differential )
Dimensionless Quantities
SRQM Symmetries:
Hamilton-Jacobi vs. Relativistic Action
Differential (4-Vector) vs. Integral (4-Scalar)
Schrödinger Relations vs. Cyclic Imaginary Time ↔ Inverse Temperature
4-Velocity:4-Position vs. Euler-Lagrange Equations
Matter-Matter Trace Identification of Lorentz Transforms, CPT
Quantum Relativity: GR is *NOT* wrong, "Never bet against Einstein" :)
Quantum Mechanics is Derivable from Special Relativity, SR → QM: SRQM

SRQM: A treatise of SR → QM by John B. Wilson (SciRealm@aol.com)
There are some paradigm assumptions that need to be cleared up:

The physical world *IS NOT* Euclidean 3-dimensional (3D) with an absolute background time

Classical and quantum 3D physics is a great approximation, but only for slow-moving objects \(|v| \ll c\).

3D physics uses 3-vectors, which are 3D (1,0)-tensors, and does have various 3D invariants, but it does not contain many of the physical relationships that we now know to be true.

This is based on a century of relativistic 4D physics experiments and observations.

The physical world *IS* locally Minkowskian 4-dimensional SpaceTime (4D), with relativistically-interconnected (1 time + 3 space) dimensions

Time and space are interconnected in a very specific way, via relativistic metrics, which give a great many special relationships and invariances that 3D physics misses entirely.

These properties are easily explained using 4-Vectors, which are Physical 4D (1,0)-Tensors.

3D physics can be obtained from 4D Physics as limiting-cases of \(|v| \ll c\).

Classical Mechanics (CM) is just a low-speed limiting-case of Special Relativity (SR)

Quantum Mechanics (QM) is is just a low-speed limiting-case of Relativistic Quantum Mechanics (RQM)
There are some paradigm assumptions that need to be cleared up:

**Minkowskian:SR 4D Physical 4-Vectors *ARE NOT* generalizations of Classical/Quantum 3D Physical 3-vectors.**

While a "mathematical" Euclidean (n+1)D-vector is the generalization of a Euclidean (n)D-vector, the "Physical/Physics" analogy ends there.

**Minkowskian:SR 4D Physical 4-Vectors *ARE* the primitive elements of 4D Minkowski:SR SpaceTime.**

Classical/Quantum Physical 3-vectors are just the **spatial** components of SR Physical 4-Vectors = 4D (1,0)-Tensors. There is also a fundamentally-related Classical/Quantum Physical scalar related to each 3-vector, which is just the **temporal** component scalar of a given SR Physical SpaceTime 4-Vector.

4-Position \( \mathbf{R} = R^\mu = (r^\mu) = (r^0, r^i) = (ct, \mathbf{r}) \rightarrow (ct, x, y, z) \)

4-Momentum \( \mathbf{P} = P^\mu = (p^\mu) = (p^0, p^i) = (E/c, \mathbf{p}) \rightarrow (E/c = p^0/c, p^x, p^y, p^z) \)

These Classical/Quantum \{scalar\}+\{3-vector\} are the dual \{temporal\}+\{spatial\} components of a single SR **TimeSpace** 4-Vector = (temporal scalar * \(c\)^\pm, spatial 3-vector) with SR LightSpeed factor \((c^\pm)\) to give correct overall dimensional measurement units.

While different observers may see different relative "values" of the Classical/Quantum components \((v_0, v_1, v_2, v_3)\) from their point-of-view/frame-of-reference in SpaceTime, each will see the same actual SR 4-Vector \(\mathbf{V}\) and its magnitude \(V \cdot V = |\mathbf{V}|^2 = (v_0^2) - \mathbf{v} \cdot \mathbf{v}\) at a given <Event> in SpaceTime. Magnitudes\(^2\) can be \(+/-/0/\) in Special Relativity, due to the Lorentzian=pseudo-Riemannian metric (non-positive-definite)
Special Relativity → Quantum Mechanics

Paradigm Background Assumptions (part 3)

There are some paradigm assumptions that need to be cleared up:

Relativistic Physics **IS NOT** the generalization of Classical or Quantum Physics.
Classical & Quantum Physics **ARE** the low-velocity { |v| << c } limiting-case approximation of Relativistic Physics.

This includes (Newtonian) Classical Mechanics and Classical QM (NRQM: meaning the non-relativistic Schrödinger QM Equation – it is not fundamental).
The rules of standard QM are just the low-velocity approx. of RQM rules. Classical EM is for the most part already compatible with Special Relativity.
However, Classical EM doesn't include intrinsic spin, even though spin is a result of SR Poincaré Invariance, not QM.

So far, in all of my research, if there was a way to get a result classically, then there was usually a much simpler way to get the result using tensorial 4-Vectors and SRQM relativistic thinking. Likewise, a lot of QM results make much more sense when approached from SRQM (ex: for Temporal vs. Spatial relations).

4-Vector formulations are all extremely easy to derive in SRQM and are all relativistically covariant and give invariant results.

Einstein Energy: Mass Eqn: \( P = m_u U \rightarrow \{ E = mc^2 = \gamma m_u c^2 = \gamma E_o : p = mu = \gamma m_u \} \)

Hamiltonian: \( H = \gamma (P_T U) \) \{ Relativistic \} \rightarrow (T + V) = (E_{\text{kinetic}} + E_{\text{potential}}) \} \} \}

Lagrangian: \( L = -(P_T U)/\gamma \) \{ Relativistic \} \rightarrow (T - V) = (E_{\text{kinetic}} - E_{\text{potential}}) \} \}

SR Wave Eqn \{inv of Phase Eqn\}: \( K_T = -\partial[\Phi_{\text{phase}}] = P_T/\hbar \rightarrow \{ \omega_T = -\partial_t[\Phi] : \kappa_T = \nabla[\Phi] \} \)

Hamilton-Jacobi Eqn \{inv of Action Eqn\}: \( P_T = -\partial[S_{\text{action}}] = \hbar K_T \rightarrow \{ E_T = -\partial_t[S] : p_T = \nabla[S] \} \)

SR Action Eqn \{inv of H-J Eqn\}: \( \Delta S_{\text{action}} = -\int_{\text{path}} P_T \cdot dX = -\int_{\text{path}} (P_T \cdot U) dt = \int_{\text{path}} L dt \)

SR/QM Phase Eqn \{inv of Wave Eqn\}: \( \Delta \Phi_{\text{phase}} = -\int_{\text{path}} K_T \cdot dX = -\int_{\text{path}} (K_T \cdot U) dt = \Delta S_{\text{action}}/\hbar \)

Euler-Lagrange Equation: \( (U = (d/d\tau)R) \rightarrow (\partial_R = (d/d\tau) \partial_U) \) \{ the easy derivation \}

Hamilton’s Equations: \( (d/d\tau)[X] = (\partial/d\tau)[P_T][H_o] \) & \( (d/d\tau)[P_T] = (\partial/d\tau)[\Phi][H_o] \)

Euler-Lagrange Equation: \( \Phi = (d/d\tau)X = (\partial/d\tau)[P_T][H_o] \) & \( (d/d\tau)[P_T] = (\partial/d\tau)[\Phi][H_o] \)

(d'Alembertian Wave Equation: \( \partial^2 - \partial^2 = (\partial/c)^2 - \nabla^2 \), with solutions \( \sim \sum_n (\lambda_n) e^{+i \lambda_n x} \)

Einstein-de Broglie Relation: \( P = \hbar \kappa \rightarrow \{ E = \hbar \omega : p = \hbar \kappa \} \)

Complex Plane-Wave Relation: \( K = i \partial \rightarrow \{ \omega = i \partial_t : k = i \partial \} \)

Schrödinger Relations: \( P = i \hbar \partial \rightarrow \{ E = i \hbar \partial_t : p = -i \partial \} \)

Canonical QM Commutation Relations inc. QM Time-Energy: \( [P^\mu, X^\nu] = \hbar \eta_{\mu \nu} \rightarrow \{ [x^\mu, p^\nu] = \hbar \delta^\mu_\nu \} \)

\( [\partial^\mu, X^\nu] = \eta_{\mu \nu} \rightarrow \{ [x^\mu, \partial^\nu] = \hbar \delta^\mu_\nu \} \)

Total Momentum: \( P_T = P + q A \rightarrow \{ E_T = E + q \Phi : p_T = p + qa \} \)

Minimal Coupling: \( P = P_T - q A \rightarrow \{ E = E_T - q \Phi : p = p_T - qa \} \)

Invariance, not QM.

{Physical Inverse Effects}

Josephson-Junction (differential 4-Vector format): \( A = -(h/q)\partial[\Delta \Phi_{\text{pot}}] \)

Aharonov-Bohm (integral 4-Scalar format): \( \Delta \Phi_{\text{pot}} = -(q/h)\partial A \cdot dX \)

Compton Scattering: \( \Delta \lambda = (\lambda' - \lambda) = (h/m_c c)(1 - \cos \phi) \)

Klein-Gordon Relativistic Quantum Wave Eqn: \( \partial^2 - \partial^2 = -(m_c^2 c^2 \hbar^2) \)

SR → QM

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

http://scirealm.org/SRQM.pdf

John B. Wilson
SciRealm@aol.com
SciRealm.org
There are some paradigm assumptions that need to be cleared up:

We will **NOT** be employing the commonly-(mis)used Newtonian classical limits \(\{c \to \infty\}\) and \(\{\hbar \to 0\}\).

Neither of these is a valid physical assumption, for the following reasons:

1. Both \(c\) and \(\hbar = \hbar/2\pi\) are **unchanging** Universal Physical Constants and Lorentz Scalar Invariants. Taking a limit where these change is non-physical. They are **CONSTANT**.
   Many, many experiments verify that these physical constants have not changed over the lifetime of the universe. This is one reason for the 2019 Redefinition of SI Base Units on Fundamental Constants \(\{c, \hbar, e, k_B, N_A, K_B, \Delta \nu_{Cs}\}\).

2. Photons/waves have energy \((E)\): via momentum \((E=pc)\) & frequency \((E=\hbar \omega)\): \((\omega = 2\pi \nu)\) \(\text{[angular [rad/s], circular[cycle/s], } 2\pi \text{ rad = 1 cycle]}\)
   Let \(E = pc\). If \(c \to \infty\), then \(E \to \infty\). Then Classical EM light rays/waves have infinite energy.
   Let \(E = \hbar \omega = \hbar \nu\). If \(\hbar \to 0\), then \(E \to 0\). Then Classical EM light rays/waves have zero energy.

Obviously neither of these is true in the Newtonian/Classical limit.
In Classical EM and Classical Mechanics, LightSpeed \((c)\) remains a large but finite constant.
Likewise, Dirac's (Planck-reduced) Constant \((\hbar = \hbar/2\pi)\) remains very small but never becomes zero.

The **correct way** to take the limits is via:
The low-velocity non-relativistic limit \(\{|v| << c\}\), which is a physically-occurring situation.
The Hamilton-Jacobi non-quantum limit \(\{\hbar |\nabla \cdot p| << (p \cdot p)\}\) or \(\{|\nabla \cdot k| << (k \cdot k)\}\), which is a physically-occurring situation.
There are some paradigm assumptions that need to be cleared up:

We will *NOT* be implementing the common {→ lazy and extremely misguided} convention of setting physical constants to the value of (dimensionless) unity, often called "Natural Units", to hide them from equations; nor using mass \((m)\) instead of \((m_o)\) as the RestMass.

Likewise for other components vs Lorentz Scalars with naughts \((o)\), like energy \((E)\) vs \((E_o)\) as the RestEnergy.

One sees this very often in the literature. The usual excuse cited is "For the sake of brevity". Well, the "sake of brevity" forsakes "clarity". There is nothing physically "natural" about "natural units".

The *ONLY* situations in which setting constants to unity (1) is practical or advisable is in numerical simulation or mathematical analysis.

When teaching physics, or trying to understand physics: it helps when equations are dimensionally correct.

In other words, the physics technique of dimensional analysis is a powerful tool that should not be disdained.

\[ \text{i.e. Brevity only aids speed of computation, Clarity aids understanding.} \]

The situation of using “naught = \(_o\)” for rest-values, such as \((m_o)\) for RestMass and \((E_o)\) for RestEnergy:

Is intrinsic to SR, is a very good idea, absolutely adds clarity, identifies Lorentz Scalar Invariants, and will be explained in more detail later.

Essentially, the relativistic gamma \((\gamma)\) pairs with an invariant (Lorentz scalar: rest value \(_o\)) to make a relativistic component: \{ \( m = \gamma m_o \); \( E = \gamma E_o \) \}

Note the multiple equivalent ways that one can write 4-Vectors of SpaceTime (TimeSpace):

\[ \begin{align*}
4\text{-Momentum } P & = P^\mu = (p^\mu) = (p^0, p^i) = (mc/Ec, p) = -\partial [ \text{S}_\text{action,free} ] \\
& = m,U = m,\gamma(c,u) = \gamma m(c,u) = m(c,u) = (mc, mu) = (mc,p) = mc(1,\beta) = (m,c)T \\
& = (E/c^2)U = (E/c^2)\gamma(c,u) = \gamma(E/c^2)(c,u) = (E/c^2)(c,u) = (E/c,Eu/c^2) = (E/c,p) = (E/c)(1,\beta) = (E/c)T
\end{align*} \]

This notation makes clear what is \{ relativistically-varying=(frame-dependent) vs. Invariant=(frame-independent) \} and \{ Temporal vs. Spatial \}

BTW, I prefer the “Particle Physics” Metric-Signature-Convention (+,-,-,-). \{Makes rest values positive, fewer minus signs to deal with\}

Show the physical constants and rest naughts \((o)\) in the work. They deserve the respect and you will benefit.

You can always set constants to unity later, when you are doing your numerical simulations.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
There are some paradigm assumptions that need to be cleared up:

Some physics books say that the Electric field \( \mathbf{E} \) and the Magnetic field \( \mathbf{B} \) are the "real" physical objects, and that the EM scalar-potential \( \varphi \) and the EM 3-vector-potential \( \mathbf{A} \) are just "calculational/mathematical" artifacts.

Neither of these statements is relativistically correct.

All of these physical EM properties: \{\mathbf{E}, \mathbf{B}, \varphi, \mathbf{A}\} are actually just the components of SR tensors, and as such, their values will relativistically vary in different observers' reference-frames.

Given this SR knowledge, to match 4-Vector notation, we demote the physical property symbols, (the tensor components) to their lower-case equivalents \{e, b, \varphi, a\}.

see Wolfgang Rindler

The truly SR invariant physical objects are:

The 4-Gradient \( \nabla \), the 4-VectorPotential \( \mathbf{A} \), their combination via the exterior (wedge=\( ^\wedge \)) product into the Faraday EM 4-Tensor \( F^\alpha{}_{\beta} = \nabla \mathbf{A}^\beta - \nabla \mathbf{A}^\alpha = (\nabla \cdot \mathbf{A}) \), and their combination via the inner (dot=\( \cdot \)) product into the Lorenz Gauge 4-Scalar \( (\nabla \cdot \mathbf{A}) = 0 \)

Temporal-spatial components of 4-Tensor \( F^\alpha{}_{\beta} \): electric 3-vector field \( \mathbf{e} = e^i \)
Spatial-spatial components of 4-Tensor \( F^\alpha{}_{\beta} \): magnetic 3-vector field \( \mathbf{b} = b^i \)
Temporal component of 4-Vector \( \mathbf{A} \): EM scalar-potential \( \varphi \)
Spatial components of 4-Vector \( \mathbf{A} \): EM 3-vector-potential \( \mathbf{a} \)

Note that the Speed-of-Light \( c \) plays a prominent role in the component definitions.

Also, QM requires the 4-VectorPotential \( \mathbf{A} \) as explanation of the Aharonov-Bohm Effect. The physical measurability of the AB Effect proves the reality of the 4-VectorPotential \( \mathbf{A} \). Again, all the lower and higher-rank SR tensors can be built from fundamental 4-Vectors.

SR 4-Tensor
\[ (2,0) \text{-Tensor } T^{\mu \nu}, \text{ or } T_{\mu \nu}, \text{ or } T^{\mu}_{\nu}, \text{ or } T_{\mu \nu} \]
SR 4-Vector
\[ (1,0) \text{-Tensor } V^\nu = V = (v^0, v), \text{ or } v^\nu \]
SR 4-CoVector:OneForm
\[ (0,1) \text{-Tensor } V_\nu = (v_0, -v^i) \]
SR 4-Scalar
\[ (0,0) \text{-Tensor } S \text{ or } S_\mu \text{ or } \nabla \text{ or } \text{Lorentz Scalar} \]
There are some paradigm assumptions that need to be cleared up:

A number of QM philosophies make the assertion that particle “properties” do not “exist” until measured. The assertion is based on the QM Heisenberg Uncertainty Principle, and more specifically on quantum non-zero commutation, in which a measurement on one property of a particle alters a different non-commuting property of the same particle.

That is an incorrect analysis. Properties define particles: what they do & how they interact with other particles. Particles and their properties “exist” as events independently of human intervention or observation. The correct way to analyze this is to understand what a measurement is: the arrangement of some number of fundamental particles in a particular manner as to allow an observer to get information about one or more of the subject particle’s properties. Typically this involves “counting” spacetime events and using SR invariant intervals as a basis-of-measurement.

Some properties are indeed non-commuting. This simply means that it is not possible to arrange a set of particles in such a way as to measure (ie. obtain “complete” information about) both of the “subject particle’s” non-commuting properties at the same spacetime event. The measurement arrangement events can be done at best sequentially, and the temporal order of these events makes a difference in observed results. EPR-Bell, however, allows one to “infer” (due to conservation:continuity laws) properties on a “distant” subject particle by making a measurement on a different "local" (space-like-separated but entangled) particle. This does *not* imply FTL signaling nor non-locality.

The measurement just updates local partial-information one already has about particles that interacted/entangled then separated.

So, a better way to think about it is this: The “measurement-updated information” of a property does not “exist” until a physical setup event is arranged. Non-commuting properties require different physical arrangements in order for the properties to be measured, and the temporally-first measurement alters that particle’s properties in a minimum sort of way, which affects the latter measurement. All observers agree on Causality, the time-order of temporally-separated spacetime events. However, individual observers may have different sets of partial information about the same particle(s).

This makes way more sense than the subjective belief that a particle’s property doesn’t exist until it is observed, which is about as unscientific and laughable a statement as I can imagine.

**Relativity is the System-of-Measurement that QM has been looking for**
There are some paradigm assumptions that need to be cleared up:

**Correct Notation is critical for understanding physics**

Unfortunately, there are a number of “sloppy” notations seen in relativistic and quantum physics.

Incorrect: Using $T_{ii}$ as a Trace of tensor $T_{ij}$, or $T_{\mu\mu}$ as a Trace of tensor $T_{\mu\nu}$

$T_{ii}$ is actually just the diagonal part of 3-tensor $T_{ij}$, the components: 

$$T_{ii} = \text{Diag}[T_{11}, T_{22}, T_{33}]$$

The Trace operation requires a paired upper-lower index combination, which then gets summed over.

Incorrect: Using $T_{\mu\mu}$ as a Trace of tensor $T_{\mu\nu}$

$T_{\mu\mu}$ is actually just the diagonal part of 4-Tensor $T_{\mu\nu}$, the components:

$$T_{\mu\mu} = \text{Diag}[T_{00}, T_{11}, T_{22}, T_{33}]$$

The Trace operation requires a paired upper-lower index combination, which then gets summed over.

Incorrect: Using $m$ instead of $m_o$ for rest mass; Using $E$ instead of $E_o$ for rest energy

$E$ & $m$ are relativistic internal components of 4-Momentum $P=(mc^2, \gamma E-o)$ which vary in different reference-frames. $E_o$ & $m_o$ are Lorentz Scalar Invariants, the rest values, which are the same, even in different reference-frames: $P=m_oU=(E_o/c^2)U$
There are some paradigm assumptions that need to be cleared up:

Incorrect: Using the same symbol for a tensor-index and a component
The biggest offender in many books for this one is quantum commutation. Unclear because \(( i )\) means two different things in the same equation. Correct way: \(( i = \sqrt{-1} )\) is the imaginary unit; \(\{ j, k \}\) are tensor-indicies
In general, any equation which uses complex-number math should reserve \(( i )\) for the imaginary, not as a tensor-index.

Incorrect: Using the 4-Gradient:Gradient One-Form notation incorrectly
The 4-Gradient is a 4-Vector, a \((1,0)\)-Tensor, which uses an upper index, and has a negative spatial component \((-\nabla)\) in SR. The Gradient One-Form, its natural tensor form, a \((0,1)\)-Tensor, uses a lower index in SR.
4-Gradient: \(\partial=\partial^{\mu}=\left(\frac{\partial}{c},-\nabla\right)=\left(\frac{\partial}{c},\nabla\right)\)
Gradient One-Form: \(\partial_{\mu}=\partial_{\mu}=\left(\frac{\partial}{c},\nabla\right)=\left(\frac{\partial}{c},\nabla\right)\)

Incorrect: Mixing styles in 4-Vector naming conventions
There is pretty much universal agreement on the 4-Momentum \(P^{\mu}=(p^{\mu})=(p^{0},p)=(E/c,p)=(mc,p)=(E/c,p)=mc\)
Do not in the same document use 4-Potential \(A=\left(\varphi,A\right):\) This is wrong on many levels, inc. dimensional units. The correct form is 4-VectorPotential \(A=A^{\mu}=\left(a^{\mu}\right)=\left(a^{0},a\right)=\left(\varphi/c,a\right)=\left(\varphi/c,a\right),\) with \(\varphi=\)the scalar-potential & \(a=the\ 3-vector-potential\)

For all SR 4-Vectors, one should use a consistent notation:
The UPPER-CASE SpaceTime (TimeSpace) 4-Vector Names match the lower-case spatial 3-vector names
There is a LightSpeed \((c)\) factor in the temporal component to give overall matching dimensional units for the entire 4-Vector
4-Vector components are typically lower-case with a few exceptions, mainly energy \((E)\) vs. energy-density \((e),\rho_e,\rho\)
Old Paradigm: QM (as I was taught...)
SR and QM as separate theories

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
- $c, G =$ physical constants

GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle (FD-statistics)
- Bose Aggregation Principle (BE-statistics)
- Hermitian Generators
- Correspondence Principle to CM
- Born Probability Interpretation
- $h, \hbar =$ physical constants

SR limiting-case: $|v| \ll c$

QM limiting-case: # particles $N \gg 1$

This was the QM paradigm that I was taught while in Grad School: everyone trying for Quantum Gravity
Old Paradigm: QM (years later...)

SR and QM still as separate theories

QM limiting-case better defined, still no QG

It is known that QM + SR “join nicely” together to form RQM, but problems with RQM + GR...

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric \( g^{\mu\nu} \)
- \( c, G \) = physical constants

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle (FD-statistics)
- Bose Aggregation Principle (BE-statistics)
- Hermitian Generators
- Correspondence Principle to CM
- Born Probability Interpretation
- \( h, \hbar = \) physical constants

SR limiting-case: \( |v| \ll c \)

GR limiting-case: \( g^{\mu\nu} \rightarrow \eta^{\mu\nu} \)
Minkowski “Flat” SpaceTime Metric = (Curvature \( \sim 0 \))

QM-limiting case: \( \hbar |\nabla \cdot p| \ll (p \cdot p) \) or \( \psi \rightarrow \text{Re}[\psi] \)

Yet another “would be” fortuitous merging??

50+ years searching for QG with no success...

A fortuitous merging?

A tensor study of physical 4-vectors
SRQM Study:
Physical Theories as Venn Diagram
Which regions are empirically real?

Many QM physicists believe that the regions outside of QM don’t exist…

SRQM Interpretation would say that the regions outside of GR probably don’t exist...

A new approach is needed: SR→QM (SRQM) fits the facts…

SR: Special Relativity

GR: General Relativity

QM: Quantum Mechanics

Reality

CM: Classical Mechanics

SR limiting-case: |v| << c

QM limiting-case: ħ|∇⋅p| << (p⋅p)

RQM: Relativistic QM

Old Wrong Idea

Hence the attempt to Quantize Gravity: Unsuccessful for 50+ years…

QM physicists think these areas, anything outside of QM, doesn’t exist…

GR limiting-case: gμν → ημν Minkowski "Flat" SpaceTime = (Curvature ~ 0)

Many-Worlds Interpretations
Non-local interactions
Instantaneous QM entangled connections
Instantaneous Physical Wavefunction Collapse
Spacetime Dimensions >4
Hidden: Alternate Dimensions
Super-Symmetry
String Theory
Alternate Gravity Theories
Slews of hypothetical new particles etc.

Quantum Mysticism

Basically lots of stuff for which there is no empirical evidence…
& loads of hype...

Many QM physicists believe that the regions outside of QM don’t exist…
SRQM Study:
Physical Limit-Cases as Venn Diagram
Which limit-regions use which physics?

- **Reality**: Quantum Gravity? Actual GR?
- **Classical SR**: Classical (non-QM) Special Relativity
- **CM**: Classical Mechanics (non-QM) (non-SR)
- **QM**: Non-relativistic (standard) Quantum Mechanics
- **RQM**: Relativistic QM
- **SR → QM (SRQM)**: Special Relativity → Relativistic QM

Instead of taking the Physical Theories as set, examine Physical Reality and then apply various limiting-conditions.

**What do we then call the various regions?**

As we move inwards from any region on the diagram, we are adding more stringent conditions which give physical limiting-cases of “larger, more encompassing” theories.

- If one is in Classical GR, one can get Classical SR by moving toward the Minkowski SpaceTime limit.
- If one is in RQM, one can get Classical SR by moving toward the Hamilton-Jacobi non-QM limit, or to standard QM by moving toward the SR low-velocity limit.

Looking at it this way, I can define SRQM to be equivalent to Minkowski SpaceTime, which contains ROM, and leads to Classical SR, or QM, or CM by taking additional limits.

**My assertion:**
There is no “Quantized Gravity” Actual GR contains SRQM and Classical GR. Perhaps “Gravitizing QM”...

SRQM: A treatise of SR → QM by John B. Wilson (SciRealm@aol.com)
Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties. Likewise, Relativistic Quantum Mechanics (RQM) and standard Quantum Mechanics (QM) have passed all tests within their realms of validity:

- Generally micro-scale systems: ex. Single particles, ions, atoms, molecules, electric circuits, atomic-force microscopes, etc., but a few special macro-scale systems: ex. Bose-Einstein condensates, super-currents, super-fluids, long-distance entanglement, etc.

To-date, however, there is no observational/experimental indication that quantum effects "alter" the fundamentals of either SR or GR. Likewise, there are no known violations, QM or otherwise, of Local Lorentz Invariance (LLI) nor of Local Position/Poincaré Invariance (LPI).

In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR, whereas SR/GR modify the results of QM. All tested quantum-level particles, atoms, isotopes, super-positions, spin-states, etc. obey GR's Universality of Free-Fall & Equivalence Principle and SR's $E = mc^2$ and speed-of-light (c) communication/signaling limit. Meanwhile, quantum-level atomic clocks are used to measure gravitational red/blue-shift effects. i.e. GR gravitational frequency-shift (gravitational time-dilation) alters atomic=quantum-level timing. Think about that for a moment...

Some might argue that QM modifies the results of SR, such as via non-commuting measurements. However, that is an alteration of CM expectations, not SR expectations. In fact, there is a basic non-zero commutation relation fully within SR: $([\mathcal{P},X]) = \eta^{\mu\nu}$ which will be derived from purely SR Principles in this treatise. The actual commutation part (Commutator [a,b]) is not about ($\hbar$) or ($i$), which are just invariant Lorentz Scalar multipliers.

On the other hand, GR Gravity *does* induce changes in quantum interference patterns and hence modifies QM:

- See the COW gravity-induced neutron QM interference experiments, the LIGO & VIRGO & KAGRA gravitational-wave detections via QM interferometry, and now also QM atomic matter-wave gravimeters via QM interferometry. Likewise, SR induces fine-structure splitting of spectral lines of atoms, "quantum" spin, spin magnetic moments, spin-statistics (fermions & bosons), antimatter, QED, Lamb shift, relativistic heavy-atom effects (liquid mercury, yellowish color of gold, lead batteries having higher voltage than classically predicted, heavy noble-gas interactions, relativistic chemistry...), etc. - essentially requiring QM to be RQM to be valid. QM is instead seen to be the limiting-case of RQM for ($|v| << c$).

Some QM scientists say that quantum entanglement is "non-local", but you still can't send any real messages/signals/information/particles faster than SR's speed-of-light (c). The only "non-local" aspect is the alteration of probability-distributions based on knowledge-changes obtained via measurement.

A local measurement can only alter the "partial information" already-known about the probability-distribution of a distant (entangled) system. Getting a Stern-Gerlach "up" here doesn't cause the distant entangled particle to suddenly start moving "down" there. One only knows "now" that it "would" go down "if" the distant experimenter actually performs the measurement.

QM respects the principles of SR/GR, whereas SR/GR modify the results of QM.
Special Relativity → Quantum Mechanics

Background: GR Principles

Known Physics ↔ Empirically Tested

Principles/Axioms and Mathematical Consequences of General Relativity (GR):

Equivalence Principle: Inertial Motion = Geodesic Motion, Universality of Free-Fall, Mass Equivalency (Mass\textsubscript{inertial} = Mass\textsubscript{gravitational})

Relativity Principle: SpaceTime (\textit{M}) has a Lorentzian= pseudo-Riemannian Metric (g\textsubscript{\mu\nu}), SR:Minkowski Space rules apply locally (g\textsubscript{\mu\nu}→\eta\textsubscript{\mu\nu})

General Covariance Principle: Tensors describe Physics, General Laws of Physics are independent of arbitrary, chosen Coordinate-System

Invariance Principle: Invariant Interval Measure comes from Tensor Invariance Properties, 4D SpaceTime from Invariant Trace[g\textsubscript{\mu\nu}]=4

Causality Principle: Minkowski Diagram/Light-Cone give \{ Time-Like (+), Light-Like (Null=0), Space-Like (−) \} Measures and Causality Conditions

Einstein:Riemann’s Ideas about Matter & Curvature:
Riemann(g) has 20 independent components → too many
Ricci(g) has 10 independent components = enough to describe/specify a gravitational field

\{c,G\} are Fundamental Physical Constants

To-date, there are no known violations of any of these GR Principles.
GR has passed EVERY observational test to-date, in both weak and strong field regimes.

It is vitally important to keep the mathematics grounded in known physics.
There are too many instances of trying to apply theoretical-only mathematics to physics (ex. String Theory, SuperSymmetry: no physical evidence to-date; SuperGravity: physically disproven).
Progress in science doesn’t work that way: Nature itself is the arbiter of what math works with physics. Tensor mathematics applies well to known physics {SR and GR}, which have been empirically extremely well-tested in a huge variety of physical situations. Tensors describe physics.

All known experiments to date comply with all of these Principles, including QM and RQM
Old Paradigm: QM Axioms (for comparison)

SR and QM still as separate theories

QM limiting-case better defined, still no QG

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Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
- $c,G =$ physical constants

Obscure QM Axioms:
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle (FD-statistics)
- Bose Aggregation Principle (BE-statistics)
- Hermitian Generators
- Correspondence Principle to CM
- Born Probability Interpretation
- $\hbar,\hbar =$ physical constants

GR limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
Minkowski “Flat” SpaceTime Metric = (Curvature ~ 0)

SR limiting-case:
$|v| << c$

QM limiting-case:
$\hbar |\nabla \cdot p| << (p \cdot p)$

or
$\psi \rightarrow \text{Re}[\psi]$ or $|\nabla \cdot k| << (k \cdot k)$

---

Yet another “would be” fortuitous merging???
50+ years searching for QG with no success...

Another fortuitous merging??

SR limiting-case:
$|v| << c$

A fortuitous merging?

---

It is known that QM + SR “join nicely” together to form RQM, but problems with RQM + GR...
*New Paradigm: SRQM or [SR → QM]*

QM derived from SR + a few empirical facts
Simple and fits the data

**Simple GR Axioms:**
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
c,G = physical constants

**GR limiting-case:** $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$
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**Derived RQM **Principles**:**
- Wave-Particle Duality
- Unitary Evolution
- Operator Formalism
- Hilbert Space Representation
- Principle of Superposition
- Canonical Commutation Relation
- Heisenberg Uncertainty Principle
- Pauli Exclusion Principle (FD-statistics)
- Bose Aggregation Principle (BE-statistics)
- Hermitian Generators
$h,\hbar$ = physical constants

**SR 4-vector:**
$R = (ct, r)$
$U = \gamma (c, u)$
$P = (E/c, p)$
$K = (\omega/c, k)$
$\delta = (\hbar/c, -\nabla)$

**SRQM**

**Derived QM **Principles**:**
- Correspondence Principle to CM
- Born Probability Interpretation

**CM**

**QM limiting-case:**
$|\psi| \ll c$

**QFT**

**Multiple Particles**

**SR limiting-case:**
$|v| \ll c$

**SRQM**

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR
**New Paradigm: SRQM w/ EM**

QM, EM, CM derived from SR + a few empirical facts

---

**Simple GR Axioms:**
- Principle of Equivalence:
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**Derived RQM **Principles**:
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---

**Derived QM **Principles**:
- Correspondence Principle to CM
- Born Probability Interpretation

---

This new paradigm explains why RQM “miraculously fits” SR, but not necessarily GR.
Classical SR w/ EM Paradigm (for comparison)

CM & EM derived from SR + a few empirical facts

Simple GR Axioms:
- Principle of Equivalence
- Invariant Interval Measure
- Tensors describe Physics
- SpaceTime Metric $g^{\mu\nu}$
- $c,G =$ physical constants

The entire classical SR→EM,CM structure is based on the limiting-case of quantum effects being negligible.

Notice that only the SR 4-Vector relation: $K=(1/\hbar)P$ is missing from the Classical Interpretation…

All of the SR 4-Vectors, including ($K$ & $\partial$), are still present in the Classical setting.

$K$ is used in the Relativistic Doppler Effect and EM waves. $\partial$ is used in the SR Conservation/Continuity Equations, Maxwell Equations, Hamilton-Jacobi, Lorenz Gauge, etc.

$\partial=(-i)K$ may be somewhat controversial, but it is the equation for complex plane-waves, which are still used in classical EM.

This (Classical=non-QM) SR→{EM,CM} approx. paradigm has been working successfully for decades…
The SRQM view: Each level (range of validity) is a subset of the larger level.

**GR**
General Relativity

**SRQM**
Special Relativity $\rightarrow$ Relativistic QM
- **GR** limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$ Minkowski “Flat” SpaceTime = (Curvature $\sim 0$)

**QM**
Non-relativistic Quantum Mechanics
- **SRQM** limiting-case: $|v| \ll c$

**CM**
Classical Mechanics
- **QM** limiting-case: $\hbar|\nabla \cdot p| \ll (p \cdot p)$
- or $\psi \rightarrow \text{Re}[\psi]$ or $|\nabla \cdot k| \ll (k \cdot k)$
- Change by a few quanta has negligible effect on overall state

SRQM: A treatise of SR$\rightarrow$QM by John B. Wilson (SciRealm@aol.com)
The SRQM view: Each level (range of validity) is a subset of the larger level.

- **GR** General Relativity
- **SRQM** Special Relativity → Relativistic QM
  - *GR* limiting-case: $g^{\mu\nu} \rightarrow \eta^{\mu\nu}$ Minkowski “Flat” SpaceTime = (Curvature ~ 0)
- **QM** Non-relativistic Quantum Mechanics
  - *SRQM* limiting-case: $|v| \ll c$
- **CM** Classical Mechanics
  - *QM* limiting-case: $\hbar|\nabla \cdot p| \ll (p \cdot p)$ or $\psi \rightarrow \text{Re}[\psi]$ or $|\nabla \cdot k| \ll (k \cdot k)$
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---

**SRQM = New Paradigm:**

**SRQM w/ EM View as Venn Diagram**

Ranges of Validity

**SR → QM** Physics

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SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SR is able to expand the concept of mathematical vectors into the Physical 4-Vector, which combines both (time) and (space) components into a single (TimeSpace) object. These 4-Vectors are elements of Minkowski 4D SR SpaceTime. They have Lorentzian (relative) components but invariant 4D Magnitudes. There is a Speed-of-Light factor (c) in the temporal component to make the dimensional units match. 

SR 4-Vector name will match up with the 3-vector name.

In this presentation:
I use the +Time (c) metric signature, giving $\mathbf{A} \cdot \mathbf{A} = A^\mu \eta_{\mu \nu} A^\nu = [(a_0)^2 - \mathbf{a} \cdot \mathbf{a}] = (a_0^2)$

4-Vectors will use Upper-Case Letters, ex: $\mathbf{A}$; 3-vectors will use lower-case letters, ex: $\mathbf{a}$. I always put the (c) dimensional factor in the temporal component.

Vectors of both types will be in bold font; components and scalars in normal font and usually lower-case. 4-Vector name will match with 3-vector name.

Tensor form will usually be normal font with tensor indices: { Greek TimeSpace index (0,1,3); ex: $\mathbf{A} = A^\mu$ } or { Latin SpaceOnly index (1,3); ex: $\mathbf{a} = a^\mathbf{a}$ }

The basis-values of these components can differ in certain relativistic ways, via Galilean transforms, yet still refer to the same overall 3-vector object.

Newton's laws of classical physics are greatly simplified by the use of physical 3-vector notation, which converts 3 separate space components, which may be different (relative) in various coordinate systems, into a single invariant object: a vector, with an invariant 3D magnitude. 

The triangle/wedge $\Delta$ (3 sides) represents splitting the components into a scalar and 3-vector.

I style classical 3D objects this way (by a triangle/wedge $\Delta$) to emphasize that they are actually just the separated components of SR 4-Vectors.
4-Vectors are 4D (1,0)-Tensors, Lorentz 4-Scalars are 4D (0,0)-Tensors, 4-CoVectors are 4D (0,1)-Tensors, (m,n)-Tensors have (m) # upper-indices and (n) # lower-indices.

Any equation which employs only Tensors, such as those with only 4-Vectors and Lorentz 4-Scalars, (ex. \( P = P^\mu = m, \mathbf{U} = m, \mathbf{U}^\mu \)) is automatically Frame-Invariant, or coordinate-frame-independent. One’s frame-of-reference plays no role in the form of the overall equations. This is also known as being “Manifestly-Invariant” when no inner components are used. This is exactly what Einstein meant by his postulate: “The laws of physics should have the same form for all inertial observers”. Use of the RestFrame-naught \((\gamma)\) helps show this.

It is seen when the spatial part \((a)\) of a magnitude can be set to zero (= at-rest). Then the temporal part \((a^0)\) would equal the rest value \((a^0)\).

The components \((a^0, a^1, a^2, a^3)\) of the 4-Vector \(\mathbf{A}\) can relativistically vary depending on the observer and their choice of coordinate system, but the 4-Vector \(\mathbf{A} = A^\mu\) itself is invariant. Equations using only 4-Tensors, 4-Vectors, and Lorentz 4-Scalars are true for all inertial observers. The SRQM Diagramming Method makes this easy to see in a visual format, and will be used throughout this treatise.

The following examples are SR TimeSpace frame-invariant equations:

\[
\mathbf{U} \cdot \mathbf{U} = (\mathbf{c})^2 \\
\mathbf{U} = \gamma \mathbf{(c, u)} \\
\mathbf{P} = (mc, p) = \mathbf{m} \mathbf{U} = \left(\frac{E}{c^2}\right) \mathbf{U} \\
\mathbf{K} = (\omega/c, \mathbf{k}) = \left(\frac{\omega}{c^2}\right) \mathbf{U} \\
\mathbf{P} \cdot \mathbf{U} = E_o \\
\mathbf{A} = A^\mu = (a^\mu) = (a^0, a) = (a^0, a^1, a^2, a^3) \rightarrow (a^t, a^x, a^y, a^z) \{\text{rectangular basis}\} \\
\rightarrow (a^0, 0) \{\text{rest-frame basis, becomes purely temporal}\}
\]

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The following examples are SR TimeSpace frame-invariant equations:
SR 4-Vectors are primitive elements of Minkowski SpaceTime 4D←(1+3)D

We want to be clear, however, that SR 4-Vectors are NOT generalizations of Classical or Quantum 3-vectors.

SR 4-Vectors are the primitive elements of Minkowski SpaceTime (TimeSpace) = 4D←(1+3)D, which incorporate both: a \{temporal scalar element\} and a \{spatial 3-vector element\} as components. Temporals and Spatially are metrically distinct, but can mix in SR.

4-Vector \( \mathbf{A} = A^\nu = (a^0, a^1, a^2, a^3) \) \rightarrow \( (a^0, a^1, a^2, a^3) \) with component scalar \( a^0 \) \& component 3-vector \( a^i = a \) \rightarrow \( (a^0, a^i, a^j, a^k) \)

It is the \{Classical (Newtonian) or Quantum\} 3-vector \((a)\) which is a limiting-case approximation of the spatial part of SR 4-Vector \((\mathbf{A})\) for \(|v| \ll c\).

i.e. The energy \((E)\) and 3-momentum \((p)\) as “separate” entities occurs only in the low-velocity limit \(|v| \ll c\) of the Lorentz Boost Transform. They are actually part of a single 4D entity: the 4-Momentum \( \mathbf{P} = (E/c, p) \) with the components: \( \text{temporal energy (E), spatial 3-momentum (p)} \), dependent on a frame-of-reference, while the overall 4-Vector \( \mathbf{P} \) is invariant. Likewise with \text{time (t), space 3-position (r)} in the 4-Position \( \mathbf{R} = (ct, \mathbf{r}) \).

SR is 4D Minkowskian: obeys Lorentz/Poincaré Invariance.

CM is 3D Euclidean: obeys Galilean Invariance.
SRQM:

**SR 4-Vectors & Lorentz Scalars**

Rest Values ("naughts"=₀) are Lorentz Scalars

\[ \mathbf{A} = (a'_{0}a'_{a}) - (a_{0}a) \text{, where (a'_{0}) is the rest-value, the value of the temporal coordinate when the spatial coordinate is zero (a=0).} \]

The "rest-values" of several physical properties are all Lorentz scalars.

\[ \mathbf{P} = (mc,p) \]

\[ \mathbf{P} \cdot \mathbf{P} = (mc)^{2} - (p^{i}p_{i}) \]

(P-P) and (K-K) are Lorentz Scalars. We can choose a frame that may simplify the expressions.

Choose a frame in which the spatial component is zero. This is known as the "rest-frame" of the 4-Vector. It is not moving spatially.

\[ \mathbf{P} \cdot \mathbf{P} = (mc)^{2} - (p^{i}p_{i}) \]

The resulting simpler expressions then give the "rest values", indicated by (₀).

RestMass (m) and RestAngularFrequency (ω₀)

They are Invariant Lorentz Scalars by construction.

This leads to simple relations between 4-Vectors.

\[ \mathbf{P} = (m_{0}) \mathbf{U} = (E_{0}/c^{2}) \mathbf{U} \]

And gives nice Scalar Product relations between 4-Vectors as well.

\[ \mathbf{P} \cdot \mathbf{U} = (m_{0}) \mathbf{U} \cdot \mathbf{U} = (m_{0}c^{2}) \mathbf{U} = (E_{0}) \]

\[ \mathbf{P} \cdot \mathbf{K} = (m_{0}c_{0}) \rightarrow \mathbf{P} = (m_{0}) \mathbf{K} \rightarrow \mathbf{P} = \text{(const)} \mathbf{K} \]

This property of SR equations is a very good reason to use the "naught" convention for specifying the difference between relativistic component values which can vary, like (m), versus Rest Value Invariant Scalars, like (m₀), which do not vary. They are usually related via a Lorentz Factor: \( \gamma = \gamma_{m_{0}} = \gamma_{E_{0}} = \omega = \gamma_{\omega_{0}} \), as seen in the relations of \( \mathbf{P} \), \( \mathbf{K} \), \( \mathbf{U} \), and \( \mathbf{T} \).

\[ \mathbf{P} = (mc,p) = (m_{0}) \mathbf{U} = (m_{0}) \gamma (c,u) = (m_{0}c_{0}) \gamma (m_{0},m_{0}u) = (mc,m_{0}) = (m_{0}c_{0}) \gamma (1,\beta) = (mc)(1,\beta) \]

4-Vector \(\mathbf{A}_{\mu} = (a_{0},a_{a}) = (a_{0}^{a},a_{a}^{2},a_{3}^{2}) \rightarrow (a_{0},0) \) (in spatial rest frame)

\[ \mathbf{A} \cdot \mathbf{B} = (a_{0}b_{0}) \rightarrow (b_{0},0) \) (in spatial rest frame)

Notation:

\(\mathbf{A} \cdot \mathbf{A} = (a_{0}^{a})^{2}\)

\(\mathbf{B} \cdot \mathbf{B} = (b_{0}^{a})^{2}\)

SRQM: 4-Vector SRQM Interpretation of QM

SciRealm.org
John B. Wilson
SciRealm@ao.com
http://scirealm.org/SRQM.pdf
SRQM Study: Manifest Invariance

Invariant SR 4-Vector Relations

Relations among 4-Vectors and Lorentz 4-Scalars are Manifestly Invariant, meaning that they are true in all inertial reference frames.

Consider a particle at a **SpaceTime (TimeSpace) <Event>** that has properties described by 4-Vectors \( \mathbf{A} \) and \( \mathbf{B} \):

One possible relationship is that the two 4-Vectors are related by a Lorentz 4-Scalar (S): \( \mathbf{B} = (S) \mathbf{A} \).

How can one determine this? Answer: Make an experiment that empirically measures the tensor invariant \( \mathbf{B} \cdot \mathbf{C} / \mathbf{A} \cdot \mathbf{C} \).

If \( \mathbf{B} = (S) \mathbf{A} \) then \( \mathbf{B} \cdot \mathbf{C} = (S) \mathbf{A} \cdot \mathbf{C} \), giving \( (S) = [\mathbf{B} \cdot \mathbf{C} / \mathbf{A} \cdot \mathbf{C}] \)

if \( \mathbf{C} = \text{other} \), Invariant result mediated by another 4-Vector \( \mathbf{C} \), always possible.

Run the experiment many times. If you always get the same result for \( (S) \), then it is likely that the relationship is true, and thus invariant.

Example: Measure \( (S_p) = [\mathbf{P} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U}] \) for a given particle type.
Repeated measurement always give \( (S_p) = m_0 \)
This makes sense because we know \( [\mathbf{P} \cdot \mathbf{U}] = \gamma(E - \mathbf{p} \cdot \mathbf{u}) = E_o \) and \( [\mathbf{U} \cdot \mathbf{U}] = c^2 \)
Thus, 4-Momentum \( \mathbf{P} = (E_o/c^2)\mathbf{U} = (m_0)\mathbf{U} = (m_0)^*4\text{-Velocity} \mathbf{U} \)

Example: Measure \( (S_k) = [\mathbf{K} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U}] \) for a given particle type.
Repeated measurement always give \( (S_k) = (\omega/c)^2 \)
This makes sense because we know \( [\mathbf{K} \cdot \mathbf{U}] = \gamma(\omega - \mathbf{k} \cdot \mathbf{u}) = \omega_o \) and \( [\mathbf{U} \cdot \mathbf{U}] = c^2 \)
Thus, 4-WaveVector \( \mathbf{K} = (\omega/c)^2\mathbf{U} = (\omega/c)^2*4\text{-Velocity} \mathbf{U} \)

Since \( \mathbf{P} \) and \( \mathbf{K} \) are both related to \( \mathbf{U} \), this would also mean that the 4-Momentum \( \mathbf{P} \) is related to the 4-WaveVector \( \mathbf{K} \) in a particular Lorentz Invariant manner for each given particle type… a major hint for later...

\[
\text{Trace}[T^{\nu}] = \eta_{\nu\mu}T^\nu = T^\nu _\nu = T
\]
\[
V \cdot V = V^\mu \eta_{\mu\nu}V^\nu = [(v^\nu)^2 - V \cdot V] = (v^\nu)^2
\]
Lorentz Scalar
Some SR Mathematical Tools
Definitions, Approximations, Misc.

- $\beta = \frac{v}{c}$; $\beta = |\beta|$: dimensionless Velocity Beta Factor
- $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-|\beta|^2}}$: dimensionless Lorentz Relativistic Gamma Factor

$\gamma = (1.\infty)$; rest at $\gamma = 1$; speed-of-light (c) at $\gamma = \infty$

$(1+x)^\gamma \sim (1 + nx + O[x^2])$ for $|x| < 1$ Approximation used for SR→Classical limiting-cases

Lorentz Transformation $\Lambda^\mu_\nu = \partial X^\mu / \partial X'^\nu = \delta_i^j[X'^\nu]$; a relativistic frame-shift, such as a rotation or velocity boost.

It transforms a 4-Vector in the following way: $X'^\nu = \Lambda^\nu_\mu X^\mu$ with Einstein summation over the paired indices, and the (') indicating an alternate frame.

A典型 Lorentz Boost Transformation $\Lambda^\nu_\nu \rightarrow B^\nu_\nu$ for a linear-velocity frame-shift $(x,t)$-Boost in the $\hat{x}$-direction:

Original $A' = (a^t, a^x, a^y, a^z)$
Boosted $A'' = (a'^t, a'^x, a'^y, a'^z) = \Lambda^\nu_\mu A^\nu = (\gamma a^t - \gamma^i a^x, a^y, a^z)$

$A'\cdot B' = (\Lambda^\nu_\mu A^\nu)(\Lambda^\nu_\mu B^\nu) = A^\nu \eta^\nu_{\mu'} B^\nu = A^\nu B^\nu$

$= \sum_{\nu=0,3,\mu=0,3} \eta^\nu_{\mu'} [a^\nu b^\mu] = \sum_{\nu=0,3,\mu=0,3} [a^\nu b^\mu] = (a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3) = (a^0b^0, a^1b^1, a^2b^2, a^3b^3)$

using the Einstein Summation Convention where upper:lower paired-indices are summed over.

$\partial X = \partial^\mu [X^\nu] = (\hat{\partial}^\mu, -\nabla)[ct, x] = \text{Diag}[\hat{\partial}^\mu/ct, -\nabla|x|] = \text{Diag}[1,-I_{03}] = \text{Diag}[1,-1,-1,-1] = \eta^\nu_{\mu'}$ Minkowski “Flat” SpaceTime Metric

SR: Minkowski Metric

$\delta[R] = \partial^\nu R^\nu = \eta^\nu_{\mu'} V^\nu V^\mu \rightarrow \text{Diag}[1,-1,-1,-1] = \text{Diag}[1,-I_{03}] = \text{Diag}[1,-\delta^\nu_{\mu'}]$

{in Cartesian form: ‘Particle Physics’ Convention

$\eta_{\mu\nu} = 1/\{\eta^{\mu\nu}\} : \eta^\nu_{\mu'} = \delta^\nu_{\mu'} \rightarrow \text{Tr}[\eta^\nu_{\mu'}] = 4$}

SR: Lorentz Transform

$\partial[R'] = \partial^\nu R'^\nu = \lambda^\nu_{\nu'}$

$\lambda^\nu_{\nu'} = (\Lambda^{-1})^\nu_{\mu'} : \lambda^\nu_{\mu'} \Lambda^\mu_{\nu'} = \eta^\nu_{\mu'} = \delta^\nu_{\mu'}$

$\text{Det}[\Lambda^\nu_{\nu'}] = \pm 1$

$\Lambda^\nu_{\mu'} \Lambda^\mu_{\nu'} = \eta_{\mu\nu}$

$\text{Tr}[\Lambda^\nu_{\nu'}] = \{-\infty, +\infty\}$

$Lorentz\ Transform\ Type$

Note: $\Lambda^\nu_{\mu'}$ not a Lorentz Transform Type.

SpaceTime

$\partial[R] = \partial^\nu R'^\nu = 4$ Dimension

Trace[$\Lambda^\nu_{\nu'}$] = $\eta^\nu_{\mu'} T^\nu_{\mu'} = T^\nu_{\mu'} = T$

$V^\nu V^\mu = V^\nu V^\nu \rightarrow [(v^\nu)^2 - v^\nu v^\nu] = (v^2)^2$

$Lorentz\ Scalar$
SRQM Study: Ordering of TimeSpace <Events>  
Temporal Causality vs. Spatial Topology  
Simultaneity vs. Stationarity  
Venn Diagram

Properties of Minkowski:SR SpaceTime <Events>

Time-Like Ordering of...

- Time-Like Separated <Events>
  - Causal: Invariant = Absolute Temporal Order (A→B→C)
  - Time-Like Invariant Interval \( \Delta R \cdot \Delta R = (c \Delta t)^2 - \Delta r \cdot \Delta r \rightarrow + (c \Delta t)^2 \)
  - Non-Topological: Relative → Relativity of Stationarity (A→?→B)

- Light-Like (Null) Separated <Events>
  - Causal: Invariant = Absolute Temporal Order (A→B→C)
  - Light-Like Invariant Interval \( \Delta R \cdot \Delta R = (c \Delta t)^2 - \Delta r \cdot \Delta r \rightarrow 0 \)
  - Topological: Invariant = Absolute Spatial Order (A→B→C)

- Space-Like Separated <Events>
  - Non-Causal: Relative → Relativity of Simultaneity (A→?→B)
  - Simultaneity: (only if in reference-frame with Same-Time occurrence)
  - Space-Like Invariant Interval \( \Delta R \cdot \Delta R = (c \Delta t)^2 - \Delta r \cdot \Delta r \rightarrow (|\Delta r|)^2 \)
  - Topological: Invariant = Absolute Spatial Order (A→B→C) or (C→B→A)

Space-Like Ordering of...

- Time-Like Separated <Events>
  - Non-Topological: Relative → Relativity of Stationarity (A→?→B)
  - Stationarity: (only if in reference-frame with Same-Place occurrence)

- Light-Like (Null) Separated <Events>
  - Topological: Invariant = Absolute Spatial Order (A→B→C)

- Space-Like Separated <Events>
  - Stationarity: (only if in reference-frame with Same-Place occurrence)

Trace\([T^{uv}] = \eta_{uv} T^{uv} = T_u^u = T\)  
\[ V \cdot V = V^i \eta_{ij} V^j = (|V|)^2 - V^0 V_0 = (v^0)^2 \]  
= Lorentz Scalar
Focus on a few of the main SR Physical 4-Vectors:

- **4-Position**
  \[ R = (c_t, c_t, c_t, c_t) \]

- **4-Velocity**
  \[ U = (u^0, u^1, u^2, u^3) = \gamma(u^0, c) \]

- **4-Gradient**
  \[ \partial = \partial^\alpha = \partial^\beta = (c_t, c_t, c_t, c_t) \]

These 4-Vectors give some of the main classical results of Special Relativity, including 4D SR Minkowski Space concepts like:

- **Relativity**: Time Dilation (clock moving), Length Contraction (ruler moving), Proper Time (clock at rest), Proper Length (ruler at rest)

- **Invariants**: Proper Time, Proper Length

Temporal 1D Ordering of Events:
- Time-like event separations: Causality is **Absolute**
- Space-like event separations: Stationarity is **Relative**

Spatial 3D Ordering of Events:
- Time-like event separations: Simultaneity is **Relative**
- Space-like event separations: Topology is **Absolute**

Use of the Lorentz Scalar Product to make Lorentz Invariants, Continuity Equations, etc. The Invariant Speed-of-Light \( c \), Invariant Proper Measurements \( (Time \ & \ Space) \), Invariant SR Wave Equations, via the d'Alembertian (Lorentz Scalar Product of 4-Gradient with itself), leads to a 4-WaveVector \( K \) solution.
SRQM Diagram: The Basis of Classical SR Physics
Special Relativity via 4-Vectors

The Basis of most all Classical SR Physics is in the SR Minkowski Metric
of “Flat” SpaceTime \( \eta^{\mu\nu} = \delta^{\mu\nu} \), which is generated from the
4-Gradient \( \partial_{\nu} \) and 4-Position \( R = R^\mu \) and determines the
invariant measurement interval \( R \cdot R = R^\mu \eta_{\mu\nu} R^\nu \) between \( \langle \text{Events} \rangle \).

This Minkowski Metric \( \eta^{\mu\nu} \) provides the relations between the 4-Vectors
of SR: 4-Position \( R = R^\mu \), 4-Gradient \( \partial_{\nu} \), 4-Velocity \( U = U^\mu \)

The Tensor Invariants of these 4-Vectors give the:
Invariant Interval Measures \( R \cdot R = R^\mu \eta_{\mu\nu} R^\nu \), from \( R \cdot R \)
Invariant Magnitude LightSpeed \( c \), from \( U \cdot U \)
Invariant d’Alembertian Wave Equation & 4-WaveVector \( K \), from \( \partial \cdot \partial 

The relation between 4-Gradient \( \partial_{\nu} \) and 4-Position \( R \)
gives the Dimension of SpaceTime = (4),
the Minkowski Metric \( \eta^{\mu\nu} \), and the Lorentz Transformations \( \Lambda^{\nu}_{\mu} \).

The relation between 4-Gradient \( \partial_{\nu} \) and 4-Velocity \( U \)
gives the invariant ProperTime Derivative \( d/dt \).
Rearranging gives the invariant ProperTime Differential \( d \tau \),
which gives relativistic \( \langle \text{Time Dilation} \rangle \rightarrow (\text{temporal}) \) & \( \langle \text{Length Contraction} \rangle \rightarrow (\text{spatial}) \).

The ProperTime Derivative \( d/dt \):
acting on 4-Position \( R \) gives 4-Velocity \( U \)
acting on the SpaceTime Dimension Lorentz Scalar

gives the Continuity of 4-Velocity Flow.

The relation between 4-Displacement \( \Delta R \) and 4-Velocity \( U \)
gives Relativity of Simultaneity:Stationarity.

One of the most important properties is the Tensor Invariant
Lorentz Scalar Product (\( \Delta t = \Delta t \)), provided by the
lowered-index form of the Minkowski Metric \( \eta_{\mu\nu} \).

From here, each object will be examined in turn...
SR → QM
Physics

A Tensor Study of Physical 4-Vectors

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $T^{\mu\nu}$, or $T_{\mu\nu}$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V = (v^\mu, v)$

SR 4-Scalar
(0,0)-Tensor $\sigma$ or $\sigma_0$

Lorentz Scalar

SRQM Diagram:
The Basis of Classical SR Physics
4-Position, 4-Displacement, 4-Differential

SRQM Diagram

ProperTime Derivative
$\partial\gamma = \gamma(c, u) = \gamma(\delta + (dx/dt)\delta_y + (dy/dt)\delta_z)$

Continuity of 4-Velocity Flow
$\delta U = 0$

Invariant Magnitude
LightSpeed
$U \cdot U = c^2$

SRQM Diagram

SR is a theory about the relations between 4D TimeSpace <Events>, i.e. how their intervals are "measured"
The Invariant Interval is the Lorentz Scalar Product of the (4-Position, 4-Displacement, 4-Differential) with itself, giving a magnitude-squared, which may be (+/10/)

The 4D SpaceTime Intervals are Invariant, meaning that all observers must agree on their magnitudes, regardless of differing reference frames. This leads to the idea of ProperTime (\(\Delta t = \Delta t_o\)), which is the time-displacement measured by a clock at-rest, and ProperLength (\(L_o = |\Delta x_o|\)), which is the space-displacement measured by a ruler at-rest. This also leads to the various Causality Conditions of SR, and the concept of the (Minkowski Diagram) Light Cone. The differential form \(d\mathbf{R} \cdot d\mathbf{R}\) is apparently also still true in the curved spacetime of GR.
These are the 4 components that appear in:

\[ \text{this Tensor Invariant Lorentz Scalar relation gives the dimension of SpaceTime.} \]

\[ \delta^\mu = \delta^\nu = \delta_\mu = \delta_\nu = I_{44} \]  
\[ \text{Diag}[1,1,1,1] = \text{Diag}[1,-\gamma^2] \]
\[ \text{Minkowski Transform} \]
\[ \partial R^\mu = 4 \text{-Position Dimension} \]

\[ \delta[R] = \delta R^\nu = \eta^\mu_\nu \]
\[ \text{Lorentz Transform} \]
\[ \partial[R] = \partial R^\nu = \Lambda^\nu_\nu \]
\[ \partial[\partial[\text{of } \text{Dimension } c \text{ at } \tau \text{ and } \Delta r] = 0} = \text{Diag}\[1,1,1,1\]
\[ \text{Time Dilation} \]
\[ \partial \tau = (1/\gamma) \partial t = \text{Time Dilation} \]

\[ \text{SRQM Diagram} \]

This Tensor Invariant Lorentz Scalar relation gives the dimension of SpaceTime. The only way there can more dimensions is if there is another SpaceTime direction available. 4-Divergence (\( \partial \cdot J \)) is also used in SR Conservation Laws, ex. (\( \partial \cdot J = 0 \))

All empirical evidence to-date indicates that there are only the 4 known dimensions: 1 temporal (\( t \)) measured in SI units = [s], with (\( ct \)) measured in SI units [m] 3 spatial (\( x, y, z \)) measured in SI units [m] These are the 4 components that appear in:

\[ \text{4-Position } R = (ct,r) \rightarrow (ct,x,y,z) \]

\[ \text{measured in SI units [m]} \]

\[ \text{SR } \rightarrow \text{QM Physics} \]

\[ \text{SR 4-Tensor } (2,0) \text{-Tensor } T^{\mu \nu} \]
\[ (1,1) \text{-Tensor } T^\gamma, \text{ or } T^\gamma_{\text{00}} \]
\[ \text{SR 4-Vector } (1,0) \text{-Tensor } V^\mu = V^\mu_\nu \]
\[ (0,1) \text{-Tensor } V_\nu = V_\nu^\mu \]

\[ \text{SR 4-Scalar } (0,0) \text{-Tensor } S_\mu^\nu = S_\mu^\nu \]
\[ \text{or } S_\mu^\nu \]

\[ \text{Lorentz Scalar} \]

\[ \delta^{\mu \nu} = \delta^\nu_\mu = \delta_{\mu \nu} = \delta_{\nu \mu} = I_{44} = \text{Diag}[1,1,1,1] \]

\[ \text{4D Kronecker Delta} = 4 \text{D Identity} \]

\[ \text{Trace}[T^\mu] = \eta^\mu_\nu T^\nu = T^\mu_\mu = T \]
\[ V^\nu = V^\nu \eta^\mu_\nu = ([V^\mu^\nu] - V^\nu V) = (V^\nu V)^\nu \]

\[ \text{Lorentz Transform} \]
SRQM Diagram: The Basis of Classical SR Physics

The Minkowski Metric ($\eta^{\mu\nu}$), Measurement

SR: Minkowski Metric

$$\partial[R] = \partial^\nu R^\mu = \eta^{\mu\nu} \mathbf{V}^{\mu\nu} + H^{\mu\nu} \rightarrow$$

Diag$[1,-1,-1,-1] = \text{Diag}[1,-1,0,0]$.

{In Cartesian form} “Particle Physics” Convention

{\begin{align*}
\eta_{\mu\nu} &= 1/\eta^{\mu\nu} : \eta^{\mu\nu} \cdot \delta^\nu_\nu = \text{tr}[\eta^{\mu\nu}] = 4
\end{align*}}

4-Gradient $\partial^\nu$

$$\partial = \partial/\partial R^\nu = (\partial/\partial c, \mathbf{V}) = (\partial/\partial c, \partial/\partial \mathbf{V})$$

4-Position $R^\mu$

$$R = (ct, \mathbf{r}) = (r^t = \text{<Event>})$$

SR: Temporal Projection

"Vertical" $V^\mu = T^T V \rightarrow$ Diag$[1,0,0,0] = \text{Diag}[1,0,0,0]$.

SR: Spatial Projection

"Horizontal" $H^\mu = \eta^{\mu\nu} T^T \mathbf{V} \rightarrow$ Diag$[0,-1,-1,-1] = \text{Diag}[0,-1,-1,-1]$.

The component representation of 4-Vectors and the Minkowski Metric $\eta^{\mu\nu}$ will differ with the chosen basis,

$$[\eta^{\mu\nu}] = 1/\eta^{\mu\nu} \text{ and } \eta^{\mu\nu} = \delta^\nu_\nu$$

SR 4-Tensor

$$(2,0)-\text{Tensor} T^{\mu\nu} = \text{Diag}[1,1,1,1]$$

SR 4-Vector

$$(1,0)-\text{Tensor} V^\mu = \mathbf{V} = (v^t, \mathbf{V})$$

SR 4-Scalar

$(0,0)-\text{Tensor} S$ or $\mathbf{S} = \text{Diag}[1,1,1,1]$.

The SR: Minkowski Metric $\eta^{\mu\nu}$ is the fundamental SR $(2,0)$-Tensor, which shows how intervals are “measured” in SR TimeSpace. It is itself the low-mass $= (\text{Curvature} \rightarrow 0)$ limiting-case of the more general GR metric $g^{\mu\nu}$. It can be divided into temporal and spatial parts. The Minkowski Metric can be used to raise/lower indices on other SR tensors, inc. 4-Vectors. The GR Metric is used in strong gravity.
SRQM Diagram:
The Basis of Classical SR Physics

The Lorentz Transform \( \partial_v[R^\mu'] = \partial R^\mu' / \partial R^\nu = \Lambda^\mu'_{\nu} \)

4-Gradient \( \partial^\mu \)
\[ \partial\partial R_v = (\partial, c, \gamma) = (\partial^\mu) \]

4-Position \( R^\nu \)
\[ R = (ct, r) = (r^\nu) = \langle Event \rangle \]

Tensorial Lorentz Transform \( \Lambda^\mu'_{\nu} \)
\{ acting on 4-Vector \( R^\nu = (\Lambda^\mu')^{\nu}_{\mu} \) \}
\[ \partial R^\nu = (\partial R^\nu)[R^\nu'] = (\partial R^\nu)\Lambda^\mu'_{\mu} \]

Invariant Interval
\[ R^\nu - R^\nu = (ct, r) = (c^2, r^2) \]

SpaceTime Dimension
\[ \partial R^4 = (ct, dr) \]

4-Displacement \( \Delta R^4 = (c \Delta t, dr) \)
\[ dR^4 = (cd, dr) \]

4-Position \( R^\nu = (ct, r) \)

4-Vector SRQM Interpretation of QM

A Tensor Study of Physical 4-Vectors

4-Gradient \( \partial^\mu \)

The Basis of Classical SR Physics

Lorentz Transform Properties:
\[ \Lambda^\mu'_{\nu} = (\Lambda^-)^\mu'_{\mu} \]
\[ \Lambda^\mu'_{\nu} = \eta^\mu'_{\nu} = \delta^\mu'_{\nu} \]
\[ \Lambda^\mu^\nu \Lambda^\nu^\nu = 4 : \text{SpaceTime Dimension} \]

Identity \[ 1(4) \]
\[ \Lambda^\mu'_{\nu} \rightarrow \eta^\mu'_{\nu} = \delta^\mu'_{\nu} \]
\[ \Lambda^\mu'_{\nu} \rightarrow T^\nu_{\nu} \]
\[ \Lambda^\mu'_{\nu} \rightarrow P^\nu_{\nu} \]
\[ \Lambda^\mu'_{\nu} \rightarrow (PT)^\nu_{\nu} \]

Parity
\[ \Lambda^\mu'_{\nu} \rightarrow \Lambda^\mu'_{\nu} \]
\[ \Lambda^\mu'_{\nu} \rightarrow -\Lambda^\mu'_{\nu} \]

Combination
\[ \eta^\mu'_{\nu} \Lambda^\nu^\nu \Lambda^\mu^\nu = \eta^\mu'_{\nu} \]
\[ \eta^\mu'_{\nu} \Lambda^\nu^\nu = \eta^\mu'_{\nu} \]

\[ \text{Det}[\Lambda^\mu'_{\nu}] = \pm 1 : (\pm) = \text{Linearity}; (-) = \text{Anti-Linearity} \]

\[ \text{Trace}[\Lambda^\mu'_{\nu}] = \pm \infty \]

Invariant Tr[ \Lambda^\mu'_{\nu} ] → \(-\infty, \ldots, -4, \ldots, -2, \ldots, 0, \ldots, +2, \ldots, +4, \ldots, +\infty \)

Trace identifies CPT Symmetry in the Lorentz Transform
The Lorentz Transform \( \partial_v[R^\mu] = \partial R^\mu / \partial R^v = \Lambda^\mu_v \)

The Lorentz transformation can also be derived empirically. In order to achieve this, it's necessary to write down coordinate transformations that include experimentally testable parameters. For instance, let there be given a single "preferred" inertial frame \((t,x,y,z)\) in which the speed of light is constant, isotropic, and independent of the velocity of the source.

It is also assumed that Einstein synchronization and synchronization by slow clock transport are equivalent in this frame. Then assume another frame \((t',x',y',z')\) in relative motion, in which clocks and rods have the same internal constitution as in the preferred frame. The following relations, however, are left undefined:

- \(a(v)\) differences in time measurements,
- \(b(v)\) differences in measured longitudinal lengths,
- \(c(v)\) differences in measured transverse lengths,
- \(\varepsilon(v)\) depends on the clock synchronization procedure in the moving frame,

Then the transformation formula (assumed to be linear) between those frames is given by:

\[
\begin{align*}
\Delta R &= (c(t,\Delta t, \Delta r)) = (c(t', \Delta r')) \\
d\Delta R &= (c(t, \Delta t, \Delta r)) = (c(t', \Delta r')) \\
\Delta R &= (c(t, \Delta t, \Delta r)) = (c(t', \Delta r')) \\
\Delta R &= (c(t, \Delta t, \Delta r)) = (c(t', \Delta r'))
\end{align*}
\]

The value of LightSpeed \((c)\) was empirically measured by Ole Rømer to be finite using the timing of Jovian moon eclipses.
SR → QM
Physics

SRQM Diagram:
The Basis of Classical SR Physics
TimeSpace Dimension = 4D = (1+3)D

A Tensor Study
of Physical 4-Vectors

∂∙R = Tr[ημν] = ΛμβΛμβ = 4
The SpaceTime Dimension Relations

4-Displacement
ΔR=(cΔt,Δr)
Tensor Invariants include: {Trace, InnerProduct, Determinant, etc.}
dR=(cdt,dr)
4-Divergence[4-Position] , Trace[Minkowski Metric] , and
4-Position
the InnerProduct[any of the Lorentz Transforms]
R=(ct,r)
Invariant Interval
give the Dimension of SR SpaceTime = 4D.
R∙R=(ct)2-r∙r=(cτ)2
2
2
Minkowski Metric 4-Divergence Lorentz Transform ΔR∙ΔR=(cΔt)2-Δr∙Δr=(cΔτ)2
Trace Invariant of 4-Position Inner Prod Invariant dR∙dR=(cdt) -dr∙dr=(cdτ)

Trace[ημν]
= Tr[ημν]
= ημν[ημν]
= ημμ
= δμμ
= (1+1+1+1)
=4

∂∙R
= ∂μ∙Rν
= ∂μημνRν
= ημν∂μRν
= ημνημν
= Tr[ημν]
=4

General Tensor
Trace Invariant

Tr[Tμν]=Tνν=(T00+T11+T22+T33)
=(T00-T11-T22-T33)=T

4-Tensor
Tμν = [T00,T01,T02,T03]
[T10,T11,T12,T13]
[T20,T21,T22,T23]
[T30,T31,T32,T33]

ημνΛμαΛνβ = ηαβ
ηαβημνΛμαΛνβ = ηαβηαβ
ηαβΛμαημνΛνβ = ηαβηαβ
(ηαβΛμα)(ημνΛνβ) = ηαβηαβ
ΛμβΛμβ = ηαβηαβ = Tr[ημν]
ΛμβΛμβ = 4
Minkowski
=4
Trace Invariant
μν

U∙∂[..]
γd/dt[..]
d/dτ[..]

Relativity of
Simultaneity:Stationarity

U∙ΔR = γ(c,u)∙(cΔt,Δr)
= γ(c2Δt - u∙Δr)
= c2Δto = c2Δτ

4

4-Gradient
∂=(∂t /c,-∇)=∂/∂Rμ
→(∂t /c,-∂x,-∂y,-∂z)
=(∂/c∂t,-∂/∂x,-∂/∂y,-∂/∂z)
Invariant
d’Alembertian
Wave Equation
ProperTime Derivative
∂∙∂=(∂t /c)2-∇∙∇
U∙∂=γ(c,u)∙(∂t /c,-∇)=γ(∂t+u∙∇)
=γ(∂t+(dx/dt)∂x+(dy/dt)∂y+(dz/dt)∂z)
= γd/dt = d/dτ
Continuity of
4-Velocity Flow
∂∙U=0

ProperTime Differential
dτ =(1/γ)dt
=Time Dilation

SRQM Diagram

Tr[η ]=ην =(1) - (-1) - (-1) - (-1)= 4

SR 4-Tensor
SR 4-Vector
SR 4-Scalar
(2,0)-Tensor Tμν
(1,0)-Tensor Vμ = V = (v0,v)
(1,1)-Tensor Tμν or Tμν SR 4-CoVector:OneForm (0,0)-Tensor S or So
Lorentz Scalar
(0,2)-Tensor Tμν
(0,1)-Tensor Vμ = (v0,-v)

SciRealm.org
John B. Wilson
SciRealm@aol.com
http://scirealm.org/SRQM.pdf

∂[R]=∂μRν=ημν
∂ν[Rμ′]
→Diag[1,-1,-1,-1]
∂∙R=4
=∂Rμ′/∂Rν=Λμ'ν
=Diag[1,-δjk]
SpaceTime
Lorentz
Minkowski
Dimension
Transform
Metric
SpaceTime Dim
ημν
Tr[ημν] = 4 = ΛμνΛμν
∙

ν

Minkowski
Metric ημν
→
[+1,0,0,0]
[0,-1,0,0]
[0,0,-1,0]
[0,0,0,-1]

4-Vector SRQM Interpretation
of QM

Conservation:Non-Divergence
of Minkowksi Metric
∂∙ημν
σ μν
= ∂ ∙η
= ∂σησμημν
= ∂σησμημν
= ∂μημν
= ∂σηαν
= ∂σδαν
ν
=0
= 0ν

4-Velocity
U=γ(c,u)
=dR/dτ
Invariant Magnitude
LightSpeed
U∙U=c2

SR : Minkowski
TimeSpace is 4D
(1+3)D = 4D

Trace[Tμν] = ημνTμν = Tμμ = T
V∙V = VμημνVν = [(v0)2 - v∙v] = (v0o)2
= Lorentz Scalar


The Tensor Invariant Lorentz Scalar Product (LSP) is the SR 4D (Dot=∙) Product. It is used to make Invariant Scalars from two 4-Vectors. 

\[
\eta_{\mu\nu} = \frac{\eta_{\mu\nu}}{c^2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

(\(\eta_{\mu\nu}\)) is itself just the lower-index form of the SR Minkowski Metric (\(\eta^\mu\nu\)), with individual components (\(\eta_{\mu\nu}\) = 1/\(c^2\)), else 0. In Cartesian basis, this gives (\(\eta_{\mu\nu} = \eta^{\mu\nu}\)) (Cartesian).

The LSP is used in just about every relation between any two interesting 4-Vectors. It also gives the Invariant Magnitude of a single 4-Vector. If the 4-Vector is temporal, the LSP is used in just about every relation between any two interesting 4-Vectors.

A Tensor Study of Physical 4-Vectors

SR 4-Tensor
(2,0)-Tensor \(T^{\mu\nu}\)
(1,1)-Tensor \(T^\nu\), or \(T^\nu_{\mu}\)
(0,2)-Tensor \(T_{\mu\nu}\)

SR 4-Vector
(1,0)-Tensor \(V^\mu = (v^\mu, v_0)\)

SR 4-4Vector:OneForm
(0,1)-Tensor \(V_\mu = (v_\mu, v^0)\)

SR 4-Scalar
(0,0)-Tensor \(S\) or \(S_{\mu\nu}\)

Lorentz Scalar

\[P - P(m, c) = \frac{(E/c)^2}{c^2} \]

4-Momentum

\[P = m c \frac{E}{c} \]

4-Position

\[R = (c t, r) \]

4-Displacement

\[\Delta R = (c \Delta t, \Delta r) \]

4-Gradient

\[\partial R = 4 \text{ SpaceTime Dimension} \]

\[\partial[R] = \partial[R^\mu] = \eta_{\mu\nu} \text{ Lorentz Transform} \]

\[\partial[R^\mu] = \partial R^\mu = \Lambda^\mu_{\nu} \]

4-Displacement

\[\Delta R = (c \Delta t, \Delta r) = (c \Delta t, \Delta r) \]

4-Position

\[R = (c t, r) \]

SRQM Diagram

The Basis of Classical SR Physics

The Tensor Invariant Lorentz Scalar Product (LSP) is the SR 4D (Dot=∙) Product. It is used to make Invariant Scalars from two 4-Vectors. 

\[A \cdot B = A' \cdot B' = A' \eta_{\mu\nu} B' = A' B' = (a'b' - a \cdot b) = (a'b' - a \cdot a) = (a' - a)^2 \]

\[A \cdot A' = A' \eta_{\mu\nu} A = A' A' = (a' - a)^2 = (a')^2 \]

The Basis of Classical SR Physics

4-WaveVector

\[K = (\omega/c, k) = (\omega/c, \omega h/\nu) \]

Invariant Magnitude LightSpeed

\[U \cdot U = c^2 \]

4-Velocity Flow

\[\partial U = 0 \]

Trace[\(T^{\mu\nu}\)] = \(\eta_{\mu\nu} T^{\mu\nu} = T_{\nu} = T \]

\[V \cdot V = V \cdot \eta_{\mu\nu} V = [(v^\mu)^2 - (v_0)^2] = (v^\nu)^2 \]

Lorentz Scalar

a\(^0\) or \(a_0\): (0)\(^{th}\) component = temporal component (can relativistically vary)

a\(_0\): (0) observer's rest-frame "naught" Invariant value (does not vary)
4-Velocity U is the ProperTime Derivative (d/dτ) of the 4-Position R or of the 4-Displacement ΔR.

It is the SR 4-Vector that describes the motion of <Events> through SpaceTime.

(a) For an un-accelerated observer, the 4-Velocity U is a constant along the WorldLine at all points.

(b) For an accelerated observer, the 4-Velocity U is still tangent to the WorldLine at each point, but changes direction as the WorldLine bends thru SpaceTime.

The 4-UnitTemporal T & 4-Velocity U are unlike most of the other SR 4-Vectors. They have 3 independent components, whereas the others usually have 4.

This is due to the constraints placed by the LSP Tensor Invariants. T·T = +1 & U·U = c² have constant magnitudes, giving the Speed-of-Light (c) in SpaceTime.

Components: 3 independent + 0 independent = 3 independent + 1 independent = 4 independent

They also usually have the Relativistic Gamma factor (γ) exposed in component form, whereas most of the other temporal 4-Vectors have it absorbed into the Lorentz 4-Scalar factor that goes into their components.

4-UnitTemporal T = γ(1,1,0,0) U = γ(c,u) = (γc,u) = T·T = γ² = c² U·U = U² = c²
4-Momentum P = (mc,p) = (E/c,p) = mU = m(U/c)

4-Velocity U = (c,0,0,0)

SR 4-Scalar γ = 1/√(1 - β·β), β = u/c
SR 4-Vector (1,0)-Tensor S or 0,0,-1 Lorentz Scalar
SR 4-Vector (0,1)-Tensor V = (v¹,0) v² = (v²,0) v³ = (v³,0) Lorentz Spinors
SR 4-Tensor Tμν (1,1)-Tensor Tμν, or Tνμ (2,0)-Tensor Tμν or Tνμ Lorentz Tensors

Invariant Interval R·R = (ct)² - r² = (ct)²
4-Displacement ΔR = (cΔt,Δr) = (cΔt)² - (Δr)²
4-Position R = (ct,r)

Invariant Magnitude LightSpeed U·U = c²

4-Gradient ∂(c²,0,0,0) = 0 & (0,0,0,0)

LightDisplacement Dτ = γ(c,0,0,0) = (c,0,0,0)

Relativity of Simultaneity: Stationarity ΔR = γ(c,0,0,0) = γ(c,0,0,0)

Site τ = 0 lorentz TimeInterval (c,t)
4-Velocity U = (c,0,0,0)

ProperTime Derivative U·∂(c²,0,0,0) = c² dτ/dτ = U τ = U·∂(c²,0,0,0) = (c,0,0,0)

ProperTime Differential dt = (1/c²)dτ = Time Dilution

Trace[Tμν] = γ(1,0,0,0) = Tμν = T

U = R'' is tangent to WorldLine

U = R' is normal to WorldLine

(E = mc², γ = E/c²)

SRQM Diagram
SRQM Diagram: The Basis of Classical SR Physics

4-Velocity \( |\text{Magnitude}| = \text{Invariant Speed-of-Light} \ (c) \)

A Tensor Study of Physical 4-Vectors

The newly made 4-Vectors thus have \( 3 + 1 = 4 \) independent components.

The Lorentz Scalar Product of the 4-Velocity leads to the Invariant \[|\text{Magnitude}|\] of Speed-of-Light \((c)\), one of the main fundamental SR physical constants of physics.

\[ P \cdot P = (\gamma(c, u) - \gamma(c, 0)) = (U \cdot \bar{U} = \gamma(\theta + u \cdot \bar{V})R = \gamma(d/dt)R = \gamma(d/dt) = \gamma(c, u) = U^2 \]

(c) is the unique maximum speed of SR causality, which all massless particles (RestMass \( m_0 = 0 \)), ex. the photon, travel at temporally & spatially. Massive particles can travel at (c) only temporarily.

\[ P = (E/c, p) = (E/c, p) = (E/c, p) = (E/c, p) = (E/c, p) \]

\[ P \cdot P = (m_0^2 + p^2) = (E/c, p) \]

\[ (\gamma(c, u))^2 = (c^2 t)^2 - (u \cdot u)^2 = (c^2 t)^2 - (u \cdot u)^2 \]

\[ \gamma(d/dt)[u] = \gamma(d/dt)[u] = \gamma(d/dt)[u] \]

\[ \gamma = \gamma(c, u) = \gamma(c, u) \]

\[ \gamma = \gamma(c, u) = \gamma(c, u) \]

SRQM Diagram

If (c) was not a constant, but varied somehow, then all 4-Vectors made from the 4-Velocity would have more than 4 independent components, which is not observed. It seems a strong, compelling argument against variable light-speed theories.
The Basis of Classical SR Physics

Relativity of Simultaneity: Time-Delay

(Simultaneity ↔ Same-Time Occurrence ↔ Δt = 0)
(Time-Delay ↔ Different-Time Occurrence ↔ Δt ≠ 0)

SRQM Diagram:

Realizing that Simultaneity (no-delay) is not an invariant concept was a breakthrough that led Einstein to Special Relativity (SR).

Temporal Ordering:

- Simultaneity (= same time occurrence) is Relative
- Time-like Separated Events: Can appear in any temporal order, depending on one’s reference frame (Boost)

Causality is Absolute ↔ Invariant Proper Time

- Time-like Separated Events: All observers agree on 1D causal ordering. Causality is an invariant concept.

Spatial Ordering:

- Stationarity (= same place occurrence) is Relative
- Space-like Separated Events: Can appear in any spatial order, depending on one’s reference frame (Boost)

Topology is Absolute ↔ Invariant Proper Length

- Space-like Separated Events: All observers agree on topology=3D spatial ordering. Topology/topological-extension is an invariant concept.

SR 4-Tensor
(2,0)-Tensor T_{\mu
\nu} ^{\nu}

SR 4-Vector
(1,0)-Tensor V_{\nu} ^{\nu}

SR 4-Vector
(0,1)-Tensor V_{\mu} ^{\mu}

SR 4-Scalar
(0,0)-Tensor S or S_{\mu
\nu}

Lorentz Scalar
SRQM Diagram:  
The Basis of Classical SR Physics  
Relativity of Stationarity: Space-Motion  
(Stationarity ↔ Same-Place Occurrence ↔ Δx=0)  
(Space-Motion ↔ Different-Place Occurrence ↔ Δx≠0)
The derivation shows that the ProperTime Derivative \(d/d\tau\) is an Invariant Lorentz Scalar. Therefore, all observers must agree on its magnitude, regardless of their frame-of-reference. \(d/d\tau\) is used to derive some of the physical 4-Vectors: 4-Velocity, 4-Acceleration, 4-Force, 4-Torque, etc.

The ProperTime Derivative can be used to make new tensors from existing tensors, as it is taking the derivative of an existing tensor by a Lorentz Scalar: the ProperTime Derivative is a Lorentz Scalar and is used to derive some of the physical 4-Vectors: 4-Velocity, 4-Acceleration, 4-Force, 4-Torque, etc.
The Basis of Classical SR Physics

ProperTime Derivative in SR:
4-Tensors, 4-Vectors, and 4-Scalars

SRQM Diagram:

4-Vector SRQM Interpretation of QM

SR → QM Physics

A Tensor Study of Physical 4-Vectors

SRQM Diagram
There are several ways to derive Time Dilation.

ProperTime Differential (Lorentz 4-Scalar): $d\tau = (1/\gamma)dt$

ProperTime Differential (Lorentz 4-Scalar): $d\tau = (1/\gamma)dt$

Now multiply both sides by the moving-frame speed $v = |\mathbf{v}|$

$v\Delta t = v\Delta\tau = \gamma vL_0$

$v\Delta t = \text{distance } L_0$, the moving clock travels wrt. frame, which is a proper (fixed-to-frame) displacement length.

$L_0 = \gamma L$

$L_0 = (1/\gamma)L_0$ : →Length Contraction← {in spatial $v$ direction}

SR 4-Vector (2,0)-Tensor $T^\mu_{\nu}$, (1,1)-Tensor $T^\nu$, or $T^\nu_{\mu}$

SR 4-Vector (0,1)-Tensor $V^\nu = (v^\nu, \mathbf{V})$

SR 4-Vector (0,0)-Tensor $V_\nu = (v_\nu, \mathbf{V})$

SR 4-Scalar (0,0)-Tensor $S$ or $S_\nu$

Lorentz Scalar

$dR = \gamma(c\Delta t, \Delta r)$

$\Delta R = (c\Delta t, \Delta r)$

$\Delta R \Delta r = (c\Delta t)^2 - (c\Delta r)^2$

$d\tau = \gamma d\tau$ in all frames
The 4-Gradient $\partial = \partial^\mu = \partial /c, \nabla$ is the index-raised version of the SR Gradient One-Form $(\partial_\mu = \partial /c, \nabla)$. It is the 4D version of the partial derivative function of calculus, one partial for each dimensional direction, just as the Del ($\nabla$) is the 3D version of the partial derivative function.

The 4-Gradient is a 4-Vector function that can act on other 4-Scalars, 4-Vectors, or 4-Tensors. The 4-Gradient tells how things change wrt $1$-time, $3$-space $= 4$D ($\text{TimeSpace}$). It is instrumental in creating the ProperTime Derivative $U\partial = \gamma d/dt = d/dt$.

The 4-Gradient plays a major role in advanced physics, showing how SR waves are formed, creating the Hamilton-Jacobi equations, the Euler-Lagrange equations, Conservation Equations $(\partial^\mu \partial_\mu = 0$), Maxwell’s Equations, the Lorenz Gauge, the d’Alembertian, etc. It gives the Dimension of SpaceTime, the Minkowski Metric, and the Lorentz Transformations.

In QM, it provides the Schrödinger relations. $P = (E/c, p) = ih\partial = ih(\partial /c, \nabla)$.

The 4-Gradient is fundamental in connecting SR to QM.
The usual mathematical (complex) plane-wave solutions apply in SR: Its solution provides for the introduction of SR 4-WaveVector it will be seen again in the Klein-Gordon RQM wave equation. However, it is seen, for example, in the SR Maxwell Equation for EM light waves. The Lorentz Scalar Product Invariant of the 4-Gradient gives the

\[ \partial \cdot \partial = (a, c)^2 \cdot \nabla \cdot \nabla \]
d’Alembertian Wave Equation, describing SR wave motion. The Lorentz Scalar Product Invariant of the 4-Gradient gives the

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\[ \partial \cdot \partial = (a, c)^2 \cdot \nabla \cdot \nabla \]
d’Alembertian Wave Equation, describing SR wave motion. The Lorentz Scalar Product Invariant of the 4-Gradient gives the
SRQM Diagram:
The Basis of Classical SR Physics
Continuity of 4-Velocity Flow (\(\partial \cdot \mathbf{U} = 0\))

Conservation of Charge, continuity eqn:
\[ \rho \partial \cdot \mathbf{U} = \partial \rho + \nabla \cdot \mathbf{J} = (\partial \rho + \nabla \cdot \mathbf{J}) = 0 \]

The Conservation Laws of SR quantities are all in the form of Continuity Equations
\[ a \partial t \mathbf{U} = \partial \rho + \mathbf{J} = (\partial \rho + \nabla \cdot \mathbf{J}) = 0 \]

Conservation of 4-Momentum Flow-Field
4-Gradient
\[ \partial \mathbf{R} = \frac{\partial \mathbf{R}}{\partial t} = \frac{\partial \mathbf{R}}{\partial x}, \frac{\partial \mathbf{R}}{\partial y}, \frac{\partial \mathbf{R}}{\partial z} \]

SR 4-Vector
\[ \mathbf{V} = \mathbf{V} = (0, v) \]
SR 4-Scalar
\[ \alpha \mathbf{A} = \alpha \mathbf{A} = (a_1, a_2, a_3, a_4) \]
SR 4-Covector
\[ \mathbf{V} = \mathbf{V} = (v_0, v_1, v_2, v_3) \]
SR 4-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]

SR 4-Scalar Lorentz Scalar
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]

SR 4-Vector Lorentz Vector
\[ \mathbf{V} = \mathbf{V} = (0, v) \]

SR 4-Covector OneForm
\[ \mathbf{V} = \mathbf{V} = (v_0, v_1, v_2, v_3) \]

SR 4-Tensor (2,0)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]
SR 4-Tensor (1,1)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]
SR 4-Tensor (0,2)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]

SR 4-Tensor (0,0)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]

SR 4-Tensor (2,0)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]
SR 4-Tensor (1,0)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]
SR 4-Tensor (0,1)-Tensor
\[ T^{\mu \nu} = T^{\mu \nu} = (\partial \mathbf{R} / \partial t) \]
Now focus on a few more of the main SR 4-Vectors.

- **4-Position** $R = (ct, r)$
- **4-Velocity** $U = dR/d\tau = (c, \mathbf{u})$
- **4-Gradient** $\partial^\mu = \partial/\partial R^\mu = (\partial{t}/c, -\nabla)$
- **4-Momentum** $P = (E/c, p)$
- **4-WaveVector** $K = (\omega/c, k)$
- **4-CurrentDensity: ChargeFlux** $J = (\rho c, j)$
- **4-(Dust)NumberFlux** $N = (n c, n)$

These 4-Vectors give more of the main classical results of Special Relativity, including SR concepts like:

- SR Particles and Waves, Matter-Wave Dispersion
- Einstein's $E = mc^2 = \gamma m_0 c^2 = \gamma E_0$, Rest Mass, Rest Energy
- Conservation of Charge ($Q$), Conservation of Particle Number ($N$), Continuity Equations

Motion of various Lorentz Scalars leads to the “Substantiation” of various physical SR 4-Vectors.
SRQM Diagram:
The Basis of Classical SR Physics
4-Momentum, Einstein’s E = mc²

4-Position R=(ct,r)
4-Gradient $\partial=(\partial/c, \nabla)$
4-Velocity $U = \gamma(c,u)$

4-Momentum $P = (E/c,p) = \mu U = \gamma m_o(c,u) = m(c,u)$

Temporal part: $E = \gamma E_o = \gamma m_o c^2 = mc^2$
{energy}
$E = m_c c^2 + (\gamma - 1)m_c c^2$
(rest) + (kinetic)

Spatial part: $p = Eu/c^2 = \gamma E_o u/c^2 = \gamma m_o u = \mu u$

{3-momentum}

4-Momentum $P = (E/c,p) = -\partial[S_{\text{action,free}}] = -\partial[1/c, \nabla[S_{\text{action,free}}]]$

4-TotalMomentum $P_T = (E_T/c=H/c,p_T) = -\partial[S_{\text{action}}] = -\partial[1/c, \nabla S_{\text{action}}]$

Temporal part: $E = -\partial[S_{\text{action,free}}]: E_T = H = -\partial[S_{\text{action}}]$

Spatial part: $p = +\nabla[S_{\text{action,free}}]: p_T = +\nabla[S_{\text{action}}]$

4-Momentum $P=(E/c,p)=(mc,p)=m(U)

SRQM Diagram

$E^2 = |p|^2 c^2 + E_o^2$
$E^2 = (|p|c)^2 + (m_o c^2)^2$

Relativistic Energy($E$):Mass($m$) vs Invariant Rest Energy($E_o$):Mass($m_o$)

$E = \gamma E_o = \gamma m_o c^2 = mc^2$

Trace[T^\mu]\ = \eta_{\mu\nu}T^{\nu}_{\mu} = T^\mu \cdot T^\nu = T^\mu_{\mu} = T$

V•V = V'[r]V = [(v^0)^2 - (v^s)^2] = Lorentz Scalar
SRQM Diagram: The Basis of Classical SR Physics

4-WaveVector, \( \mathbf{u} \times \mathbf{v}_{\text{phase}} = c^2 \)

\[ \omega = \gamma \omega_0 \]

Temporal part:
\[ \omega = \gamma \omega_0 \]

Spatial part:
\[ \mathbf{k} = \gamma (\omega_0/c^2) \mathbf{u} = (\omega/c^2) \mathbf{u} = \omega \hat{n}/v_{\text{phase}} \]

\[ |\mathbf{u} \times \mathbf{v}_{\text{phase}}| = c^2 = |\mathbf{v}_{\text{group}} \times \mathbf{v}_{\text{phase}}| \]

4-WaveVector \( \mathbf{K} = (\omega/c, \mathbf{k}) = -\partial (\Phi_{\text{phase,free}})/c - \mathbf{V} \Phi_{\text{phase,free}} \)

4-TotalWaveVector \( \mathbf{K}_T = (\omega_0/c, \mathbf{k}_T) = -\partial (\Phi_{\text{phase}}) - (\partial/c, \mathbf{V}) \Phi_{\text{phase}} \)

Temporal part:
\[ \omega = -\partial_t [\Phi_{\text{phase,free}}] : \omega_T = -\partial_t [\Phi_{\text{phase}}] \]

Spatial part:
\[ \mathbf{k} = +\nabla [\Phi_{\text{phase,free}}] : \mathbf{k}_T = +\nabla [\Phi_{\text{phase}}] \]

\[ (\mathbf{K} \cdot \mathbf{K}) = (\omega/c)^2 - (\mathbf{k} \cdot \mathbf{k}) = (\omega_0/c^2) \]

\[ \omega^2 = (|\mathbf{k}|^2 + (\omega_0)^2) : \text{Matter-Wave Dispersion Relation} \]

Relativistic AngFreq(\( \omega \)) vs Invariant Rest AngFreq(\( \omega_0 \))

\[ \omega = \gamma \omega_0 \]

Trace[\( T^{\mu\nu} \)] = \( h_{\mu\nu} T^{\mu\nu} = T^{\mu\nu} \cdot T_{\mu\nu} = T \)

\[ \mathbf{V} \cdot \mathbf{V} = \mathbf{V}^\dagger \mathbf{V} = [(v^\dagger)^2 - \mathbf{v} \cdot \mathbf{v}] = (\mathbf{v}^\dagger \cdot \mathbf{v})^2 = \text{Lorentz Scalar} \]
4-Position \( R = (ct, r) \)
4-Gradient \( \partial = \partial /c \cdot \nabla \)
4-Velocity \( U = \gamma (c, u) \)

4-CurrentDensity \( J = (pc, j) = \rho_0 U = \gamma \rho_0 (c, u) = \rho (c, u) \)
4-ChargeFlux \( J \)

Temporal part: \( \rho = \gamma \rho_0 \)  
\{charge-density\}

Spatial part: \( j = \gamma \rho_0 u = \rho u \)  
\{3-current-density\}

Conservation of Charge (Q)
\[
\partial \cdot J = (\partial /c \cdot \nabla) \cdot (pc, j) = (\partial \rho + \nabla \cdot j) = 0
\]

Continuity Equation: Noether’s Theorem
The temporal change in charge density is balanced by the spatial change in current density. Charge is neither created nor destroyed. It just moves around as charge currents...

\[
\rho_0 = \gamma \rho_0
\]

Relativistic ChargeDensity(\( \rho \)) vs Invariant Rest ChargeDensity(\( \rho_0 \))

SRQM Diagram: The Basis of Classical SR Physics

4-Vector SRQM Interpretation of QM
A Tensor Study of Physical 4-Vectors

SciRealm.org
John B. Wilson
SciRealm@aol.com
http://scirealm.org/SRQM.pdf
4-Position $R = (ct, r)$
4-Gradient $\partial = (\partial_t/c, -\nabla)$
4-Velocity $U = \gamma(c, u)$

4-NumberFlux $N = (nc, n) = n_0U = \gamma n_0(c, u) = n(c, u)$

Temporal part: $n = \gamma n_0$
{number-density}

Spatial part: $n = \gamma n_0u = nu$
{3-number-flux}

Conservation of Particle # ($N$)

$\partial \cdot N = (\partial / c, -\nabla) \cdot (nc, n) = (\partial_t n + \nabla \cdot n) = 0$

Continuity Equation: Noether’s Theorem
The temporal change in number density is balanced by the spatial change in number-flux. Particle # is neither created nor destroyed. It just moves around as number currents...

Relativistic NumberDensity ($n$) vs Invariant Rest NumberDensity ($n_0$)

$n = \gamma n_0$

SR 4-Tensor $(2,0)$-Tensor $T^{\mu\nu}$
SR 4-Vector $(1,0)$-Tensor $V^\mu = v^\mu$
SR 4-CoVector: OneForm $(0,1)$-Tensor $V_\nu = (v_\nu)$
SR 4-Scalar $(0,0)$-Tensor $S$ or $S_0$ Lorentz Scalar
### Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]$

(Continuous) vs (Discrete)

(Proper Det=+1) vs (Improper Det=-1)

The main idea that makes a generic 4-Vector into an SR 4-Vector is that it must transform correctly according to an SR Lorentz Transformation $\{\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu'}]\}$, which is basically any linear, unitary or antiunitary, transform (Determinant[$\Lambda^{\mu'}_{\nu}$] = ±1) which leaves the Invariant Interval unchanged.

The SR continuous transforms (variable with some parameter) have (Det = +1, Proper) and include:

- "Rotation" (a mixing of space-space coordinates) and "(Velocity) Boost" (a mixing of time-space coordinates).

The SR discrete transforms can be (Det = +1, Proper) or (Det = -1, Improper) and include:

- "Space Parity-Inversion" (reversal of the all space coordinates), "Time-Reversal" (reversal of the temporal coordinate), "Identity" (no change), various single dimension "Flips", "Fixed Rotations", and combinations of all of these discrete transforms.

#### Typical Lorentz Boost Transformation

For a linear-velocity frame-shift $x$-Boost:

$A^\prime = (a^\prime, a^\prime, a^\prime, a^\prime)$

$A^\prime = (a^\prime, a^\prime, a^\prime, a^\prime)$

$= B^\prime A^\nu$

$= (\gamma a^\prime - \gamma \beta a^\prime - \gamma \beta a^\prime + \gamma a^\prime, a^\prime, a^\prime, a^\prime)$

#### Lorentz Parity-Inversion Transformation

$A^\prime = (a^\prime, a^\prime, a^\prime, a^\prime)$

$A^\prime = (a^\prime, a^\prime, a^\prime, a^\prime)$

$= P^\nu A^\nu$

$= (a^\prime - a^\prime, -a^\prime - a^\prime, -a^\prime, -a^\prime)$

#### Continuous: Boost depends on variable parameter $\beta$, with $\gamma=1/\sqrt{1-\beta^2}$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$\gamma$</td>
<td>$-\beta \gamma$</td>
<td>0</td>
</tr>
</tbody>
</table>

$Lorentz$  $x$-Boost

$A^\nu \rightarrow B^\nu = A^\nu - \beta A^\nu$

$Det[B^\nu] = +1$, Proper $\gamma^2 - \beta^2 \gamma^2 = +1$

Proper: preserves orientation of basis

#### Discrete: Parity has no variable parameters

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$Lorentz$ Parity Transform

$A^\nu \rightarrow P^\nu = A^\nu$

$Det[P^\nu] = -1$, Improper $(-1)^3 = -1$

Improper: reverses orientation of basis
**Lorentz Transforms** $\Lambda_{\nu}^{\mu'} = \partial_{\nu}[X^{\mu'}]$

**Proper Lorentz Transforms (Det=+1):** Continuous: (Boost) vs (Rotation)

$\beta = v/c$: dimensionless Velocity Beta Factor $\{\beta=(0..1), \text{with speed-of-light (c) at } (\beta=1)\}$

$\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-\beta\cdot\beta}$: dimensionless Lorentz Relativistic Gamma Factor $\{\gamma=(1..\infty)\}$

**Typical Lorentz Boost Transform (symmetric):** for a linear-velocity frame-shift $(x,t)$-Boost in the $\hat{x}$-direction:

$A'^{\mu} = (a', a', a', a') = B^{\nu}A^{\nu} = (\gamma a' - \beta a^x, -\beta a^y + \gamma a^x, a^y, a^z)$

$A'^{\mu} = (a', a', a', a') = B^{\nu}A^{\nu} = (\gamma a' - \beta a^x, -\beta a^y + \gamma a^x, a^y, a^z)$

**Lorentz Transforms:**

- Lambda ($\Lambda$) for Lorentz
- B ($\mathbf{B}$) for Boost
- R ($\mathbf{R}$) for Rotation

**Proper Transforms**

Determinant $= +1$

- $\{\cos^2 + \sin^2 = +1\}$
- $\{\gamma^2 - \beta^2 = +1\}$
- $\{\cosh^2 - \sinh^2 = +1\}$

$\zeta = \text{rapidity} = \text{hyperbolic angle}$

$\gamma = \cosh[\zeta] = 1/\sqrt{1-\beta^2}$

$\beta = \tanh[\zeta]$

$\gamma$ = speed of light $c$

**Typical Lorentz Rotation Transform (non-symmetric):** for an angular-displacement frame-shift $(x,y)$-Rotation about the $z$-direction:

$A'^{\mu} = (a', a', a', a') = R^{\nu}A^{\nu} = (a', \cos[\theta]a^x - \sin[\theta]a^y, \sin[\theta]a^x + \cos[\theta]a^y, a^z)$

$A'^{\mu} = (a', a', a', a') = R^{\nu}A^{\nu} = (a', \cos[\theta]a^x - \sin[\theta]a^y, \sin[\theta]a^x + \cos[\theta]a^y, a^z)$

**SR:Lorentz Transform**

$\delta[R^{\nu}'] = \partial R^{\mu'}/\partial R^{\nu'} = \Lambda^{\mu'}_{\nu}$

$\Lambda^{\mu'}_{\nu} = (\Lambda^{-1})^{\mu'}_{\nu} : \Lambda^{\mu'}_{\nu} \Lambda_{\mu'}^{\nu} = \eta_{\nu\nu}$

$\eta_{\nu\nu} \Lambda^{\mu'}_{\alpha} \Lambda_{\mu'}^{\rho} = \eta_{\nu\nu}$

$\text{Det}[\Lambda^{\mu'}_{\nu}] = \pm 1 < 0 < \text{Det}[\Lambda^{\mu'}_{\nu}] = 4 = \Lambda^{\mu'}_{\nu} \Lambda_{\mu'}^{\nu}$

$\text{Trace}[\Lambda^{\mu'}_{\nu}] = \eta_{\nu\nu} T^{\mu'} = T^{\nu'} = T$

$V \cdot V = V^\nu \eta_{\nu\nu} V^\mu = (V^\mu)^2 - V^\nu V^\nu = (V^\nu)^2$

$= \text{Lorentz Scalar}$

The Lorentz Rotation $R^{\nu'}$, is a 4D rotation matrix. It simply adds the time component, which remains unchanged (1), to the standard 3D rotation matrix.

**Space-Time**

- Rotated 4-Vector Circularly-Rotated $A'=A''=(a^0, a')$
- Boosted 4-Vector Hyperbolically-Rotated $A'=A''=B^{\mu'}A^{\nu'}=(a^0, a')$
Lorentz Transforms $\Lambda^\mu_\nu = \partial_\nu [X^\mu]$ 

Proper Lorentz Transforms (Det=+1): (Boost) vs (Rotation) vs (Identity)

**General Lorentz Boost Transform** (symmetric, continuous):
for a linear-velocity frame-shift (Boost) in the $v/c=\beta=(\beta',\beta,\beta^2)$-direction:

$\Lambda^\mu_\nu \rightarrow B^\mu_\nu = \frac{\gamma}{\gamma-\gamma\beta^2} (\gamma-1)\beta\beta'/(\beta\cdot\beta)+\delta^\mu_\nu$

**General Lorentz Rotation Transform** (non-symmetric, continuous):
for an angular-displacement frame-shift (Rotation) about the $\hat{n}=(n^1,n^2,n^3)$-direction:

$\Lambda^\mu_\nu \rightarrow R^\mu_\nu = \text{Diag}[1,\delta_{ij}]=I(\theta)$

**Lorentz Identity Transform** (symmetric, "discrete, continuous")
for a non-frame-shift (Identity) in any direction:

$\Lambda^\mu_\nu \rightarrow \eta^\mu_\nu = \delta^\mu_\nu = \text{Diag}[1,\delta_{ij}]=I(0)$

The Lorentz Identity Transform is the limit of both the Rotation and Boost Transforms when the respective "rotation angle" is 0.

**SR: Lorentz Transform**

$\delta_{(\alpha)} = \partial_{R^\alpha}/\partial R^\alpha = \Lambda^\alpha_\nu$$

$\Lambda^\mu_\nu = (\Lambda^{-1})^\nu_\mu : \Lambda^\mu_\nu A^\nu = A^\mu$

$\eta^\mu_\nu \Lambda^\mu_\nu = \delta^\mu_\nu \Rightarrow \eta^\mu_\nu \Lambda^\mu_\nu = 4$

$\text{Tr}[\Lambda^\mu_\nu] = \{\infty, 0\}$

$\text{Det}[\Lambda^\mu_\nu] = \pm 1$

$\beta = v/c$: dimensional Velocity Beta Factor $\{\beta=0..1\}$, with speed-of-light $(c)$ at $(\beta=1)$

$\gamma = 1/\sqrt{1-\beta^2}$: dimensional Lorentz Relativistic Gamma Factor $\{\gamma=1..\infty\}$

Identity transformation for zero relative motion:boost/rotation: $B[0] = R[0] = I(\theta)$

Proper Transformation = positive unit determinant: $\text{det}[B] = \text{det}[R] = \text{det}[\eta] = +1$.

Inverses: $B(v)^{-1} = B(-v)$ (relative motion in the opposite direction), and $R(\theta)^{-1} = R(-\theta)$ (rotation in the opposite sense about the same axis)

Matrix symmetry: $B$ is symmetric (equals transpose, $B=B^T$), while $R$ is nonsymmetric but orthogonal (transpose equals inverse, $R^T = R^{-1}$)

**4-Vector SRQM Interpretation of QM**

$V = V^i\Lambda^i_\nu = (\gamma v^i - \beta^i v) = (\gamma v^i)^2 = \text{Lorentz Scalar}$

$\text{Trace}[T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T^\mu_\mu = T$

$V^\mu = V^i\Lambda^i_\mu = T^\mu_\nu = T^\mu_\nu = T$

For the 4D Identity $I(0)=\delta^\mu_\nu \Rightarrow \text{Det}[I(0)]=1$
Lorentz Transforms $\Lambda^\mu_\nu = \partial_\nu [\mathcal{X}^\mu]$  

Discrete (non-continuous) (Parity-Inversion) vs (Time-Reversal) vs (Identity)

<table>
<thead>
<tr>
<th>Transform Type</th>
<th>No mixing</th>
<th>Time</th>
<th>Space</th>
<th>TimeSpace</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Lorentz Parity-Inversion (Space-Reversal) Transform: $\Lambda^\mu_\nu \rightarrow P^\mu_\nu$</td>
<td>$\Lambda^\mu_\nu = \begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$\text{Det}[\Lambda^\mu_\nu] = +1$</td>
<td>$\text{Det}[\Lambda^\mu_\nu] = +4$</td>
<td>$\text{Det}[\Lambda^\mu_\nu] = +4$</td>
</tr>
<tr>
<td>General Lorentz Time-Reversal Transform: $\Lambda^\mu_\nu \rightarrow T^\mu_\nu$</td>
<td>$\Lambda^\mu_\nu = \begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\text{Det}[\Lambda^\mu_\nu] = +1$</td>
<td>$\text{Det}[\Lambda^\mu_\nu] = +4$</td>
<td>$\text{Det}[\Lambda^\mu_\nu] = +4$</td>
</tr>
</tbody>
</table>

General Lorentz Identity Transform: $\Lambda^\mu_\nu \rightarrow \eta^{\mu}_{\nu} = \delta^{\mu}_{\nu} = I(4)$ (Proper, symmetric, discrete)

SR: Lorentz Transform $\Lambda^\mu_\nu = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\forall \mu, \nu$, $\partial_{\nu}[\mathcal{X}^\mu] = \mathcal{X}_\nu^\mu = \delta^\mu_\nu$ 

Both the Parity-Inversion (P) and Time-Reversal (T) have a Determinant of -1, which is an improper transform. However, combinations (PP), (TT), (PT) have overall Determinant of +1, which is proper. 

Classical SR Time Reversal neglects spin and charge. When included, there is also a Charge-Conjugation(C) transform. Then one gets (CC),(PP),(TT),(PT),(PT),(PT), & permutations of (CPT) transforms all leading back to the Identity ($I_4$).

Note that the Trace of Discrete Lorentz Transforms goes in steps from {-4,-2,2,4}. As we will see in a bit, this is a major hint for SR antimatter and CPT Symmetry. 

$\text{Trace}[\Lambda^\mu_\nu] = \eta_{\mu\nu} \Lambda^\mu_\nu = T^\mu_\nu = T^\nu_\mu$ 

$V\cdot V = V^\mu V^\nu = \left((V^\mu)^2 - V^2\right) = (V^\nu)^2 = \eta_{\mu\nu} V^\mu V^\nu$ 

$= \text{Lorentz Scalar}$
SRQM Lorentz Transforms $\Lambda_{\nu}^{\mu'} = \partial_{\nu}[X^{\mu'}]$

Discrete & Fixed Rotation → Particle Exchange

Lorentz Coordinate-Flip Transforms

SR 4-Vector $\nu \rightarrow F_{\nu}$

$\nu \rightarrow F_{\nu}$ = (0,1)-Tensor $V_{\nu}$, Lorentz Parity Transform

SR 4-CoVector:One Form $\psi^{\nu}$

$\psi^{\nu}$ = (1,0)-Tensor $V^{\nu}$, Lorentz Coordinate-Flip Transform

SR 4-Scalar $\psi$

$\psi$ = (0,0)-Tensor, Lorentz Scalar

SR 4-Tensor

$T_{\mu\nu}$, Lorentz Coordinate-Flip Transform

SR 4-Scalar

$\eta_{\nu} = \delta_{\nu 0}$

$\text{Trace}[T_{\mu\nu}] = \eta_{\nu} T_{\mu\nu} = T_{\mu\nu} = T$

$V \cdot V = V^{\nu} \eta_{\nu} V^{\mu} = (V^{\nu})^2 - V \cdot V = (V^{\nu})^2$

$= \text{Lorentz Scalar}$

Any single Lorentz Flip Transform is Improper, with a Determinant of -1. However, pairwise combinations are Proper, with a Determinant of +1. All single flips have Trace of +1.

The combination of any two Spatial Flips is the equivalent of a Spatial Rotation by (π) about the associated rotational axis.

Since this is a Proper transform, it is also the equivalent of a particle location exchange.

The combination of all three Spatial Flips, Flip-xyz, gives the Lorentz Parity Transform, which is again Improper, with Trace of -2.

The Flip-t is the standard Lorentz Time-Reversal, Improper.
SR: Lorentz Transform
\[ \partial_v [X^\mu'] = \Lambda^\mu'_v \]
Lorentz Transform Connection Map

- Continuous Various Rotations
- Continuous Various Boosts
- Continuous Rotate-z
- Continuous Rotate-z
- Identity: \( \Lambda^\mu'_v \rightarrow \eta^\mu'_v \rightarrow B^\mu'_v \)
- Tr[\(B^\mu'_v]\) = \{4..Infinity\} \[ Det[\(B^\mu'_v\)] = +1 \]
- Tr[\(\eta^\mu'_v\)] = 4 \[ Det[\(\eta^\mu'_v\)] = +1 \]
- \( \Lambda^\mu'_v \rightarrow R^\mu'_v \) (unitary)
- Tr[\(R^\mu'_v\)] = 2 \[ Det[\(R^\mu'_v\)] = +1 \]
- \( \Lambda^\mu'_v \rightarrow R^\mu'_v \) (anti-unitary)
- Tr[\(F^\mu'_v\)] = 2 \[ Det[\(F^\mu'_v\)] = -1 \]
- Rotations, \( \Lambda^\mu'_v \rightarrow \) Rotations
- Flip-x, \( \Lambda^\mu'_v \rightarrow Fx^\mu'_v \) (x \rightarrow x)
- Flip-y, \( \Lambda^\mu'_v \rightarrow Fy^\mu'_v \) (y \rightarrow y)
- Rotation-z, \( \Lambda^\mu'_v \rightarrow R^\mu'_v \) (3π/2)
- Time-reversal, \( \Lambda^\mu'_v \rightarrow T^\mu'_v \)
- Boost (any Axis), \( \Lambda^\mu'_v \rightarrow B^\mu'_v \)
- t \rightarrow -t^* \[ time parity anti-unitary \]
- Continuous Various Flips
- Continuous Rotate-z
- \( \Lambda^\mu'_v \rightarrow \) Other Axis Flips
- Tr[\(\eta^\mu'_v\)] = 2 \[ Det[\(\eta^\mu'_v\)] = -1 \]
- Tr[\(\eta^\mu'_v\)] = 2 \[ Det[\(\eta^\mu'_v\)] = -1 \]
- \(-\eta^\mu'_v\) \[ Neg Identity \]
SRQM Lorentz Transforms $\Lambda^{\mu'}_{\nu} = \partial_{\nu}[X^{\mu}]$

Lorentz Transform Connection Map – Discrete Transforms
CPT, Big-Bang, Matter$\leftrightarrow$AntiMatter, Arrow(s)-of-Time

Examine all possible combinations of Discrete Lorentz Transformations which are Linear (Determinant of $\pm 1$).

A lot of the standard SR texts only mention (P)arity-Inverse and (T)ime-Reversal. However, there are many others, including (F)lips and (R)otations of a fixed amount. However, the (T)imeReversal and Combo(P)arity(T)ime take one into a separate section of the chart. Taking into account all possible discrete Lorentz Transformations fills in the rest of the chart. The resulting interpretation is that there is CPT Symmetry (Charge:Parity:Time) and Dual TimeSpace (with reversed timeflow). In other words, one can go from the Identity Transform (all +1) to the Negative Identity Transform (all -1) by doing a Combo PT Lorentz Transform or by Negating the Charge (Matter$\leftrightarrow$AntiMatter). The Feynman-Stueckelberg CPT Interpretation (AntiMatter moving spacetime-backward = NormalMatter moving spacetime-forward) aligns with this as a Dual-Universe “AntiMatter” Side.

This is similar to Dirac’s prediction of AntiMatter, but without the formal need of Quantum Mechanics, or Spin. In fact, it is more general than Dirac’s work, which was about the electron. This is from general Lorentz Transforms for any kind of particle: event.

Discrete NormalMatter (NM) Lorentz Transform Type

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
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<td>$+1$</td>
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<td>$+1$</td>
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</table>

Discrete AntiMatter (AM) Lorentz Transform Type

<table>
<thead>
<tr>
<th>$t$</th>
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<tr>
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</tbody>
</table>

Note that the (T)imeReversal and Combo (P)arityInverse & (T)imeReversal take NormalMatter $\leftrightarrow$ AntiMatter.
SRQM Lorentz Transforms $\Lambda_{\mu'}^\nu = \partial_\nu [X_{\mu'}^\nu]$

Lorentz Transform Connection Map – Trace Identification
CPT, Big-Bang, [Matter ↔ AntiMatter], Arrow(s)-of-Time

All Lorentz Transforms have Tensor Invariants: Determinant = ±1 and InnerProduct = 4. However, one can use the Tensor Invariant Trace to Identify CPT Symmetry & AntiMatter

Trace : Determinant
$\text{Tr}[\text{NM-Rotate}] = \{0...+4\}$  $\text{Tr}[\text{NM-Identity}] = +4$  $\text{Tr}[\text{NM-Boost}] = \{+4...+\infty\}$
$\text{Tr}[\text{AM-Rotate}] = \{0...-4\}$  $\text{Tr}[\text{AM-Identity}] = -4$  $\text{Tr}[\text{AM-Boost}] = \{-4...-\infty\}$

Discrete NormalMatter (NM) Lorentz Transform Type
Minkowski-Identity : AM-Flip-xyz=AM-ComboPT
Flip-xy=TimeReversal, Flip-x, Flip-y, Flip-z
AM-Flip-xyz=AM-ParityInverse
AM-Flip-xyz=AM-Rotate-xy(\pi), AM-Flip-xyz=AM-Rotate-xz(\pi), AM-Flip-xyz=AM-Rotate-yz(\pi)

AM-Flip-xy=AM-Rotate-xy(\pi), AM-Flip-xz=AM-Rotate-xz(\pi), AM-Flip-yz=AM-Rotate-yz(\pi)

Discrete AntiMatter (AM) Lorentz Transform Type
AM-Minkowski-Identity : Flip-xyz=ComboPT
"Discrete AntiMatter (AM) Lorentz Transform Type"

Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:

- Trace = Sum (∑) of EigenValues : Determinant = Product (Π) of EigenValues

As 4D Tensors, each Lorentz Transform has 4 EigenValues (EV's).
Create an Anti-Transform which has all EigenValue Tensor Invariants negated.
$\Lambda_{\mu'}^\nu$ = (\Lambda^\nu_\mu')^{-1}$
$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta}$
$\det[\Lambda_{\mu'}^\nu] = \pm 1$
$\text{Tr}[\Lambda_{\mu'}^\nu] = \{-\infty, +\infty\}$

The Trace Invariant identifies a "Dual" Negative-Side for all Lorentz Transforms.
Based on the Lorentz Transform properties of the last few pages, here is an interesting observation about Lorentz Transforms:

They all have Determinant of \(\pm 1\), and Inner Product of \(+4\), but the Trace varies depending on the particular Transform.

The Trace of the Identity is at \(+4\). Assume this applies to normal matter particles.

The Trace of normal matter particle Rotations varies continuously from \(0\) to \(+4\).

The Trace of normal matter particle Boosts varies continuously from \(+4\) to \(+\infty\).

So, one can think of Trace \(= +4\) being the connection point between normal matter Rotations andBoosts.

Now, various Flip Transforms (inc. the Time Reversal and Parity Transforms, and their combination as PT transform), take the Trace in discrete steps from \(-4, -2, 0, +2, +4\). Applying a bit of symmetry:

The Trace of the Negative Identity is at \(-4\). Assume this applies to anti-matter particles.

The Trace of anti-matter particle Rotations varies continuously from \(-4\) to \(-\infty\).

The Trace of anti-matter particle Boosts varies continuously from \(-\infty\) to \(-4\).

So, one can think of Trace \(= -4\) being the connection point between anti-matter Rotations and Boosts.

This observation would be in agreement with the CPT Theorem (Feynman-Stueckelberg) idea that (normal/anti)-matter particles moving backward in SpaceTime are CPT symmetrically equivalent to (anti/normal)-matter particles moving forward in SpaceTime.

Now, scale this up to Universe size: The Baryon Asymmetry problem (aka. The Matter-AntiMatter Asymmetry Problem). If the Universe was created as a huge chunk of energy, and matter-creating energy is always transformed into matter-antimatter mirrored pairs, then where is all the antimatter?

Turns out this is directly related to the Arrow-of-Time Problem as well.

Answer: It is temporally on the “Other/Dual-Side” of the Big-Bang! The antimatter created at the Big-Bang is travelling in the negative-time \((-t)\) direction from the Big-Bang creation point, and the normal matter is travelling in the positive-time direction \((+t)\).

Universal CPT Symmetry. So, what happened “before” the Big-Bang? It “is” the AntiMatter Dual to our normal matter universe!

Pair-production is creation of AM-NM mirrored pairs within SpaceTime. The Big-Bang is the creation of SpaceTime itself.

This also resolves the Arrow-of-Time Problem. If all known physical microscopic processes are time-symmetric, why is the flow of Time experienced as uni-directional?? (see Wikipedia “CPT Symmetry”, “CP Violation”, “Andrei Sakharov”)

**Answer:** Time flow on This-Side of the Universe is \((+t)\) direction, while time flow on the Dual-Side of the Universe is \((-t)\) direction. The math all works out. Time flow is bi-directional, but on opposite sides of the Big-Bang! Universal CPT Symmetry.

This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=AntiMatter, NM=NormalMatter):

- Trace Various (AM_Flips):
  - Trace Various (NM_Flips)
  - Infinity...AM_Boosts...AM_Identity...-4...AM_Rotations...0...NM_Rotations...+4...NM_Identity...NM_Boosts...+Infinity

**SR:Lorentz Transform**

\[ \partial_v[R^\mu] = \partial R^\mu/\partial R^v = \Lambda^\mu_v \]

\[ \Lambda^\mu_v = (\Lambda^{-1})^\nu_{\alpha} : \Lambda^\mu_{\alpha} \Lambda^\alpha_v = \eta^\mu_v = \delta^\mu_v \]

\[ \eta_{\mu\alpha} \Lambda^\alpha_{\beta} = n_{\delta\beta} \]

\[ \text{Det}[\Lambda_{\mu\nu}] = \pm 1 \]

\[ \text{Tr}[\Lambda^\mu_{\nu}] = \{ -\infty \ldots +\infty \} \]

\( = \text{Lorentz Transform Type} \)
This idea of Universal CPT Symmetry also gives a Universal Dimensional Symmetry as well.

Consider the well-known “balloon” analogy of the universe expansion. The “spatial” coordinates are on the surface of the balloon, and the expansion is in the (+t) direction. There is symmetry in the (+/-) directions of the spatial coordinates, but the time flow is always uni-directional, (+t), as the balloon gets bigger → inflates.

By allowing a “Dual-Side”, it provides a universal dimensional symmetry. One now has (+/-) symmetry for the temporal directions.

The “center” of the Universe is, literally, the Big Bang Singularity. It is the “center = zero” point of both time and space directions.

The expansion gives time-flow always AWAY FROM the Big Bang singularity in both the Normal-Side (+t) and the Dual-Side (-t).

Note that this gives an unusual interpretation of what came “before” the Big Bang. The “past” on either side extends only to the BB singularity, not beyond. Time flow is always away from this creation singularity.

This is also in accord with known black hole physics, in that all matter entering a BH event horizon ends at the BH singularity. Time and space coordinates both come to a stop at either type of singularity, from the point of view of an observer that is in the spacetime but not at one of these singularities.

So, the Big Bang is a “starting” singularity, and black holes are “ending” singularities. This also provides for idea of “white holes” actually just being black holes on the Dual-Side. White hole = time-reversed black hole. Always confusion about stuff coming out. This way, the mass is still attractive. Time-flow is simply reversed on the alternate side so stuff still goes INTO the hole… which makes way more sense than stuff that can only come out of the “massive=attractive” white-hole.

So, Universal CPT Symmetry = Universal Dimensional Symmetry.

And, going even further, I suspect this is the reason there is a duality in Metric conventions.

In other words, physicists have wondered why one can use Metric signature {+, -,-,-} or {−, +, +, +}.

I submit that one of these metrics applies to the Normal Matter side, while the other complementarily applies to the Dual side. This would allow correct causality conditions to apply on either side. Again, this is similar to the Dirac prediction of antimatter based on a duality of possible solutions.

This gives total CPT Symmetry to all of the possible Lorentz Transforms (AM=AntiMatter, NM=NormalMatter):

\[
\text{Trace Various (AM_Flips)} : \text{Trace Various (NM_Flips)}
\]

\[-\text{Infinity...}(AM_{Boosts})...(AM_{Identity}=4)...(AM_{Rotations})...0...(NM_{Rotations})...(4+NM_{Identity})...(NM_{Boosts})...+\text{Infinity}\]

This solves the: Baryon (Matter↔AntiMatter) Asymmetry Problem & Arrow(s)-of-Time Problem (+t / -t )
A Klein geometry is a pair (G,H) where G is a Lie group and H is a closed Lie subgroup of G such that the (left) coset space X = G/H is connected.

G acts transitively on the homogeneous space X.

We may think of H\cdot G as the stabilizer subgroup of a point in X.
Lie Groups

de Sitter Group SO(1,4)
de Sitter invariant relativity
(?maybe?)

Poincaré Group ISO(1,3)
\{ r << r_{ds} = \text{de Sitter Radius} \}
\quad r_{ds} = \sqrt{3/\Lambda} = L_d/\sqrt{\Omega\Lambda}

SR & GR Physics
(** currently thought correct **)
SRQM Transforms: Venn Diagram

**Poincaré = Lorentz + Translations**

(10) (6) (4)

**Lorentz Transform** \( \Lambda^{\mu\nu} \)

4-Tensor \{mixed type-(1,1)}

**Translation Transform** \( \Delta X^{\mu} \)

1+3=4

**Discrete**

Time-reversal \( \Lambda^{\mu\nu}_{v} \rightarrow T^{\mu\nu}_{v} \)

(0)

t \rightarrow -t*

time parity anti-unitary

Parity-Inversion \( \Lambda^{\mu\nu}_{v} \rightarrow P^{\mu\nu}_{v} \)

(0)

r \rightarrow -r

space parity unitary

Charge-Conjugation \( \Lambda^{\mu\nu}_{v} \rightarrow C^{\mu\nu}_{v} \)

(0)

\( R \rightarrow -R^* \), q \rightarrow -q

charge parity anti-unitary

CPT Symmetry \{Charge\}

\{Parity\}

\{Time\}

**Continuous**

Rotation \( \Lambda^{\mu\nu}_{v} \rightarrow R^{\mu\nu}_{v} \)

(0)

\{x|y|z \} \rightarrow \{-x|y|z\}

Identity \( I_{4(0)} \)

\( \Lambda^{\mu\nu}_{v} \rightarrow \eta^{\mu\nu}_{v} = \delta^{\mu\nu}_{v} \)

no mixing unitary

Boost \( \Lambda^{\mu\nu}_{v} \rightarrow B^{\mu\nu}_{v} \)

(3)

tx | ty | tz

no motion

Isotropy \{same all directions\}

Temporal

\( \Delta X^{\mu} \rightarrow (c \Delta t, 0) \)

(1)

\( \Delta t \)

Spatial

\( \Delta X^{\mu} \rightarrow (0, \Delta x) \)

(3)

\( \Delta x | \Delta y | \Delta z \)

**Discrete**

Temporal

\( \Delta X^{\mu} \rightarrow (0,0) \)

(0)

no motion

Spatial

**Continuous**

Homogeneity

\{same all points\}

Det[\( \Lambda^{\mu\nu}_{v} \)] = +1 for Proper Lorentz Transforms

Det[\( \Lambda^{\mu\nu}_{v} \)] = -1 for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix M with Tr[M]=0 which gives:

\{ \( \Lambda = e^\hat{M} = e^\hat{(+\theta J - \zeta K)} \) \}

\{ \( \Lambda^T = (e^\hat{M})^T = e^\hat{-M} \) \}

\{ \( \Lambda^{-1} = (e^\hat{M})^{-1} = e^\hat{-M} \) \}

Rotations \( J_\mu = -\epsilon_{\alpha\beta\gamma\delta} M^{\alpha\mu}/2 \), Boosts \( K_\mu = M^{\mu\gamma}* \)

\( \{ (R \rightarrow -R^*) \} \) or \( (t \rightarrow -t^*) \) & \( (r \rightarrow -r) \) imply \( q \rightarrow -q \)

Feynman-Stueckelberg Interpretation

Amusingly, Inhomogeneous Lorentz adds homogeneity.

SR: Lorentz Transform

\[ \partial[R^{\mu\nu}_v] = \partial R^{\mu\nu}_v / \partial R^{\mu\nu}_v = \Lambda^{\mu\nu}_v \]

\( \Lambda^{\mu\nu}_v = (\Lambda^{-1})_\alpha^\beta \Lambda^{\alpha\beta}_v = \eta^{\mu\nu}_v = \delta^{\mu\nu}_v \)

\[ \eta_{\mu\nu} \Lambda^\alpha_{\nu} \Lambda^\beta_{\mu} = \eta_{\mu\nu} \Lambda^\alpha_{\nu} \Lambda^{\alpha\beta}_v \]

\[ Det[\Lambda^{\mu\nu}_v] = \pm 1 \]

\[ \Lambda^{\mu\nu}_v \Lambda^{\mu\nu}_v = 4 \]

\[ \Delta[\Lambda^{\mu\nu}_v] = \pm 1 \]

\[ \Lambda^{\mu\nu}_v \Lambda^{\mu\nu}_v = 4 \]
Review of SR Transforms

10 Poincaré Symmetries, 10 Conservation Laws

10 Generators: Noether’s Theorem

Lagrange “Shift Operator” version of Taylor’s Theorem: \( e^{\text{Shift}\, f(x)}=f(x+a) \)

Bloch Theorem: Translation Operator: \( e^{i\mathbf{K}\cdot\mathbf{r}} \), with \( \mathbf{K} \) as reciprocal lattice

Conservation of relativistic mass-moment 3-vector

Conservation of angular-momentum (3 + 3) = (6) Laws

Conservation of angular-momentum (3 + 3)

General Linear,Affine Transform \( \mathbf{X}' = \Lambda \mathbf{X} + \Delta \mathbf{X} \) with \( \det[\Lambda] = \pm 1 \)

4-AngularMomentum Generator

\( \Delta \mathbf{X} \rightarrow (\mathbf{c}\Delta t, \mathbf{A}) \)

Generated by energy \( E = cp^0 \)

\( \Delta \mathbf{X} \rightarrow (0, \Delta x) \)

Generated by 3-momentum \( \mathbf{p} = p' \)

4-Position \( \mathbf{X} = (ct, \mathbf{x}) \)

\( \mathbf{U} \cdot \partial [\mathbf{\ldots}] = \partial \mathbf{U} / \partial [\mathbf{\ldots}] \)

\( \mathbf{d}/dt [\mathbf{\ldots}] \)

\( \mathbf{U} = (c, \mathbf{u}) \)

\( \mathbf{d}/d\tau [\mathbf{\ldots}] \)

\( \mathbf{d}/d\tau [\mathbf{\ldots}] \)

\( \gamma d/dt [\mathbf{\ldots}] \)

4-AngularMomentum Tensor

\( \mathbf{M}^{\mu\nu} = \mathbf{X}^\mu \mathbf{P}^\nu - \mathbf{X}^\nu \mathbf{P}^\mu = \mathbf{X} \wedge \mathbf{P} \)

\( \Lambda^\mu \mathbf{P}^\nu - \mathbf{X}^\nu \mathbf{P}^\mu = \mathbf{X} \wedge \mathbf{P} \)

\( \Lambda^\mu \mathbf{P}^\nu = \mathbf{X} \wedge \mathbf{P} \)

\( \mathbf{P}^{\mu} = \mathbf{P} = (mc, \mathbf{p}) = (E/c, \mathbf{p}) \)

\( \mathbf{M}^{\mu\nu} + \mathbf{P}^{\mu} = \mathbf{X} \wedge \mathbf{P} = \mathbf{X}^\mu \mathbf{P}^\nu - \mathbf{X}^\nu \mathbf{P}^\mu = \mathbf{X} \wedge \mathbf{P} \)

\( \text{Jacobi’s Formula for Complex Square Matrix A:} \ 
\det(\text{Exp}[A])=\text{Exp}(\text{Tr}[A]) \)

\( \det(A)_{ij} = \begin{vmatrix} (tr A) - 6 tr(A^2) (tr A)^2 + 3 (tr A^2)^2 + 8 tr(A^3) tr A - 6 tr(A^4) \end{vmatrix} / 24 \)
Review of SR Transforms

Poincaré Algebra & Generators

Casimir Invariants

The (10) one-parameter groups can be expressed directly as exponentials of the generators:

\[ U(l, (a^0, 0)) = e^{a^0 l - i a^0 \cdot p} \]

These are Operators that commute with all of the Poincaré Generators

The Poincaré Algebra is the Lie Algebra of the Poincaré Group:

The (10) one-parameter groups can be expressed directly as exponentials of the generators:

\[ \phi^\Lambda(\lambda, 0, v) = e^{i\lambda \phi^\Lambda} \]

Total of (1+3+3+3 = (1+3)+(3+3) = 4+6 = 10) Invariances from Poincaré Symmetry

Covariant form:

These are the commutators of the poincaré Algebra :

\[ [X^\mu, X^\nu] = 0_{\mu\nu} \]

\[ [P^\mu, P^\nu] = -i\hbar q(F^{\mu\nu}) \]

\[ [M^\mu, P^\nu] = \eta^{\mu\nu}P^\rho - \eta^{\nu\rho}P^\mu \]

\[ [M^\mu, M^\nu] = i\hbar(\eta^{\mu\nu}M^\rho + \eta^{\nu\rho}M^\mu + \eta^{\rho\mu}M^\nu) \]

Component form: Rotations J = -ε_mnM^\mu\nu/2, Boosts K = M_o

\[ J_{mn}, P_{mn} = i(\epsilon_{mn})P^\rho \]

\[ J_{mn}, P_{mn} = 0 \]

\[ K_{mn}, P_{mn} = i\hbar qP^\rho \]

\[ K_{mn}, P_{mn} = 0 \]

\[ K_{mn}, J_{mn} = i(\epsilon_{mn})J^\rho \]

\[ K_{mn}, J_{mn} = 0 \]

\[ K_{mn}, K_{mn} = -i(\epsilon_{mn})J^\rho \]

A Wigner Rotation resulting from consecutive boosts

\[ J_{m} + iK_{m}, J_{n} - iK_{n} = 0 \]

Poincaré Algebra has 2 Casimir Invariants = Operators that commute with all of the Poincaré Generators

These are \( P^2 = P^\mu P_\mu = (mc)^2 \), \( W^2 = W^\mu W_\mu = -(mc)^2(j + 1) \), and \( W^\mu = (-1/2)\epsilon^{\mu\nu\rho\sigma}J_\nu P_\rho \) as the Pauli-Lubanski Pseudovector

\[ P^2 = P^\mu P_\mu = [P^2, J^2] = [P^2, K^2] = 0 \]

Total of (1+3+3+3 = (1+3)+(3+3) = 4+6 = 10) Invariances from Poincaré Symmetry

The Poincaré Algebra is the Lie Algebra of the Poincaré Group:
SR Study:

10 Poincaré Symmetry Invariances

Noether’s Theorem: 10 SR Conservation Laws

d’Alembertian Invariant Wave Equation: \( \partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla = (\partial/c)^2 \)

Time Translation:
Let \( X_T = (ct + \Delta t, x) \), then \( \partial [X_T] = (\partial/c, -\nabla)(ct + c\Delta t, x) = \text{Diag}[1, -1] = \partial[X] = \eta^{\mu\nu} \)
so \( \partial [X_T] = \partial[X] \) and \( \partial[K] = [0] \)
\( (\partial \partial)[K \cdot X] = \partial(\partial[K \cdot X]) = \partial[K] \cdot X + K \cdot \partial[X] = \partial(\partial[K \cdot X]) = (\partial \partial)[K \cdot X] \)

Space Translation:
Let \( X_S = (ct, x + \Delta x) \), then \( \partial [X_S] = (\partial/c, -\nabla)(ct, x + \Delta x) = \text{Diag}[1, -1] = \partial[X] = \eta^{\mu\nu} \)
so \( \partial [X_S] = \partial[X] \) and \( \partial[K] = [0] \)
\( (\partial \partial)[K \cdot X_S] = \partial(\partial[K \cdot X_S]) = \partial[K] \cdot X_S + K \cdot \partial[X_S] = 0 + K \cdot \partial[X] = \partial[K] \cdot X + K \cdot \partial[X] = \partial(\partial[K \cdot X]) = (\partial \partial)[K \cdot X] \)

Lorentz Space-Space Rotation:
Let \( X_R = (ct, R[x]) \), then \( \partial [X_R] = (\partial/c, -\nabla)(ct, R[x]) = \text{Diag}[1, -1] = \partial[X] = \eta^{\mu\nu} \)
so \( \partial [X_R] = \partial[X] \) and \( \partial[K] = [0] \)
\( (\partial \partial)[K \cdot X_R] = \partial(\partial[K \cdot X_R]) = \partial[K] \cdot X_R + K \cdot \partial[X_R] = 0 + K \cdot \partial[X] = \partial[K] \cdot X + K \cdot \partial[X] = \partial(\partial[K \cdot X]) = (\partial \partial)[K \cdot X] \)

Lorentz Time-Space Boost:
Let \( X_B = (\gamma ct - \beta x, \beta ct + x) \), then \( \partial [X_B] = (\partial/c, -\nabla)(\gamma ct - \beta x, \beta ct + x) = [\gamma y, -\gamma \beta, y] = \Lambda^{\mu\nu} \)
\( \partial[K \cdot X_B] = \partial[K] \cdot X_B + K \cdot \partial[X_B] = \Lambda^{\mu\nu} K = K_B = \text{a Lorentz Boosted } K, \text{ as expected} \)
\( \partial[K_B] = \partial(\partial[K \cdot X]) = (\partial \partial)[K \cdot X] \)

SR Waves:
Let \( \Psi = \psi^\phi \cdot (i[K \cdot X]), \psi_\perp = \psi^\phi \cdot (i[K \cdot X]) \), \( \Psi_S = \psi^\phi \cdot (i[K \cdot X_S]) \), \( \Psi_R = \psi^\phi \cdot (i[K \cdot X_R]) \), \( \Psi_B = \psi^\phi \cdot (i[K \cdot X_B]) \)
\( (\partial \partial)[K \cdot X] = (\partial \partial)[K \cdot X_S] = (\partial \partial)[K \cdot X_R] = (\partial \partial)[K \cdot X_B] \): Wave Equation Invariant under all Poincaré transforms

Total of \((1+3+3+3) = 10\) Invariances from Poincaré Symmetry
There are at least three 4-Vector relations which use the Exterior (Wedge=\^) Product.

\[ \partial^\wedge A = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} : \text{the Faraday EM 4-Tensor} \]

\[ R^P = R^{\mu}P^{\nu} = R^{\mu}P^{\nu} - R^{\nu}P^{\mu} = M^{\mu\nu} : \text{the 4-Angular-Momentum} \]

\[ R^F = R^{\mu}F^{\nu} = R^{\mu}F^{\nu} - R^{\nu}F^{\mu} = \Gamma^{\mu\nu} : \text{the 4-(Angular-)Torque} \]

This gives the components of each remarkably similar properties.

Likewise, each of these has a physical (Dot=\cdot) Product relation as well.

\[ \partial A = \partial_\nu A^\mu = 0 : \text{the Lorenz Gauge, a conservation of 4-EMVectorPotential} \]

\[ R \cdot P = R^{\mu}P^{\mu} = -S_{\text{action,free}} : \text{the Action Scalar} \]

\[ R \cdot F = R^{\mu}F^{\mu} = ???? : \text{probably something important} \]
SRQM Study: 4-Momentum → 4-Force

4-AngularMomentum → 4-Torque

Linear:
4-Force is the ProperTime Derivative of 4-Momentum.

Angular:
4-Torque is the ProperTime Derivative of 4-AngularMomentum.

\[
\frac{d}{d\tau}[M^{\mu\nu}] = \frac{d}{d\tau}[X^\mu P^\nu - X^\nu P^\mu] = \left[ U^\mu P^\nu + X^\nu F^\mu - U^\nu P^\mu - X^\mu F^\nu \right]
\]

\[
\frac{d}{d\tau}[P^\mu] = \gamma \frac{d}{d\tau}[E/c^2, \mathbf{v}]
\]

\[
\frac{d}{d\tau}[M^{\mu\nu}] = \Gamma^{\mu\nu} = \left[ X^\mu F^\nu - X^\nu F^\mu \right]
\]

\[
\frac{d}{d\tau}[P^\mu] = (mc, \mathbf{p}) = (Ec, \mathbf{p})
\]

\[
\frac{d}{d\tau}[\mathbf{F}] = \gamma (\dot{\mathbf{E}}/c + \dot{\mathbf{p}})
\]
All \{4-Vectors:4-Tensors\} have an associated \{Lorentz Scalar Product:Trace\}

Each 4-Vector has a “magnitude” given by taking the Lorentz Scalar Product of itself.

\[ V \cdot V = V^\mu \eta_{\mu \nu} V^\nu = V_0^2 + v_1^2 + v_2^2 + v_3^2 = (v^0 v^0 - \mathbf{v} \cdot \mathbf{v}) = (\mathbf{v}^0)^2 \]

The absolute magnitude of \( V \) is \( \sqrt{|V \cdot V|} \)

Each 4-Tensor has a “magnitude” given by taking the Tensor Trace of itself.

\[ \text{Trace}[T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T_0^0 - T_1^1 - T_2^2 - T_3^3 \]

Note that the Trace runs down the diagonal of the 4-Tensor.

Notice the similarities. In both cases there is a tensor contraction with the Minkowski Metric Tensor \( \eta_{\mu \nu} \rightarrow \text{Diag}[+1,-1,-1,-1] \) \( \{ \text{Cartesian basis} \} \)

\[ \text{ex. } P \cdot P = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 = (m_0 c)^2 \]

which says that the “magnitude” of the 4-Momentum is the RestEnergy/c = RestMass*c

\[ \text{ex. Trace}[\eta^\mu_\nu] = (\eta^{00} - \eta^{11} - \eta^{22} - \eta^{33}) = 1 -(-1) -(-1) -(-1) = 1+1+1+1 = 4 \]

which says that the “magnitude” of the Minkowski Metric = SpaceTime Dimension = 4
Some other SR Invariants include:

- **Lorentz Scalar Invariant**
  
  \[ V \cdot V = V^\mu V_\mu = (v^0)^2 \]

- **4-Momentum**
  
  \[ P = (mc,p) = (E/c,p) \]

- **Phase Space Invariant**
  
  \[ d^4p/E \]

- **Rest Volume**
  
  \[ V_o = \int d^4V = \int \gamma d^4x = -cN/\int dT \cdot N \]

- **d^4X**
  
  \[ d^4X = c dt \cdot dx \cdot dy \cdot dz = c \gamma d^4x \]

- **d^4P**
  
  \[ d^4P = (dE/c) dp^x \cdot dp^y \cdot dp^z = (dE/c) d^3p \]

- **d^4K**
  
  \[ d^4K = (d\omega/c) dk^x \cdot dk^y \cdot dk^z = (d\omega/c) d^3k \]

- **Particle #**
  
  \[ N = (-V_o/c) \int dT \cdot N = \int nd^3x = \int \gamma n_o d^3x \rightarrow n_o V_o \]

- **EM Charge**
  
  \[ Q = (-V_o/c) \int dT \cdot J = \int \rho d^3x = \int \gamma \rho_o d^3x \rightarrow \rho_o V_o \]

- **d^3p d^3x**
  
  \[ d^3p d^3x = dp^x \cdot dp^y \cdot dp^z \cdot dx \cdot dy \cdot dz \]

- **d^3k d^3x**
  
  \[ d^3k d^3x = dk^x \cdot dk^y \cdot dk^z \cdot dx \cdot dy \cdot dz \]

- **SR 4-Tensor**
  
  \[ (2,0)\text{-Tensor } T^{\mu
\nu} \]

- **SR 4-Vector**
  
  \[ (1,0)\text{-Tensor } V = (v^0,v) \]

- **SR 4-CoVector: OneForm**
  
  \[ (0,1)\text{-Tensor } V_\mu = (v_0\cdot v) \]

- **SR 4-Scalar**
  
  \[ (0,0)\text{-Tensor } S \text{ or } S_\alpha \]

- **SR 4-Vector & 4-Tensors**

- **SR 4-Vector-based Invariants**

- **Lorentz Scalar**
  
  \[ \eta_{\mu\nu} T^{\mu\nu} = T_{\mu} = T \]

- **Trace**
  
  \[ \text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu} T^{\mu\nu} = (v^0)^2 = \text{Lorentz Scalar} \]
Some 4-Vectors have an alternate form of Tensor Invariant: \((d\mathbf{v}'/v_0') = (d\mathbf{v}/v_0)\) or \((d^3\mathbf{v}'/v_0') = (d^3\mathbf{v}/v_0)\) in addition to the standard Lorentz Invariant \(V \cdot V = V_\mu V_\mu = (v_0 v_0 - \mathbf{v} \cdot \mathbf{v}) = (v_0^2)\)

If \(V \cdot V = \text{constant}\): with \(V = (v_0, \mathbf{v})\)
then \(d(V \cdot V) = 2(\mathbf{V} \cdot d\mathbf{V}) = d(\text{constant}) = 0\)
hence \((\mathbf{V} \cdot d\mathbf{V}) = 0 = v_0 dv_0 - \mathbf{v} \cdot d\mathbf{v}\)

Generally: with \(\Lambda = \Lambda_{\nu}^\mu\), Lorentz Boost Transform in the \(\beta\)-direction
\(\mathbf{V}' = \Lambda \mathbf{V}\) : from which the temporal component \(v_0' = (\gamma v_0 - \gamma \beta \mathbf{v})\)
\(d\mathbf{V}' = \Lambda d\mathbf{V}\) : from which the spatial component \(d\mathbf{v}' = (\gamma d\mathbf{v} - \gamma \beta d\mathbf{v}')\)

Combining:
\(d\mathbf{v}' = (\gamma d\mathbf{v} - \gamma \beta (\mathbf{v} \cdot d\mathbf{v}/v_0'))\)
\(d\mathbf{v}' = (1/v_0') (v_0' d\mathbf{v} - \gamma \beta (\mathbf{v} \cdot d\mathbf{v}))\)
\(d\mathbf{v}' = (1/v_0') (v_0' d\mathbf{v} - \gamma \beta (\mathbf{v} \cdot d\mathbf{v}))\)
\(d\mathbf{v}' = (\gamma v_0' - \gamma \beta \mathbf{v}) (1/v_0') d\mathbf{v}\)
\(d\mathbf{v}' = (v_0' /v_0) d\mathbf{v}\)
\(d\mathbf{v}'/v_0' = d\mathbf{v}/v_0\) Invariant of \(\mathbf{V}\) for \(\mathbf{V} \cdot \mathbf{V}\) = (constant)

So, for example:
\(P \cdot P = (mc)^2 = \text{constant}\)

Thus, \(d\mathbf{p}'/(E'/c) = dp/(E/c)\) Invariant
Or: \(d\mathbf{p}'/E' = dp/E \rightarrow d^3p/E = dp^4dp'/E = \text{Invariant}, \text{usually seen as} \int F(\text{various invariants}) d^3p/E = \text{Invariant}\)
Invariant $d^4X = -(V_o) dT \cdot dX = -(dV_o) T \cdot dX = dV_o c dt \cdot dx \cdot dy \cdot dz$

The 4D Position coords that are integrated to give a 4D volume: SI units $[m^4]$

4-Differential $dX = (cdt, dx)$; $dR = (cdt, dr)$

4-UnitTemporal $T = \gamma(1, \beta) = (\gamma, \gamma \beta)$

4-UnitTemporal Differential $dT = d[(\gamma, \gamma \beta)] = (d\gamma, d\gamma \beta)$

$V = \int dV = \int dx \int dy \int dz = \int \int \int dx \ dy \ dz = \int d3x$

$V = V_o/\gamma = 3D Spatial Volume$:

$3D Spatial Volume Element$

$\gamma = V_o/V$

$\gamma dV = -(V_o)^2 dV c dt \cdot dx \cdot dy \cdot dz = -(dV_o) dT \cdot dX = -(V_o) (d\gamma c dt - d\gamma \beta \cdot dx)$

It is sort of quirky though, that the temporal ($cdt$) comes from the $dT$ part, and the spatial ($d^3x$) comes from the $dX$ part.

SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants

Phase Space Integration

SR 4-Vector $(1,0)$-Tensor $V$ or $V_o = (v_0,v)$

SR 4-CoVector: OneForm $(0,1)$-Tensor $V_\mu = (v_\mu, v)$

SR 4-Scalar $(0,0)$-Tensor $S$ or $S_o$ Lorentz Scalar

SR 4-Tensor $(2,0)$-Tensor $T_{\mu \nu}$

SR 4-CoVector: OneForm $(0,1)$-Tensor $V_\mu = (v_\mu, v)$

SR 4-Scalar $(0,0)$-Tensor $S$ or $S_o$ Lorentz Scalar

Trace $[T_{\mu \nu}] = \eta_{\mu \nu} T_{\mu \nu} = T_{\mu \nu} = T$

$Lorentz \ Scalar$

http://scirealm.org/SRQM.pdf

SciRealm.org
John B. Wilson
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More 4-Vector-based Invariants

**Phase Space Integration**

- **4-Velocity:** 
  \[ \mathbf{U} = \mathbf{V} = (c, 0, 0, 0) \]
  \[ \mathbf{U} = \gamma (c, \mathbf{u}) \]

- **4-Charge Flux:** 
  \[ \mathbf{J} = \gamma (c, \mathbf{J}) \]

- **4-Density Flux:** 
  \[ \mathbf{N} = \gamma (c, \mathbf{N}) \]

**Tensor Invariants**

- **Trace:**
  \[ \text{Trace}[\mathbf{T}] = \eta_{\mu \nu} T^{\mu \nu} = T^\gamma_{\gamma} = T \]

- **Lorentz Scalar:**
  \[ V \cdot V = V^0 V^0 = [ (v^0)^2 - \mathbf{v} \cdot \mathbf{v} ] = (v^0)^2 \]
  \[ = \text{Lorentz Scalar} \]

**Physical Vectors & Tensors**

- **4-Vector SRQM Interpretation of QM**

**4-Vectors & 4-Tensors**

- **SR 4-Vectors & 4-Tensors**

- **SR 4-Vector & 4-Scalar**

- **SR 4-Vector One Form**

**A Tensor Study of Physical 4-Vectors**

- Lorentz Scalar Invariant:
  \[ \eta_{\mu \nu} = \text{Lorentz Scalar} \]

- Lorentz Scalar Invariant:
  \[ \gamma (c, \mathbf{u}) \]

- Lorentz Scalar Invariant:
  \[ \gamma (c, \mathbf{J}) \]

- Lorentz Scalar Invariant:
  \[ \gamma (c, \mathbf{N}) \]

**Total Charge**

\[ Q = \int_\mathbb{V} \rho d^3x = \int_\mathbb{V} \rho d^3x = \text{Lorentz Scalar Invariant} \]

**Total Particle #**

\[ N = \int_\mathbb{V} n d^3x = \text{Lorentz Scalar Invariant} \]

**Total Rest Volume**

\[ V_0 = \int_\mathbb{V} d^3x = \text{Lorentz Scalar Invariant} \]

This also gives an alternate way to define the RestVolume Invariant \( V_0 \).

\[ (-V/c) dT \cdot J = \text{Lorentz Scalar Invariant} \]

\[ N = (n/c) = n(c, u) = n_0 U \]

\[ Q = (-V/c) dT \cdot N = \int_\mathbb{V} (n/c)d^3x \rightarrow n_0 V_0 \]

\[ V_0 = \int_\mathbb{V} d^3x = -cN/(dT \cdot N) = -cQ/(dT \cdot J) \]

\[ \text{Trace}[T] = \eta_{\mu \nu} T^{\mu \nu} = T^\gamma_{\gamma} = T \]

\[ V \cdot V = V^0 V^0 = [ (v^0)^2 - \mathbf{v} \cdot \mathbf{v} ] = (v^0)^2 \]

\[ = \text{Lorentz Scalar} \]
The 4D Momentum coords that are integrated to give a 4D Momentum Volume: SI Units \([(\text{kg} \cdot \text{m/s})^4]\)

The 4D Wavevector coords that are integrated to give a 4D Wavevector Volume: SI Units \([(1/\text{m})^4]\)

4-UnitTemporalDifferential \(dT = (d[\gamma], d[\gamma \beta])\)

4-MomentumDifferential \(dP = dP'' = (dE/c, dp)\)

\(\int d^4P = \left(\frac{dE}{c}\right) \int dp_x \int dp_y \int dp_z = \left(\frac{dE}{c}\right) \int d^3p\)
SR 4-Vectors & 4-Tensors

More 4-Vector-based Invariants
Phase Space Integration

\[ d^3p \ d^3x = (V_{Po})dT \cdot (-V_o)dT = (-V_o)(V_{Po})dT \cdot dT \]
\[ d^3k \ d^3x = (V_{po})dT \cdot (-V_o)dT = (-V_o)(V_{po})dT \cdot dT \]

4-UnitTemporal \( T = \gamma (1, \beta) = (\gamma, \gamma \beta) \)

4-UnitTemporalDifferential \( dT = d[(\gamma, \gamma \beta)] = (d[\gamma], d[\gamma \beta]) \)

\[ (V_{po})dT \cdot (-V_o)dT = \text{Invariant} \]
\[ (V_{po})dT \cdot (-V_o)dT = \int F[\text{various Invariants}] d^3p \ d^3x \]

Likewise, \( d^3k \ d^3x = \text{Invariant} \)

\[ \int F[\text{various Invariants}] d^3k \ d^3x \]

SR 4-Tensor

(2,0)-Tensor \( T^{\mu}_{\nu} \), or \( T^\mu_\nu \)

(1,1)-Tensor \( V^\mu = (v^0, v^\mu) \)

(0,2)-Tensor \( T_{\mu \nu} \)

SR 4-Vector

(1,0)-Tensor \( V^\mu = (v^0, v^\mu) \)

SR 4-CoVector:OneForm

(0,1)-Tensor \( V^\mu = (v^0, -v^\mu) \)

SR 4-Scalar

(0,0)-Tensor \( S \) or \( S_0 \)

Lorentz Scalar

\[ \text{Lorentz Scalar} = \sqrt{-g} \]

\[ \text{SR} \rightarrow \text{QM} \]

Physics

A Tensor Study

of Physical 4-Vectors

SRQM Interpretation of QM

SciRealm.org

John B. Wilson
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http://scirealm.org/SRQM.pdf
SRQM Study: SR 4-Tensor Properties

**General → Symmetric & Anti-Symmetric**

Any SR Tensor $T^{\mu\nu} = (S^{\mu\nu} + A^{\mu\nu})$ can be decomposed into parts:

- **Symmetric**: $S^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2$ with $S^{\mu\nu} = +S^{\nu\mu}$
- **Anti-Symmetric**: $A^{\mu\nu} = (T^{\mu\nu} - T^{\nu\mu})/2$ with $A^{\mu\nu} = -A^{\nu\mu}$

\[ S^{\mu\nu} + A^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2 + (T^{\mu\nu} - T^{\nu\mu})/2 = T^{\mu\nu}/2 + T^{\nu\mu}/2 + T^{\nu\mu}/2 - T^{\nu\mu}/2 = T^{\mu\nu} + 0 = T^{\mu\nu} \]

**Symmetric 4-Tensor**

\[ S^{\mu\nu} = \begin{bmatrix} S^{00}, S^{01}, S^{02}, S^{03} \\ S^{10}, S^{11}, S^{12}, S^{13} \\ S^{20}, S^{21}, S^{22}, S^{23} \\ S^{30}, S^{31}, S^{32}, S^{33} \end{bmatrix} \]

- Max 16 possible
- Independent components: \( \{ 4^2 = 16 = 10 + 6 \} \)

**Anti-Symmetric 4-Tensor**

\[ A^{\mu\nu} = \begin{bmatrix} A^{00}, A^{01}, A^{02}, A^{03} \\ A^{10}, A^{11}, A^{12}, A^{13} \\ A^{20}, A^{21}, A^{22}, A^{23} \\ A^{30}, A^{31}, A^{32}, A^{33} \end{bmatrix} \]

- Max 6 possible
- Proof:
  - $S^{\mu\nu} A_{\mu\nu} = 0$
  - Physically, the anti-symmetric part contains rotational information and the symmetric part contains information about isotropic scaling and anisotropic shear.

\[ \text{Tr}[S^{\mu\nu}] = S^{\mu\nu} \]
\[ \text{Tr}[A^{\mu\nu}] = 0 \]
### SRQM Study: SR 4-Tensor Properties

**Symmetric → Isotropic & Anisotropic**

Any Symmetric SR Tensor $S^{\mu\nu} = (T_{iso}^{\mu\nu} + T_{aniso}^{\mu\nu})$ can be decomposed into parts:

- **Isotropic** $T_{iso}^{\mu\nu} = \frac{1}{4} \text{Trace}[S^{\mu\nu}] \eta^{\mu\nu} = (T) \eta^{\mu\nu}$
- **Anisotropic** $T_{aniso}^{\mu\nu} = S^{\mu\nu} - T_{iso}^{\mu\nu}$

The Anisotropic part is Traceless by construction, and the Isotropic part has the same Trace as the original Symmetric Tensor. The Minkowski Metric is a symmetric, isotropic 4-tensor with $T=1$.

#### Independent components:

<table>
<thead>
<tr>
<th>Max 10 possible</th>
<th>Max 1 possible</th>
<th>Max 9 possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric 4-Tensor $S^{\mu\nu} =$</td>
<td>Symmetric Isotropic 4-Tensor $T_{iso}^{\mu\nu} =$</td>
<td>Symmetric Anisotropic 4-Tensor $T_{aniso}^{\mu\nu} =$</td>
</tr>
<tr>
<td>$[S^{00}, S^{01}, S^{02}, S^{03}] \quad [S^{00}, S^{01}, S^{02}, S^{03}] \quad [S^{00}, S^{01}, S^{02}, S^{03}]$</td>
<td>$[T, 0, 0, 0] \quad [0, -T, 0, 0] \quad [0, 0, 0, -T]$</td>
<td>$[S^{00} - T, S^{01}, S^{02}, S^{03}] \quad [S^{01}, S^{11} + T, S^{12}, S^{13}] \quad [S^{03}, S^{31}, S^{32}, S^{33} + T]$</td>
</tr>
<tr>
<td>$+S^{02}, S^{11}, S^{12}, S^{13}$</td>
<td>$+T, 0, 0, 0$</td>
<td>$+S^{03}, +S^{12}, S^{22} + T, S^{23}$</td>
</tr>
<tr>
<td>$+S^{03}, +S^{13}, S^{23}, S^{33}$</td>
<td>$\text{with } T = (1/4) \text{Trace}[S^{\mu\nu}]$</td>
<td>$\text{aka Deviatoric}$</td>
</tr>
</tbody>
</table>

$\text{Tr}[S^{\mu\nu}] = S^{\mu\nu} = 4T$

$\text{Tr}[T_{iso}^{\mu\nu}] = 4T$

$\text{Tr}[T_{aniso}^{\mu\nu}] = 0$

Importantly, the Contraction of any Symmetric tensor with any Anti-Symmetric tensor on the same index is always 0.

*Note* These don’t have to be composed from a single general tensor.

**S^\mu_\nu A^\nu_\nu = 0**

**Proof:**

$S^\mu_\nu A^\nu_\nu = S^\mu_\nu A^\nu_\nu$: because we can switch dummy indices

$S^\mu_\nu A^\nu_\nu$: because of symmetry

$S^\mu_\nu A^\nu_\nu$: because of anti-symmetry

$T^\mu_\nu A^\nu_\nu = 0$: because the only solution of $\{c = -c\}$ is 0

Physically, the isotropic part represents a direction independent transformation (e.g., a uniform scaling or uniform pressure); the deviatoric part represents the distortion.

An Isotropic Tensor has the same components in all possible coordinate-frames.

- **Rank 0:** All Scalars are isotropic
- **Rank 1:** There are no non-zero isotropic vectors
- **Rank 2:** Most general isotropic 2nd rank tensor must equal to $\lambda \delta_{\nu}^\mu = \Lambda \eta_{\nu}^\mu$, for some scalar $\Lambda$.
- **Rank 3:** Most general isotropic 3rd rank tensor must equal to $\lambda \epsilon_{\nu_1 \nu_2}^{\mu}$ for some scalar $\lambda$.
- **Rank 4:** Most general isotropic 4th rank tensor must equal to $a_0 \delta_{\nu}^\mu b_0 \delta_{\nu}^\mu c_0 \delta_{\nu}^\mu d_0 \delta_{\nu}^\mu$ for scalars $(a, b, c)$.

$\text{Trace}[T^\mu_\nu] = \eta^\mu_\nu T^\mu_\nu = T^\mu_\nu = T$

$v \cdot v = v^\mu \eta^\mu_\nu v^\nu = ([v^\nu]^2) - v \cdot v = (v \cdot v)^2$

$Lorentz \ Scalar$
SRQM Study: SR 4-Tensors

4-Tensor Decomposition

General (rank=2) 4-Tensor $T^{\mu\nu}$

$= T_{\text{symm}}^{\mu\nu} + T_{\text{anti-symm}}^{\mu\nu}$

**Symmetric 4-Tensor**

$T_{\text{symm}}^{\mu\nu} = \frac{(T^{\mu\nu} + T^{\nu\mu})}{2}$

$= T_{\text{iso}}^{\mu\nu} + T_{\text{aniso}}^{\mu\nu}$

**Isotropic Symm 4-Tensor**

$T_{\text{iso}}^{\mu\nu} = \frac{\text{Tr}[T_{\text{symm}}^{\mu\nu}]}{4}$

**Anisotropic Symm 4-Tensor**

$T_{\text{aniso}}^{\mu\nu} = T_{\text{symm}}^{\mu\nu} - T_{\text{iso}}^{\mu\nu}$

**Tr$[T_{\text{aniso}}^{\mu\nu}] = 0$**

**Tr$[T_{\text{symm}}^{\mu\nu}]$ max DoF = 10**

**Anisotropic Symm 4-Tensor**

$T^{\text{anti-symm}}_{\mu\nu} = \frac{(T^{\mu\nu} - T^{\nu\mu})}{2}$

**Anti-Symmetric 4-Tensor**

$= T_{\text{iso}}^{\mu\nu} + T_{\text{aniso}}^{\mu\nu}$

$T_{\text{aniso}}^{\mu\nu} = -T_{\text{aniso}}^{\nu\mu}$

**Tr$[T_{\text{anti-symm}}^{\mu\nu}] = 0$**

**Tr$[T_{\text{symm}}^{\mu\nu}]$ max DoF = 6**

Maximum Degrees of Freedom (DoF)

$= \# \text{ of possible independent components}$

$= (\text{Tensor dimension})^{\text{(Tensor rank)}}$
SRQM Study: SR 4-Tensors

SR Tensor Invariants

(0,0)-Tensor = Lorentz Scalar $S$: Has either (0) or (1) Tensor Invariant, depending on exact meaning.
(S) itself is Invariant

(1,0)-Tensor = 4-Vector $V^\mu$: Has (1) Tensor Invariant = The Lorentz Scalar Product $V \cdot V = (v_0^2 - v_i v^i)$

(2,0)-Tensor = 4-Tensor $T^{\mu \nu}$: Has (4+) Tensor Invariants (though not all independent)

SR 4-Tensor

(2,0)-Tensor $T^{\mu \nu}$: Has (4+) Tensor Invariants (though not all independent)

a) $T^{\mu \nu} = V \cdot V = \sum_{\mu \nu} V^\mu V^\nu$, for symmetric inner product;

b) $T^{\mu \nu} T_{\rho \sigma} = 0$, for anti-symmetric inner product:

$c)$ Antisymmetric Inner Product $T^{\mu \nu}$ = $3(Tr[T^{\mu \nu}])^2 - 3\sum_{\mu \nu} T^{\alpha \beta} T^{\nu \alpha} T^{\nu \beta}$

d) Determinant

The lowered-indices form of a tensor just negativizes the (time-space) and (space-time) sections of the upper-indices sections of the upper-indices tensor

The lowered-indices form of a tensor just negativizes the (time-space) and (space-time) sections of the upper-indices tensor

а) $\text{Trace}[T^{\mu \nu}] = Tr[T^{\mu \nu}] = \eta_{\mu \nu} T^{\mu \nu} = T^{\mu \nu} = (T_0^0 + T_1^1 + T_2^2 + T_3^3) = (T_0^0 - T_1^1 - T_2^2 - T_3^3) = (T)$

for anti-symmetric: $= 0$

b) $\text{Inner Product } T^{\mu \nu}_0 = T_{00} T^{00} + T_{01} T^{01} + T_{02} T^{02} + T_{03} T^{03} = T^{(00)}^2 + \Sigma_{i<j} T^{(ij)}^2$

for symmetric | anti-symmetric: $= (T^{(00)})^2 - 2\Sigma_{i<j} T^{(ij)}^2$

Symmetric $\Sigma_{i<j} T^{(ij)}^2$ $\Sigma_{i<j} T^{(ij)}^2$ $2\Sigma_{i<j} T^{(ij)}^2$ $2\Sigma_{i<j} T^{(ij)}^2$

for anti-symmetric: $= 0$

d) Determinant $\det[T^{\mu \nu}] = (-1)^{\frac{1}{2} \epsilon_{\mu \nu \rho \sigma} T^{\mu \rho} T^{\nu \sigma}}$

for anti-symmetric: $\det[T^{\mu \nu}] = Pfaffian[T^{\mu \nu}]^2$ (The Pfaffian is a special polynomial of the matrix entries)

Set of 4 Eigenvalues $[T^{\mu \nu}]_1$

4-Tensor $T^{\mu \nu}$

$\Sigma_{i<j} T^{(ij)}^2$ $\Sigma_{i<j} T^{(ij)}^2$ $2\Sigma_{i<j} T^{(ij)}^2$ $2\Sigma_{i<j} T^{(ij)}^2$

- Determinant $\det[T^{\alpha \beta}] = \Pi_{\mu \nu} \{\lambda_\mu \}$, with $\{\lambda_\mu \}$ = Set of Eigenvalues Characteristic Eqns: $\det[T^{\mu \nu} - \lambda_i I_{\mu \nu}] = 0$

SR 4-Vector

(1,0)-Tensor $V^\mu = V = (v_0^2, v_1, v_2, v_3)$

$T^{\mu \nu} = [T_0^0, T_0^1, T_0^2, T_0^3]$

Trace $T^{\mu \nu} = \eta_{\mu \nu} T^{\mu \nu}$

$\Sigma_{i<j} T^{(ij)}^2$ $\Sigma_{i<j} T^{(ij)}^2$ $2\Sigma_{i<j} T^{(ij)}^2$ $2\Sigma_{i<j} T^{(ij)}^2$

SR 4-Scalar

(0,0)-Tensor $S$ or $S_0$

$\det[T^{\mu \nu}] = \Pi_{\mu \nu} \{\lambda_\mu \}$, with $\{\lambda_\mu \}$ = Set of Eigenvalues Characteristic Eqns: $\det[T^{\mu \nu} - \lambda_i I_{\mu \nu}] = 0$
SRQMQ Study: SR 4-Tensors

SR Tensor Invariants

Tensor Gymnastics

Some Tensor Gymnastics:

Matrix $\mathbf{A}$ = Tensor $A^c_d$

with rows denoted by “r", columns by “c"

Example with dim=4: r,c=0...3

Matrix $\mathbf{A} = A^c_d$

$= \begin{bmatrix}
A^{c_1}_{d_1} & A^{c_1}_{d_2} & A^{c_1}_{d_3} & A^{c_1}_{d_4} \\
A^{c_2}_{d_1} & A^{c_2}_{d_2} & A^{c_2}_{d_3} & A^{c_2}_{d_4} \\
A^{c_3}_{d_1} & A^{c_3}_{d_2} & A^{c_3}_{d_3} & A^{c_3}_{d_4} \\
A^{c_4}_{d_1} & A^{c_4}_{d_2} & A^{c_4}_{d_3} & A^{c_4}_{d_4}
\end{bmatrix}$

$\mathbf{M} = \mathbf{A} \times \mathbf{B} = A^c_d \cdot B^d_c = M^{c_d}$

with the rows of $\mathbf{A}$ multiplied by the columns of $\mathbf{B}$

due to the summation over index “c”

If we have sums over both indices:

$\begin{aligned}
A^c_d \cdot A^d_c = (\mathbf{A} \mathbf{A})^c_d &= (\mathbf{N})^c_d = \text{Trace}[\mathbf{M}] = \text{Trace}[\mathbf{A}^2] = \text{Tr}[\mathbf{A}^2] \\
A^c_d \cdot (\mathbf{n} \mathbf{A})^c_c &= (\mathbf{n} \mathbf{A}^2) = \eta^{c_d} (\mathbf{A}^2) = \delta^{c_d} (\mathbf{N}^2) = \text{Tr}[\mathbf{N}] = \text{Tr}[\mathbf{A}^2] \\
A^c_d \cdot (\mathbf{A} \mathbf{A})^c_d &= A^c_d - A^d_c = (\text{Tr}[\mathbf{A}])^2 - \text{Tr}[\mathbf{A}^2]
\end{aligned}$

with brackets [.] around the indices indicating anti-symmetric product

The Trace formula’s are independent of tensor dimension.

$A^{\mu}_a = \text{Tr}[\mathbf{A}]$

$A_{\mu a}^{\mu b} = A^{\mu a} A^{\mu b} - A^{\mu b} A^{\mu a} = (\text{Tr}[\mathbf{A}])^2 - \text{Tr}[\mathbf{A}^2]$

$A_{\mu a}^{\mu b} A_{\mu c}^{\mu d} = A_{\mu a}^{\mu b} A_{\mu c}^{\mu d} - A_{\mu a}^{\mu d} A_{\mu c}^{\mu b} + A_{\mu a}^{\mu c} A_{\mu d}^{\mu b} - A_{\mu a}^{\mu b} A_{\mu c}^{\mu d} + A_{\mu a}^{\mu d} A_{\mu c}^{\mu b} - A_{\mu a}^{\mu b} A_{\mu c}^{\mu d} - A_{\mu a}^{\mu c} A_{\mu d}^{\mu b} + A_{\mu a}^{\mu d} A_{\mu c}^{\mu b}$

$= + (\text{Tr}[\mathbf{A}])^4 - 6* (\text{Tr}[\mathbf{A}])^2 + 8* (\text{Tr}[\mathbf{A}]) + 3* (\text{Tr}[\mathbf{A}])^2 - 6* (\text{Tr}[\mathbf{A}])$

$= \text{Trace}[\mathbf{V}^\mu V_\nu] = \eta_{\mu \nu} T^\nu = T_{\mu}^\nu T_{\nu}^\mu$

$\mathbf{V} \cdot \mathbf{V} = \mathbf{V}^\nu V_\nu = (\mathbf{v}^2)^2 - \mathbf{v} \cdot \mathbf{v} = (\mathbf{v}^2)^2$

$= \text{Lorentz Scalar}$
SRQM Study: SR 4-Tensors

SR Tensor Invariants

Cayley-Hamilton Theorem

General Cayley-Hamilton Theorem

\[ A^n + c_{n-1} A^{n-1} + \ldots + c_0 A^0 = 0, \] with \( A \) = square matrix, \( d \) = dimension, \( A^0 \) = Identity(\( d \)) = \( I_d \)

Characteristic Polynomial: \( p(\lambda) = \det(A - \lambda I_d) \)

The following are the Principle Tensor Invariants for dimensions 1..4

**dim = 1:**

\[ A^3 + c_2 A^2 + c_0 A^0 = 0 : A - I_d \quad I_d(1) = 0 \]

\( I_1 = \text{tr}[A] = \det[I_d[A] = \lambda_1 \)

**dim = 2:**

\[ A^4 + c_3 A^3 + c_2 A^2 + c_0 A^0 = 0 : A^2 - I_d \quad I_d(2) = 0 \]

\[ I_1 = \text{tr}[A] = \sigma[\text{Eigenvalues}] = \lambda_1 + \lambda_2 \]

\[ I_2 = (\text{tr}[A]^2 - \text{tr}[A]) / 2 = \det[A] = \sigma[\text{Eigenvalues}] = \lambda_1 \lambda_2 \]

**dim = 3:**

\[ A^5 + c_4 A^4 + c_3 A^3 + c_2 A^2 + c_0 A^0 = 0 : A^3 - I_d \quad I_d(3) = 0 \]

\[ I_1 = \text{tr}[A] = \sigma[\text{Eigenvalues}] = \lambda_1 + \lambda_2 + \lambda_3 \]

\[ I_2 = (\text{tr}[A]^2 - 3 \text{tr}(A)) / 2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \]

\[ I_3 = [\text{tr}(A)^3 - 3 \text{tr}(A^2) \text{tr}(A) + 2 \text{tr}(A^3)] / 6 = \det[A] = \sigma[\text{Eigenvalues}] = \lambda_1 \lambda_2 \lambda_3 \]

**dim = 4:**

\[ A^6 + c_5 A^5 + c_4 A^4 + c_3 A^3 + c_2 A^2 + c_1 A + c_0 A^0 = 0 : A^4 - I_d \quad I_d(4) = 0 \]

\[ I_1 = \text{tr}[A] = \sigma[\text{Eigenvalues}] = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \]

\[ I_2 = (\text{tr}[A]^2 - \text{tr}[A]) / 2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \]

\[ I_3 = \left[ \text{tr}(A)^3 - 3 \text{tr}(A^2) \text{tr}(A) + 2 \text{tr}(A^3) \right] / 6 = \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 \]

\[ I_4 = [\text{tr}(A)^4 - 6 \text{tr}(A^2) \text{tr}(A) + 3 \text{tr}(A^3)]^2 / 8 = \det[A] = \sigma[\text{Eigenvalues}] = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \]

Each dimension gives the number of elements from it’s row in Pascal’s Triangle ;}

**Characteristic Eqns:** \( \det[T^\alpha - \lambda_k I_d] = 0 \)

\[ I_0 = \sigma[\text{Unique Eigenvalue Naughts}] = 1 \quad (1) \]

\[ I_1 = \sigma[\text{Unique Eigenvalue Singles}] = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \quad (4) \]

\[ I_2 = \sigma[\text{Unique Eigenvalue Doubles}] = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 + \lambda_3 \lambda_4 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \quad (6) \]

\[ I_3 = \sigma[\text{Unique Eigenvalue Triples}] = \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 \quad (4) \]

\[ I_4 = \sigma[\text{Unique Eigenvalue Quadruples}] = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \quad (1) \]
SRQM Study: SR 4-Tensors

SR Tensor Invariants

Cayley-Hamilton Theorem

<table>
<thead>
<tr>
<th>General Cayley-Hamilton Theorem</th>
<th>Dim = 1</th>
<th>Dim = 2</th>
<th>Dim = 3</th>
<th>Euclidean 3-Space</th>
<th>Dim = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ A^d + c_{d-1}A^{d-1} + \ldots + c_0A^0 = 0 ] with ( A ) = square matrix, ( d ) = dimension, ( A^0 = \text{Identity}(d) = I_{(d)} )</td>
<td>( A = { a } )</td>
<td>( A = { a \ b \ c \ d } )</td>
<td>( A = { a \ b \ c \ d } )</td>
<td>( A = { a \ b \ c \ d } )</td>
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</tr>
<tr>
<td>Tensor Invariants ( I_\alpha )</td>
<td>( I_\alpha : j,k={1} )</td>
<td>( I_\alpha : j,k={1,2} )</td>
<td>( I_\alpha : j,k={1,2,3} )</td>
<td>( I_\alpha : \mu,\nu={0,1,2,3} )</td>
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<td>( I_0 = 1/0! = 1 )</td>
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<tr>
<td>( I_1 = \text{tr}[A]/1! )</td>
<td>( \lambda_1 )</td>
<td>( \lambda_1 + \lambda_2 )</td>
<td>( \lambda_1 + \lambda_2 + \lambda_3 )</td>
<td>( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 )</td>
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<tr>
<td>( = A^a_a )</td>
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<tr>
<td>( = \Sigma[\text{Unique Eigenvalue Singles}] )</td>
<td>( = \text{Det}_0[A] )</td>
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<td>( = \Sigma[\text{Eigenvalues}] )</td>
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<tr>
<td>( I_2 = (\text{tr}[A]^2 - \text{tr}[A^2])/2! )</td>
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<td>( = A^\alpha_{\beta \gamma} / 2 )</td>
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<td>( = \Sigma[\text{Unique Eigenvalue Doubles}] )</td>
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<tr>
<td>( I_3 = [ (\text{tr}[A^3] - 3 \text{tr}[A^2])\text{tr}[A] + 2 \text{tr}[A^3] ]/3! )</td>
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<td>( = A^\alpha_{\beta \gamma} A^\beta_{\gamma \delta} / 6 )</td>
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<td>( = \Sigma[\text{Unique Eigenvalue Triples}] )</td>
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<tr>
<td>( I_4 = [ ((\text{tr}[A^4] - 6 \text{tr}[A^3])\text{tr}[A]^2 + 3(\text{tr}[A^2])^2 + 8 \text{tr}[A^3])\text{tr}[A] - 6 \text{tr}[A^4])]/4! )</td>
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<tr>
<td>( = A^\alpha_{\beta \gamma} A^\beta_{\gamma \delta} A^\delta_{\epsilon \gamma} / 24 )</td>
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<tr>
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</tbody>
</table>
SRQM Study: SR 4-Tensors

SR Tensor Invariants for Faraday EM Tensor

The Faraday EM Tensor \( F^{αβ} = \frac{∂A^α}{∂t} - \frac{∂A^β}{∂t} \) is an anti-symmetric tensor that contains the Electric and Magnetic Fields, defined by the Exterior “Wedge” Product (\(^^\lor\)). The 3-electric components (\( e = e^i \)) are in the temporal-spatial sections. The 3-magnetic components (\( b = b^i \)) are in the only-spatial section.

\[ (2,0)\)-Tensor = 4-Tensor \( T^{αβ}_{\text{Faraday}} \) Has (4+) Tensor Invariants (though not all independent)

- a) \( T^{αβ}_{\text{Faraday}} = \text{Trace} = \text{Sum of EigenValues for (1,1)-Tensors} \) (mixed)
- b) \( T^{αβ}_{\text{Faraday}} T^{γδ}_{\text{Faraday}} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product} \)
- c) \( T^{αβ}_{\text{Faraday}} T^{γδ}_{\text{Faraday}} T^{ιυ}_{\text{Faraday}} = \text{Asymm Tri-Product} \rightarrow \text{?Name?} \)
- d) \( T^{αβ}_{\text{Faraday}} T^{γδ}_{\text{Faraday}} T^{ιυ}_{\text{Faraday}} T^{κλ}_{\text{Faraday}} = \text{Asymm Quad-Product} \rightarrow \text{4D Determinant} = \text{Product of EigenValues for (1,1)-Tensors} \)

Importantly, the Faraday EM Tensor has only (2) linearly-independent invariants:

- a) Faraday Trace:\[ F^{αγ} = (F^{αβ} F^{βγ} - F^{βα} F^{αγ}) = (0 -0 -0 -0) = 0 \]
- b) Faraday Inner Product:\[ F^{αβ} F^{γδ} = 2Σ [F^{αβ}]^2 + 2Σ [F^{γδ}]^2 = (0 -2(e\cdot e/c^2) + 2(b\cdot b) = 2((b\cdot b) - (e\cdot e/c^2) \]
- c) Faraday AsymmTri:\[ F^{αβ} = 3(Tr[F^{αβ}])F^{αβ}_0 + F^{αβ}_0 F^{αβ} + F^{α0} F^{0α} + F^{β0} F^{0β} = 0 -3(0 + F^{αβ}_0) F^{αβ}_0 = 0 \]
- d) Faraday Det[anti-symmetric \( \tilde{F}^{αβ} \)] \( = \text{Pfaffian} = [[(e\cdot e/c^2) - (e\cdot e/c^2)]^2 = [(b\cdot b)/c + (e\cdot e/c^2)]^2 = [(e\cdot b)/c]^2 ] \)

The 4-Gradient and 4-EMVectorPotential have (4) independent components each, for a total of (8). Subtract the (2) invariants which give constraints to get a total of (6) independent components = (6) independent components of a 4x4 anti-symmetric tensor = (3) 3-electric \( e \) + (3) 3-magnetic \( b \) = (6) independent EM field components

Note: It is possible to have non-zero \( e \) and \( b \), yet still have zeroes in the Tensor Invariants.

If \( e \) is orthogonal to \( b \), then \( \text{Det}[\tilde{F}^{αβ}] = (b\cdot e)/c^2 = 0 \)

If \( (b\cdot b) = (e\cdot e/c^2) \), then \( \text{InnerProd}[\tilde{F}^{αβ}] = 2((b\cdot b) - (e\cdot e/c^2)) = 0 \)

These conditions lead to the properties of EM waves = photons = null 4-vectors, which have fields \( |b| = |e|/c \) and \( b \) orthogonal to \( e \), travelling at velocity \( c \).

4-EMVectorPotential
\[ A^α = (φ/c, a) \]

SR 4-Vector
- \( (2,0)\)-Tensor \( T^{αβ}_{\text{Faraday}} \)
- \( (1,1)\)-Tensor \( T^{αβ} \) or \( T^{αβ}_{\text{Faraday}} \)
- Lorentz Scalar
- Lorentz Tensor

SR 4-CoVector
- OneForm
- Tensor \( V^α = \nabla α \phi \)
- Tensor \( V^{αβ} \)

SR 4-Scalar
- Tensor \( S_α = S^{α} \)
- Tensor \( S_α β = S^{α β} \)
- Tensor \( S_α β γ = S^{α β γ} \)
- Tensor \( S_α β γ δ = S^{α β γ δ} \)

Fundamental EM Invariants:
- Pressure
\[ P = ((1/2) F^{αβ} F_{αβ} = (1/2) F^{αβ} F_{αβ} = ((b\cdot b) - (e\cdot e/c^2)) \]
- Electric Quadrupole
\[ Q = (1/4) F^{αβ} F_{αβ} = (1/4) e \alpha β γ δ F^{αβ} F_{αβ} = (e\cdot b)/c \]

4-Vector SRQM Interpretation of Quadratic Mass Terms:
\[ M^α_μ = M^μ_α = (M_0/c) - (M\cdot e/c^2) b \]
The 4-AngularMomentum Tensor $M^\alpha{}_{\beta}$ is an anti-symmetric tensor. The 3-mass-moment components $(n = n)$ are in the temporal-spatial sections. The 3-angular-momentum components $(l = l)$ are in the only-spatial section.

$$n \langle-k/r\rangle = m(k/r) = m(k/r)$$

where $n, l, k$ are classical conserved vectors. The invariance is shown here to be relativistic in origin.

The 4-Position and 4-Momentum have $(4)$ independent components each, for total of $(8)$.

$$d) \quad \text{Importantly, the 4-AngulMom Tensor has only (2) linearly-independent invariants:}$$

- $\text{Tr}[M^\mu{}_{\nu}] = M^\nu{}_{\nu} = 0$
- $\text{M}_{\alpha\beta}M^{\alpha\beta} = 2(s(l)-(c^n\cdot n))$
- $\text{AsymmTri}[M^\mu{}_{\nu}] = 0$
- $\text{Determinant Tensor Invariant}$

For $4$-AngularMomentum Tensor

$$M^\alpha{}_{\beta} = X^\alpha P^\beta - X^\beta P^\alpha = X \wedge P$$

The $(2,0)$-Tensor $T^\alpha{}_{\beta}$: Has $(4+)$ Tensor Invariants (though not all independent)

- $a) \quad \text{Tr}[M^\mu] = M^\nu = (M^0-M^1-M^2-M^3) = (0-0-0-0) = 0$
- $b) \quad \text{Inner Product} M_{\mu\nu}M^{\mu\nu} = \sum_{\mu} (M^\mu)^2 - 2\sum_{\mu} (M^\mu)^2 = (0) - 2(c^2\cdot n) + 2(l) = 2((l)-(c^n\cdot n))$
- $c) \quad \text{AsymmTri}[M^\mu{}_{\nu}] = \text{Tr}[M^\mu{}_{\nu}] - 3(M^\mu{}_{\nu}M^\nu{}_{\mu}) = M^\mu{}_{\nu} + M^\nu{}_{\mu} = 0$
- $d) \quad \text{AsymmTri}[M^\mu{}_{\nu}] = \text{Pfaffian}[M^\mu{}_{\nu}] = [- (c^n)(+l) - (c^n)(-l) + (c^n)(+l)]^2 = [c(n)(l)]^2$

Importantly, the 4-AngulMom Tensor has only $(2)$ linearly-independent invariants:

- $(2(l)-(c^n\cdot n))$: see Wikipedia Laplace-Runge-Lenz_vector, sec. Casimir Invariants
- $(c(n)(l))^2$

$\text{a)} \quad (c(n)(l))^2$

and $\text{c)} \quad (c(n)(l))^2$, and do not provide additional constraints

The 4-Position and 4-Momentum have $(4)$ independent components each, for total of $(8)$. Subtract the $(2)$ invariants which provide constraints to get a total of $(6)$ independent components: $(6)$ independent components of a $4 \times 4$ anti-symmetric tensor

$\text{= (3) 3-mass-moment} n + (3) 3$-angular-momentum $l = (6)$ independent $4$-AngularMomentum components

$3$-massmoment $n = x m - t p = m(x - tu) = m(r - t(w x r))$: Tangential velocity $u_r = (\omega \times r)$

$$\langle-k-r\rangle = m(k/r) = m(k/r) = t^* d/dt(p) x L = mk(x x r)$$

which is another classical conserved vector. The invariance is shown here to be relativistic in origin. Wikipedia article: Laplace-Runge-Lenz vector shows these as Casimir Invariants.

See Also: Relativistic Angular Momentum.
The Minkowski Metric Tensor $\eta^{\mu\nu}$ is the tensor all SR 4-Vectors are measured by.

$$(2,0)\text{-Tensor} = 4\text{-Tensor} T^{\mu\nu}$$. Has (4x4) Tensor Invariants (though not all independent)

a) $T^{\alpha}_\alpha = \text{Trace} = \text{Sum of EigenValues for } (1,1)\text{-Tensors (mixed)}$
b) $T^{\alpha}_{\beta\alpha} = \text{Asymm Bi-Product} \rightarrow \text{Inner Product}$
c) $T^{\alpha}_{\beta\gamma\alpha} = \text{Asymm Tri-Product} \rightarrow \text{Name?}$
d) $T^{\alpha}_{\beta\gamma\delta\alpha} = \text{Asymm Quad-Product} \rightarrow 4D \text{Determinant} = \text{Product of EigenValues for } (1,1)\text{-Tensors}$

**SRQM Study: SR 4-Tensors**

**SR Tensor Invariants for Minkowski Metric Tensor**

<table>
<thead>
<tr>
<th>$\eta^{\mu\nu}$</th>
<th>4-Gradient $\partial = \partial^\mu = (\partial/c,-V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial$</td>
<td>$\partial^\mu = (\partial/c,-V)$</td>
</tr>
<tr>
<td>Tensor Invariant</td>
<td>In GR</td>
</tr>
<tr>
<td>$\text{Diag}[1,-1,-1,-1]$</td>
<td>$\text{Diag}[1,-\delta^{\mu\nu}]$</td>
</tr>
<tr>
<td>$[+1\ 0\ 0\ 0\ ]$</td>
<td>$[0\ -1\ 0\ 0\ ]$</td>
</tr>
<tr>
<td>$[0\ 0\ -1\ 0\ ]$</td>
<td>$[0\ 0\ 0\ -1\ ]$</td>
</tr>
<tr>
<td>(in Cartesian form)</td>
<td></td>
</tr>
<tr>
<td>$[\eta_{\mu\nu}] = 1/</td>
<td>\eta^{\mu\nu}</td>
</tr>
<tr>
<td>SR: Minkowski Metric</td>
<td>Invariant</td>
</tr>
<tr>
<td>Determinant Tensor Invariant</td>
<td>Asymm Tri-Product Tensor Invariant</td>
</tr>
<tr>
<td>$\text{Det}[\eta_{\mu\nu}] = +1$</td>
<td>$\text{Det}[\eta^{\mu\nu}] = -1$</td>
</tr>
</tbody>
</table>

**4-Position**

$$R = R^{\mu\nu} = (ct,r)$$

$\Lambda^{\mu\nu}$, $\eta^{\mu\nu} = \eta_{\mu\nu}$

| $\text{Det}[\text{Exp}[A]] = \text{Exp}[\text{Tr}[A]]$ |
|------------------|-----------------------------------------------|
| a) $T^{\alpha}_\alpha /1! = 4/1 = 4$ | b) $T^{\alpha}_{\beta\alpha} /2! = 12/2 = 6$ |
| c) $T^{\alpha}_{\beta\gamma\alpha} /3! = 24/6 = 4$ | d) $T^{\alpha}_{\beta\gamma\delta\alpha} /4! = 24/24 = 1$ |

Determinant

$\text{Det}[\text{Exp}[A]] = \text{Exp}[\text{Tr}[A]]$

$\text{Det}(\text{Exp}[A]) = (\text{tr} A)^4 - 6 \text{tr}(A^2) \text{tr} A^2 + 3(\text{tr} A^2)^2 + 8 \text{tr}(A^3) \text{tr} A - 6 \text{tr}(A^4)/24$

**EigenValues** not defined for the standard Minkowski Metric Tensor since it is a type (2,0)-Tensor, all upper indices. However, they are defined for the mixed form (1,1)-Tensor.

- **EigenValues** are defined for the Lorentz Transforms since they are type (1,1)-Tensors, mixed indices

- **SR 4-Tensor**
  - $(2,0)$-Tensor $T^{\mu\nu}$
  - $(1,1)$-Tensor $T^\mu_\mu$, or $T^\nu_\nu$
  - $(0,2)$-Tensor $T^\mu_{\alpha\beta}$

- **SR 4-Vector**
  - $(1,0)$-Tensor $V^\mu = (v^\mu,0)$
  - $(0,1)$-Tensor $V_\mu = (v_\mu,0)$

- **SR 4-Scalar**
  - $(0,0)$-Tensor $S$ or $S_0$

**Characteristic Eqns:**

**Det**[**$T^\mu_{\alpha\beta}$**] = $\Pi_4[\lambda_\alpha]$; with $\{\lambda_\alpha\}$ = EigenValues

$\text{SRQM Study: SR 4-Tensors}$

**SR 4-CoVector:** One Form $(0,1)$-Tensor $V_\mu = (v_\mu,0)$

**SR 4-Scalar:** $(0,0)$-Tensor $S$ or $S_0$

**Lorentz Scalar**

$T^\nu = \eta^{\mu\nu} T^\mu_\mu = T$

**V-V = $V^\mu V_\nu = [(v^\mu)^2 - (v^\nu)^2 = (v^\mu - v^\nu)^2$**

| $\text{Lorentz Scalar}$ |
|------------------|-----------------------------------------------|
| $\text{SR\ Fusion}$ |
| $\eta^{\mu\nu}$ | $\eta_{\mu\nu}$ |
| $\delta^{\mu\nu}$ | $\delta^{\mu\nu}$ |
| $\text{SRQM Interpretation}$ |
| $\text{of QM}$ |

**http://scirealm.org/SRQM.pdf**

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http://scirealm.org/SRQM.pdf
The Perfect Fluid Stress-Energy Tensor $T^{\mu\nu}$ is the tensor of a relativistic fluid.

$(2,0)$-Tensor $= 4$-Tensor $T^{\mu\nu}$: Has $(4 + 1)$ Tensor Invariants (though not all independent)

- a) Perfect Fluid Trace $[T^{\mu\nu}]$ = $\rho_o - 3p_o$
- b) Perfect Fluid Inner Product $T^{\mu\nu} = (\rho_o)^2 + 3(p_o)^2$
- c) Perfect Fluid Asymmetric Tri-Product $[T^{\mu\nu}]$ =
- d) Perfect Fluid Det $[T^{\mu\nu}] = \rho_o (p_o)^3$

SR Conservation of Stress Energy $T^{\mu\nu}$

If $F_{\text{density}}^{\mu\nu} = 0$

Dets $[\text{Exp}[A]] = \text{Exp}[\text{Tr}[A]]$

$T_{\text{perfect fluid}}^{\mu\nu} = (\rho_o) V^{\mu\nu} + (p_o) H^{\mu\nu}$

$\text{Trace Tensor Invariant}$

$\text{Signature Tensor Invariant}$

$\text{Equation of State Tensor Invariant}$

$\text{Det}[T^{\mu\nu}] = \rho_o (p_o)^3$

$\text{Asymmetric Tri-Product Tensor Invariant}$

$\text{Determinant Tensor Invariant}$

EigenValues not defined for the standard Perfect Fluid Tensor since it is a type $(2,0)$-Tensor, all upper indices. However, they are defined for the mixed form $(1,1)$-Tensor $\text{EigenValues}$ are defined for the Lorentz Transforms since they are type $(1,1)$-Tensors, mixed indices.

$\text{SR 4-Tensor}$

$(2,0)$-Tensor $T^{\mu\nu}$

$\text{SR 4-Vector}$

$(1,0)$-Tensor $V^{\mu} = V = (\hat{v}^\mu, \nu)$

$\text{SR 4-CoVector: One Form}$

$(0,1)$-Tensor $V_\mu = (\nu, \hat{v}_\mu)$

$\text{SR 4-Scalar}$

$(0,0)$-Tensor $S$ or $S^\alpha$ Lorentz Scalar

$\text{SR 4-Vector SRQM Interpretation}$

$\text{SRQM Study: SR 4-Tensors}$

$\text{SR Tensor Invariants}$

for Perfect Fluid Stress-Energy Tensor
**SRQTM Study: SR 4-Tensors**

**SR Tensor Invariants for Continuous Lorentz Transform Tensors**

The Lorentz Transform Tensor \( \{ \Lambda^\alpha_\beta \} \) is the tensor all SR 4-Vectors must transform by.

\[
(2,0)-\text{Tensor} = 4-\text{Tensor} \; T^{\alpha\mu} \; \text{Has (4+) Tensor Invariants (though not all independent)}
\]

1. \( T^{\alpha}_\alpha = \text{Trace} = \text{Sum of EigenValues for (1,1)-Tensors (mixed)} \)
2. \( T^{\alpha\beta}_\mu T^{\mu\beta}_\gamma = \text{Asymm Bi-Product} \to \text{Inner Product} \)
3. \( T^{\alpha\beta}_\mu T^{\gamma\mu}_\delta = \text{Asymm Tri-Product} \to \text{Name?} \)
4. \( T^{\alpha\beta}_\mu T^{\gamma\beta}_\nu = \text{Asymm Quad-Product} \to 4\text{ Determinant} = \text{Product of EigenValues for (1,1)-Tensors} \)

**SR Lorentz Invariant (0)**

\[
\Lambda^{\mu}_\nu \Lambda^{\nu}_\mu = 4 = \Lambda^{\mu}_\mu \Lambda^{\nu}_\nu
\]

**Rotation(0)**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos[\theta] & -\sin[\theta] & 0 \\
0 & \sin[\theta] & \cos[\theta] & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**Identity**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**Boost(0)**

\[
\begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**Lorentz SR Lorentz Boost meets Rotation at Identity of 4**

A more general version would be a & b as arbitrary complex values:

- **Lorentz Transform**
  \[ T^{\alpha\beta}_\mu T^{\gamma\beta}_\nu = \text{EigenValues} \]
  \[ \Lambda^{\mu}_\nu \Lambda^{\nu}_\mu = 4 = \Lambda^{\mu}_\mu \Lambda^{\nu}_\nu \]

- **Trace Tensor Invariant**
  \[ \text{Tr}[\Lambda^{\alpha\beta}_\mu T^{\mu\beta}_\gamma] = \text{EigenValues} \]
  \[ \text{Tr}[\Lambda^{\alpha\beta}_\mu T^{\mu\beta}_\gamma] = \text{Set} \{ e^a, e^b, e^c, e^d \} \]

- **Product of EigenValues**
  \[ \text{Det}[\Lambda^{\alpha\beta}_\mu T^{\mu\beta}_\gamma] = \text{Set} \{ e^a, e^b, e^c, e^d \} \]

- **Proper Transform always +1**

**SR Lorentz Transform**

\[
\Lambda^{\mu}_\nu = \Lambda^{\mu\nu}_\nu = \delta^{\mu}_\nu
\]

**SR Lorentz Transform**

\[
\delta^{\mu}_\nu = \Lambda^{\alpha}_\nu \Lambda^{\alpha}_\mu = \eta^{\alpha}_\alpha = \delta^{\mu}_\nu
\]

**Lorentz SR Lorentz Invariant (0)**

\[
\Lambda^{\mu}_\nu \Lambda^{\nu}_\mu = 4 = \Lambda^{\mu}_\mu \Lambda^{\nu}_\nu
\]

**EigenValues**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos[\theta] & -\sin[\theta] & 0 \\
0 & \sin[\theta] & \cos[\theta] & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**EigenValues**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**EigenValues**

\[
\begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**EigenValues**

\[
\begin{bmatrix}
1 & e^a & e^b & e^c \\
e^a & 1 & e^b & e^c \\
e^b & e^c & 1 & e^a \\
e^c & e^a & e^b & 1
\end{bmatrix}
\]

**EigenValues**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & e^a & e^b & e^c \\
e^a & 1 & 0 & 0 \\
e^b & e^c & 0 & 1
\end{bmatrix}
\]
SRQM Study: SR 4-Tensors
SR Tensor Invariants for Discrete Lorentz Transform Tensors

The Trace of various discrete Lorentz transforms varies in steps from \{-4,-2,0,2,4\}

This includes Mirror Flips, Time Reversal, and Parity Inversion –
especially taking all combinations of ±1 on the diagonal of the transform.

Trace Tensor Invariant

\[ \text{Trace} \{\Lambda'\} = \{ -4, -2, 0, 2, 4 \} \]

Determinant Tensor Invariant

\[ \text{Det}[\Lambda'] = \pm 1 \]

Proper Transform = +1

Improper Transform = -1

LR Tensor SR

\[ \Lambda_\alpha' \rightarrow \Lambda_\nu' \]

Tensor Invariants

- TP combo
- Parity-Inversion
- Flip-xy-Combo
- Time-Reversal
- Identity

Characteristic Eqns:

\[ \text{Det}[\Lambda'] = \pm 1 \]

SR 4-Tensor

- (2,0)-Tensor \( T^{\mu \nu}_s \)
- (1,0)-Tensor \( V^\nu = (v^\nu, v^0) \)
- (0,1)-Tensor \( V_\nu = (v_0, -v^\nu) \)
- Lorentz Scalar

SR 4-CoVector: OneForm

- (0,0)-Tensor \( S \) or \( \omega \)

\[ \text{Det}[T^{\mu \nu}_s] = \Pi_s[\lambda_k]; \text{ with } \{ \lambda_k \} = \text{EigenValues} \]

Trace\[T^\nu_\nu] = \eta_{\nu \nu} T^\nu_\nu = T^\nu_\nu \]

4-Vector SRQM Interpretation of QM

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SRQM Study: SR 4-Tensors

More SR Tensor Invariants for Discrete Lorentz Transform Tensors

SR: Lorentz Transform
\[ \delta_{\mu \nu} = \partial R_{\mu}^i \partial R_{\nu}^j = \Lambda_{\mu \nu}^iv \]
\[ \Lambda_{\mu \nu} = (\Lambda^{iv})_\mu \Lambda_{\nu}^v = \eta_{\nu \mu} = \delta_{\nu \mu} \]

\[ \text{Det} \{ \Lambda^{iv} \} = \pm 1 \]
\[ \Lambda_{\mu \nu} \Lambda^{iv} = \delta_{\mu \nu} \]

The Flip-xy-Combo is the equivalent of a π-Rotation-z.

I suspect that this may be related to exchange symmetry and the Spin-Statistics idea that a particle-exchange is the equivalent of a spin-rotation.

A single Flip would not be an exchange because it leaves a mirror-inversion of <right-|-left>.

But the extra Flip along an orthogonal axis corrects the mirror-inversion, and would be an overall exchange because the particle is in a different location.

SR 4-Tensor
\[ T^{iv}_{\mu} \]
\[ (2,0)-Tensor T^{iv}_{\mu} \]
\[ (1,1)-Tensor T^{iv}_{\mu} \]
\[ (0,2)-Tensor T^{iv}_{\mu} \]

SR 4-Vector
\[ V^i = (v_0^-, v_0^+) \]
\[ V_{\mu} = (v_0^-, v_0^+) \]

SR 4-Scalar
\[ S_{\mu} = (S_0, S_\mu) \]

SR 4-CoVector: OneForm
\[ \eta^{iv}_{\mu} \]

Characteristic Eqns: \[ \text{Det}[T^{iv}_{\mu} - \lambda_k I_{(4)}] = 0 \]

Trace \[ [T^{iv}_{\mu}] = \eta_{\mu \nu} T^{iv}_{\mu} = T^{iv}_{\mu} = T \]

\[ VV = v_0^(-) v_0^+ = (v_0^+)^2 - v_0^(-) v_0^+ = (v_0^+)^2 = \text{Lorentz Scalar} \]

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Interpretation of QM

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SR 4-Scalars, 4-Vectors, 4-Tensors

Elegantly join many dual physical properties and relations

SR 4-Scalars, 4-Vectors, and 4-Tensors beautifully and elegantly display the relations between lots of different physical properties and relations. Their notation makes navigation through the physics very simple.

They also devolve very nicely into the limiting/approximate Newtonian cases of \( |v| \ll c \) by letting \( \gamma \rightarrow 1 \) and \( \gamma' = \frac{1}{\sqrt{1 - v^2/c^2}} \rightarrow 0 \).

SR tells us that several different physical properties are actually dual aspects of the same thing, with the only real difference being one's point of view, or reference frame.

Examples of 4-Vectors = (1,0)-Tensors include:

- \((\text{Time, Space}), (\text{Energy, Momentum}), (\text{Power, Force}), (\text{Frequency, WaveNumber}), (\text{Time Differential, Spatial Gradient}), (\text{NumberDensity, NumberFlux}), (\text{ChargeDensity, CurrentDensity}), (\text{EM-ScalarPotential, EM-VectorPotential})\), etc.

One can also examine 4-Tensors, which are type (2,0)-Tensors. The Faraday EM Tensor similarly combines EM fields:

- Electric \( \{ e = e^i = (e^x,e^y,e^z) \} \)
- Magnetic \( \{ b = b^i = (b^x,b^y,b^z) \} \)

\[
F^{\alpha\beta} = \begin{pmatrix} 0 & -e^i/c \\ +e^i/c & -(\varepsilon^i_k b^k) \end{pmatrix}
\]

Also, things are even more related than that. The 4-Momentum is just a constant times 4-Velocity. The 4-WaveVector is just a constant times 4-Velocity.

In addition, the very important conservation/continuity equations seem to just fall out of the notation. The universe apparently has some simple laws which can be easy to write down by using a little math and a super notation.

[Diagram of SR 4-Scalars, 4-Vectors, 4-Tensors and their relations]
SRQM Study:

SR Gradient 4-Vectors = (1,0)-Tensors
SR Gradient One-Forms = (0,1)-Tensors

4-Vector = Type (1,0)-Tensor
4-Position \( R = R^\mu = (ct,r) \)
4-Gradient \( \partial_R = \partial = \partial^\mu = \partial/\partial R_\mu = (\partial/ct, \nabla) \)

[Temporal : Spatial] components
[Time (t) : Space (r)]
[Time Differential (\( \partial_t \)) : Spatial Gradient(\( \nabla \))]

Standard 4-Vector
4-Position \( R = R^\mu = (ct,r) \)
4-Velocity \( U = U^\mu = \gamma(c,u) \)
4-Momentum \( P = P^\mu = (E/c, p) \)
4-WaveVector \( K = K^\mu = (\omega/c, k) \)

Related Gradient 4-Vector (from index-raised Gradient One-Form)
4-PositionGradient \( \partial_R = \partial_R^\mu = \partial/\partial R_\mu = (\partial/ct, \nabla_R) = \partial = \partial^\mu = 4-Gradient \)
4-VelocityGradient \( \partial_u = \partial_u^\mu = \partial/\partial U_\mu = (\partial/ct, \nabla_u) \)
4-MomentumGradient \( \partial_p = \partial_p^\mu = \partial/\partial P_\mu = (\partial/ct, \nabla_p) \)
4-WaveGradient \( \partial_K = \partial_K^\mu = \partial/\partial K_\mu = (\partial/ct, \nabla_K) \)

In each case, the (Whichever)Gradient 4-Vector is derived from an SR One-Form or 4-CoVector, which is a type (0,1)-Tensor
ex. One-Form PositionGradient \( \partial_R^\nu = \partial/\partial R^\nu = (\partial/ct, \nabla_R) \)

The (Whichever)Gradient 4-Vector is the index-raised version of the SR One-Form (Whichever)Gradient
ex. 4-PositionGradient \( \partial_R^\mu = \partial/\partial R_\mu = (\partial/ct, \nabla_R) = \eta^{\mu\nu} \partial_R^\nu = \eta^{\mu\nu} (\partial/ct, \nabla_R) = \eta^{\mu\nu} (One-Form PositionGradient) \)

This is why the 4-Gradient is commonly seen with a minus sign in the spatial component, unlike the other regular 4-Vectors, which have all positive components.

In the 4-Gradient, the (Whichever)Gradient 4-Vector is derived from an SR One-Form or 4-CoVector, which is a type (0,1)-Tensor. For example, the PositionGradient is given by \( \partial_R = \partial_R^\mu = \partial/\partial R_\mu = (\partial/ct, \nabla_R) \). The index-raised version of this gradient, known as the Gradient 4-Vector, is \( \partial_R^\nu = \partial/\partial R^\nu = (\partial/ct, \nabla_R) \) for the PositionGradient. This is why the 4-Gradient is commonly seen with a minus sign in the spatial component, unlike the other regular 4-Vectors, which have all positive components.

4-Tensors can be constructed from the Tensor Outer Product of 4-Vectors.
Some Basic 4-Vectors

Minkowski SpaceTime Diagram

Events & Dimensions

past  

future  

now  

here  

elsewhere  

c  

1/c  

time displacement  

\( \Delta t \) time-like interval  

\( \Delta r \) space-like interval  

Note the matching dimensional units: (4D SpaceTime)  

\( (c \Delta t) \) is \([\text{length/time}] \cdot [\text{time}] = [\text{length}] \),  

\( |\Delta r| \) is \([\text{length}] \)

\( \tau \) is the Proper Time = “rest-time”, time as measured by something not moving spatially

The Minkowski Diagram provides a great visual representation of SpaceTime

Classical Mechanics

- 3-displacement
- \( \Delta r = \Delta r^i \rightarrow (\Delta x, \Delta y, \Delta z) \)

Special Relativity

- Time-Like \((+\)
- Light-like: Null \((0\)
- Space-like \((-\)

Note the separate dimensional units: (time + 3D space)  

\( \Delta t \) is \([\text{time}] \),  

\( |\Delta r| \) is \([\text{length}] \)

SR 4-Tensor

\( T^{\mu\nu} \)

\( (2,0) \)-Tensor \( T^{\mu\nu} \)

\( (1,1) \)-Tensor \( T^\nu \), or \( T_{\mu}^\nu \)

\( (0,2) \)-Tensor \( T_{\mu\nu} \)

SR 4-Vector

\( V^\nu = V = (V_0, V) \)

\( (1,0) \)-Tensor \( V^\nu = \mathbf{V} = (V_0, V) \)

\( (0,1) \)-Tensor \( V_\mu = (V_0, -V) \)

SR 4-Scalar

\( S \) or \( S_0 \)

Lorentz Scalar

Classical (scalar)

Galilean Invariant

3-vector

Not Lorentz Invariant

Trace \( T^{\mu\nu} = \eta_{\mu\nu}T^{\mu\nu} = T^\mu_\mu = T \)

\( \mathbf{V} \cdot \mathbf{V} = V^\mu V_\mu = (V^0)^2 - (V \cdot V) = (V^0)^2 \)

= Lorentz Scalar

4-Vector SRQM Interpretation

of QM

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http://scirealm.org/SRQM.pdf
Some Basic 4-Vectors

Minkowski SpaceTime Diagram, WorldLines, LightSpeed to the Future!

- **4-Displacement**
  \[ \Delta R = (c \Delta t, \Delta r) \]

- **4-Position**
  \[ R = (ct, r) = \text{<Event>} \]

\[ 4\text{-Displacement} \Delta R \cdot \Delta R = \left( c \Delta t \right)^2 - \Delta r \cdot \Delta r = 0 \]

- **Light-like interval** \((0)\)
  \[ (\Delta r_o)^2 = 0 \]

- **Space-like interval** \((-\rangle\)
  \[ - (\Delta r_o)^2 \]

**4-Velocity**

- **Inertial motion**
  \[ U = \gamma(c, u) = \frac{dR}{d\tau} \]
  \[ U \cdot U = c^2 \]

- **Photonic**
  \[ U_c = \gamma(c, c\hat{n}) \]
  \[ U_c \cdot U_c = c^2 \]

\[ U \cdot U = \gamma(c, u) \cdot \gamma(c, u) = \gamma^2 (c^2 - u \cdot u) = c^2 \]

\[ \gamma = \frac{1}{\sqrt{1 - (u/c)^2}} = \frac{1}{\sqrt{1 - \beta^2}} \]

Massive particles move temporally into the future at the speed-of-light \((c)\) in their own rest-frame.

Massless particles (photonic) move nullly into the future at the speed-of-light \((c)\), and have no rest-frame.

The 4-Position is a particular type of 4-Displacement, for which the vector base is at the origin \((0,0,0,0) = \text{4-Zero}\).

4-Position is Lorentz Invariant, but not Poincaré Invariant. A standard 4-Displacement is both.

\[ \Delta R \cdot \Delta R = [(c \Delta t)^2 - \Delta r \cdot \Delta r] = 0 \]

for time-like \((+\rangle\)

- \((c \Delta t)^2\)

for light-like \((0)\)

- \(- (\Delta r_o)^2\)

for space-like \((-\rangle\)

An Event \((*)\) is a point in SpaceTime

The 4-Position points to an Event.

A WorldLine is a series of connected Events which trace out a path in SpaceTime, such as the track of a moving particle.

4-Vector SRQM Interpretation of QM

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http://scirealm.org/SRQM.pdf

SR 4-Vector

(2,0)-Tensor \(T^\mu_{\nu}\)

(1,1)-Tensor \(T^\mu_{\nu}, \text{ or } T_{\nu}^\mu\)

SR 4-Covector: OneForm

(0,1)-Tensor \(V^\mu = (v_0, v)\)

SR 4-Scalar

(0,0)-Tensor \(S\) or \(S_0\)

Lorentz Scalar
SR Invariant Intervals

Minkowski Diagram: Lorentz Transform

Since the SpaceTime magnitude of \( \mathbf{U} \) is a constant (c), changes in the components of \( \mathbf{U} \) are like rotating the 4-Vector without changing its length. It keeps the same magnitude. Rotations, purely spatial changes, (eg. along x,y) result in circular displacements. Boosts, or temporal-spatial changes, (eg. along x,t) result in hyperbolic displacements. The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

\[
\mathbf{U} \cdot \mathbf{U} = \gamma(c,u) \cdot \gamma(c,u) = \gamma^2(c^2-u \cdot u) = (c^2)
\]

Rotation (x,y): Purely Spatial

Boost (x,t): Spatial-Temporal

The Light Cone / Minkowski Diagram provides a great visual representation of SpaceTime
The Minkowski Diagram provides a great visual representation of SpaceTime.

The SpaceTime magnitude of \( \mathbf{U} \) is a constant (c), changes in the components of \( \mathbf{U} \) are like rotating the 4-Vector without changing its length. It keeps the same magnitude (c). Rotations, purely spatial changes, {eg. along x,y} result in circular displacements. Boosts, or temporal-spatial changes, {eg. along x,t} result in hyperbolic displacements. The interval between the origin and a given topograph-line is a Lorentz Invariant Constant.

\[
\begin{align*}
\Delta \mathbf{R} \cdot \Delta \mathbf{R} &= [(c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r}] = (c\Delta t)^2 - (\Delta r)^2 \\
&= (0) \text{ Time-like:Temporal} \quad (0) \text{ Light-like:Null:Photonic} \quad (+) \text{ causal = 1D temporally-ordered, spatially relative} \\
&= (0) \text{ causal & topological, maximum signal speed (|\Delta \mathbf{r}|/c = c)} \quad (-) \text{ temporally relative, topological = 3D spatially-ordered}
\end{align*}
\]

\[\text{SR:Minkowski Metric} \quad \partial [\mathbf{R}] = \partial^\mu \mathbf{R}^\nu = \eta^{\mu \nu} = V^\mu + H^\mu \to \]

\[
\begin{align*}
\text{Diag}[1,-1,-1,-1] &= \text{Diag}[1,-\delta^k] \\
\{\eta_{\mu \mu}\} &= 1/\{\eta^{\mu \mu}\} : \eta_{\nu}^{\nu} = \delta_{\nu}^{\nu} \quad \text{in Cartesian form} \quad \text{"Particle Physics" Convention}
\end{align*}
\]

\[
\text{Tr}[\eta^{\nu}] = 4
\]

\[
\Delta \mathbf{R} = [(c\Delta t)^2 - \Delta \mathbf{r} \cdot \Delta \mathbf{r}] = (0)
\]

\[
\text{Light-like:Null:Photonic} \quad (0) \text{ causal & topological, maximum signal speed (|\Delta \mathbf{r}|/c = c)}
\]

\[
\text{Space-like:Spatial} \quad (-) \text{ temporally relative, topological = 3D spatially-ordered}
\]
SRQM: Some Basic 4-Vectors

4-Position, 4-Velocity, 4-Acceleration, (RestMass), 4-Momentum, 4-Force

SpaceTime Dynamics

**4-Vectors:**

\[ R = (c t, r) \]

\[ U = \gamma (c, u) = \frac{dR}{d\tau} \]

\[ A = \gamma c \gamma' u + \gamma a = \frac{dU}{d\tau} \]

**4-Momentum**

\[ P = m_o U \]

\[ F = \frac{m_o U}{c^2} = \frac{dP}{d\tau} \]

**4-Position**

\[ R \]

**4-Velocity**

\[ U \]

**4-Acceleration**

\[ A \]

**4-Force**

\[ F \]

**4-Gradient**

\[ \partial = (\partial_t, -\nabla) = \frac{d}{d\tau} \]

**Trace**

\[ \text{Trace}[T_{\mu\nu}] = \eta_{\mu\nu}T_{\mu\nu} = T_{\mu\nu} = T \]

\[ V \cdot V = V^\mu V_{\mu} = (v^2)^2 - V \cdot V = (v^2)^2 \]

Lorentz Scalar

---

This group of 4-Vectors are the main ones that are connected by the ProperTime Derivative.

\[ U \cdot \partial U/\tau = \gamma (c, u) / c^2 \]

\[ = \gamma (c^2 t - r \cdot u) / c^2 = (c^2 t_0) / c^2 \]

\[ = t_0 = \tau \]

**The classical part of it, the convective derivative,**

\[ (\partial_t + u \cdot \nabla) \]

**is known by many different names:**

The convective derivative is a derivative taken with respect to a moving coordinate system. It is also called the advective derivative, derivative following the motion, hydrodynamic derivative, Lagrangian derivative, material derivative, particle derivative, substantial derivative, substantive derivative, Stokes derivative, or total derivative.
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Momentum, $E=mc^2$

Special Relativity

$|v| = |u| = 0 \leftrightarrow c$

Classical Mechanics

$|v| = |u| \ll c$

Temporal part:

$E = mc^2 = \gamma m_o c^2 = \gamma E_o$

$E = m_o c^2 + (\gamma - 1)m_o c^2$

$E = E_o + (\gamma - 1)E_o$

(rest) + (kinetic)

Spatial part:

$p = \gamma m_o u = m u$

Temporal part:

$E \sim (1+(v/c)^2/2)m_o(c,u)$

Spatial part:

$p \sim (1)m_o u = m_o u \rightarrow mu$

Since time:space don't mix in CM,
Typically use energy $E$ & 3-momentum $p$ separately

The relativistic Gamma factor $\gamma = 1/\sqrt{1-(v/c)^2}$

The 1st order Newtonian Limit gives $\gamma \sim 1 + O[(v/c)^2]$

The 2nd order Newtonian Limit gives $\gamma \sim 1 + (v/c)^2/2 + O[(v/c)^4]$

For historical reasons, velocity can be represented by either $(v)$ or $(u)$

4-Velocity

$U = \gamma(c,u)$

$p = (E/c,p) = (m,c)$

4-Momentum

$P = (E,c,p)$

Temporal part:

$E = m_o c^2$

Spatial part:

$p = m_o u$

4-Vector SRQM Interpretation of QM

SR 4-Tensor

$(2,0)$-Tensor $T^{\mu\nu}$

$(1,1)$-Tensor $T^\nu$, or $T^\nu_{\mu}$

$(0,2)$-Tensor $T_{\mu
\nu}$

SR 4-Vector

$(1,0)$-Tensor $V^\nu = (v^\nu,0)$

SR 4-CoVector: OneForm

$(0,1)$-Tensor $V_\nu = (v_\nu,0)$

SR 4-Scalar

$(0,0)$-Tensor $S$ or $S_0$

Lorentz Scalar

Classical (scalar)

Galliean Invariant

3-vector

Not Lorentz Invariant

Trace $[T^\nu] = \eta_{\mu\nu} T^\nu = T^\nu_{\mu} = T$

$V.V = V^\nu \eta_{\nu\rho} V^\rho = [(v^\nu)^2 - v^2] = (v^2)^2$

$Lorentz$ Scalar
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Acceleration, SpaceTime Orthogonality

4-Velocity
\[ U = \gamma (c, u) \]

4-Acceleration
\[ A = \gamma (c' \gamma', u + \gamma a) \]

ProperTime Derivative
\[ U \cdot \partial = \gamma (c, u) \cdot (\partial / c, -\nabla) = \gamma (\partial t + u \cdot \nabla) = \gamma d/dt = \gamma d/d\tau \]

4-Gradient
\[ \partial = (\partial t/c, -\nabla) \rightarrow (\partial t/c, -\partial_x, -\partial_y, -\partial_z) \]

4-Gradient
\[ \partial \cdot R = 4 \text{ SpaceTime Dimension} \]
\[ \partial [R] = \eta^{\mu\nu} \rightarrow \text{Diag}[1,-1,-1,-1] \text{ Minkowski Metric} \]

ProperTime Derivative
\[ U \cdot \partial \cdot \partial = \gamma (c, u) \cdot (\partial / c, -\nabla) = \gamma (\partial t + u \cdot \nabla) = \gamma d/dt = \gamma d/d\tau \]

SpaceTime Orthogonality
4-Velocity is orthogonal to its 4-Acceleration.
\[ U \cdot U = c^2 \]
\[ d/d\tau[U \cdot U] = d/d\tau[c^2] = 0 \]
\[ d/d\tau[U \cdot U] = d/d\tau[U] \cdot U + U \cdot d/d\tau[U] = A \cdot U + U \cdot A = 2(U \cdot A) = 0 \]
\[ U \cdot A = U \cdot U' = 0 \leftrightarrow U \perp A \]

4-Vectors
\[ R = (ct, r) \]

4-Position
\[ R = (ct, r) \]

The Lorentz Scalar Product can be used to show SpaceTime orthogonality when the result is zero.
\[ U \cdot U = c^2 \]
\[ d/d\tau[U \cdot U] = d/d\tau[c^2] = 0 \]

U moves along Worldline.

4-Acceleration is the thing which causes a WorldLine to bend/curve.
\[ A = U' = R'' \text{ is normal to WorldLine} \]
\[ A = U' = R'' \text{ is normal to WorldLine} \]

U is Temporal
\[ U = R' \text{ is tangent to WorldLine} \]

WorldLine

\[ V \cdot V = V^\mu V_\mu = (v^0)^2 - \nabla \cdot V = (v^0)^2 \]
\[ = \text{Lorentz Scalar} \]

SR 4-Tensor
\[ (2,0)-Tensor T^{\mu \nu} \]

SR 4-Vector
\[ (1,0)-Tensor V^\mu = V = (v^0, v) \]
\[ (1,1)-Tensor T^\mu_\nu, \text{ or } T^\mu_\nu \]

SR 4-CoVector: OneForm
\[ (0,1)-Tensor V_\mu = (v_0, -v) \]

SR 4-Scalar
\[ (0,0)-Tensor S \text{ or } S_\mu = \text{Lorentz Scalar} \]
SRQM: Some Basic 4-Vectors

4-Displacement, 4-Velocity, Relativity of Simultaneity

If Lorentz Scalar ($\mathbf{U} \cdot \Delta \mathbf{X} = 0 = c^2 \Delta t$), then the ProperTime displacement ($\Delta \tau$) is zero, and the event separation ($\Delta \mathbf{X} = \mathbf{X}_2 - \mathbf{X}_1$) is orthogonal to the worldline $\mathbf{U}$. $\mathbf{X}_1$ and $\mathbf{X}_2$ are therefore simultaneous for the observer on this worldline $\mathbf{U}$.

Examining the equation we get $\gamma(c^2 \Delta t - \mathbf{u} \cdot \Delta \mathbf{x}) = 0$. The coordinate time difference between the events is ($\Delta t = \mathbf{u} \cdot \Delta \mathbf{x} / c^2$). The condition for simultaneity in an alternate frame (moving at 3-velocity $\mathbf{u}$ wrt. the worldline $\mathbf{U}$) is $\Delta t = 0$, which implies ($\mathbf{u} \cdot \Delta \mathbf{x} = 0$).

This can be met by:

- ($|\mathbf{u}| = 0$), the alternate observer is not moving wrt. the events, i.e. is on worldline $\mathbf{U}$ or on a worldline parallel to $\mathbf{U}$.
- ($|\Delta \mathbf{x}| = 0$), the events are at the same spatial location (co-local).
- ($\mathbf{u} \cdot \Delta \mathbf{x} = 0$), the alternate observer's motion is perpendicular (orthogonal) to the spatial separation $\Delta \mathbf{x}$ of the events in that frame.

If none of these conditions is met, then the events will not be simultaneous in the alternate reference frame. This is the mathematics behind the concept of Relativity of Simultaneity.

**SR 4-Scalar**

- $(0,0)$-Tensor $S$ or $S_0$
- Lorentz Scalar

**SR 4-Vector**

- $(0,1)$-Tensor $V$, or $V^\mu = (v^0, v^1, \ldots, v^n)$

**SR 4-CoVector: OneForm**

- $(1,0)$-Tensor $V^\mu = (v^0, -v^1, \ldots, -v^n)$

**SR 4-Tensor**

- $(2,0)$-Tensor $T^\mu_\nu$
- $(1,1)$-Tensor $\Lambda^\mu_\nu$
- $(0,2)$-Tensor $T_\mu^\nu$

- Lorentz Transform

- Minkowski Metric

- ProperTime Derivative

- Rest-Frame

- Lorentz Boost-Frame

- Lorentz Scalar

- Lorentz Transform

- Lorentz Boost-Frame
Interesting note:

Most 4-Vectors have 4 independent components. (1 temporal, 3 spatial)

The 4-Velocity has only the 3 spatial however, due to its invariant magnitude $c^2 \cdot U \cdot U = c^2$.

This fact allows one to multiply it by a Lorentz Scalar to make a new 4-Vector with 4 independent components, as shown in the diagram.

Proof of non-varying $c$.

$$P \cdot P = (m \cdot c)^2 = (E/c)^2$$

$$J \cdot J = (\rho \cdot c)^2$$

$$N \cdot N = (n \cdot c)^2$$

$$A \cdot A = (\varphi \cdot c)^2$$

$$K \cdot K = (w \cdot c)^2$$

$$\text{Trace}[T_{\mu \nu}] = \eta_{\mu \nu}T_{\mu \nu} = T_{\mu \nu} = T$$

$$V \cdot V = V^\mu V_\nu \eta^{\mu \nu} = (v^\mu)^2 - v \cdot v = (v^\mu)^2$$

$$= \text{Lorentz Scalar}$$
SRQM Diagram:

SRQM Motion * Lorentz Scalar = Interesting Physical 4-Vector

4-Displacement $\Delta R = (c\Delta t, \Delta r)$
4-Position $R = (ct, r)$

4-NumberFlux $N = (nc, n) = n(c, u)$
4-ProbCurrDensity $J = (pc, j) = \rho(c, u)$
4-ProbabilityFlux $J_{\text{prob}} = (\rho_{\text{prob}}, j_{\text{prob}})$

4-Velocity $U = \gamma(c, u)$
4-Acceleration $A = \gamma(cy', y' u + ya)$

Interesting note:
Most 4-Vectors have 4 independent components.
{1 temporal, 3 spatial}

This fact allows one to multiply it by a Lorentz Scalar Invariant to make a new 4-Vector with 4 independent components, as shown in the diagram.

Proof of non-varying (c)

SR 4-Tensor (2,0)-Tensor $T^{iv}$
 SR 4-Vector (1,0)-Tensor $V^v = V = (v^0, v)$
 SR 4-CoVector:OneForm (0,1)-Tensor $V_i = (\omega_0, \omega)$
 SR 4-Scalar (0,0)-Tensor $S$ or $\phi_0$ Lorentz Scalar

$\partial \cdot R = 4$ SpaceTime Dimension
$\partial [\ldots] = n^{iv} \rightarrow \text{Diag}[1,-1,-1,-1]$ Minkowski Metric

$\partial \cdot \delta[\ldots] = \gamma d/dt[\ldots] = d/d\tau[\ldots]$ ProperTime Derivative

Rest Number Density
Rest Probability Density

EM Charge

$J = (\rho c, j)$

$\varepsilon_0 c^2 / \mu_0$

$(\partial \cdot A) - \partial (\partial \cdot A) = \mu_0 J$
Maxwell EM Wave Eqn

$E \cdot /\omega_0$

$K \cdot (\omega/c, \hbar)$

$\text{EM Frequency}$

$\omega /c^2$

$\text{Rest Angular Frequency}$

$\text{Wave Velocity}$

$\text{Group Velocity}$

$\text{Phase Velocity}$

$\text{Rest Mass:Energy}$

$E = mc^2$

$P = hK$

$\text{Einstein de Broglie}$

http://scirealm.org/SRQM.pdf

SciRealm.org
John B. Wilson
SciRealm@aol.com

Existing SR Rules
Quantum Principles

4-Vector SRQM Interpretation
of QM
SRQQM Diagram: ProperTime Derivative
Very Fundamental Results

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
4-Position
\[ R = (ct, r) \]
\[ \partial \cdot R = 4 \text{ SpaceTime Dimension} \]

\[ \partial[R] = \eta^{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1] \]
Minkowski Metric

\[ \gamma \frac{d}{dt} \] [ ]
4-Velocity
\[ U = \gamma (c, u) \]
\[ U \cdot U = c^2 \]

4-Momentum
\[ P = (E/c, p) = (mc, p) \]
4-Force
\[ F = \gamma (\dot{E}/c, \dot{f} = \dot{p}) \]

\[ \eta_{\mu\nu} T^{\mu\nu} = T \]
4-Vectors:
\[ R = <\text{Event}> \]
\[ U = dR/d\tau \]
\[ A = dU/d\tau \]
\[ P = m_0 U \]
\[ F = dP/d\tau \]

SR 4-Tensor
\[(2, 0)\]-Tensor \[ T^{\mu\nu} \]
\[(1, 1)\]-Tensor \[ T^\mu \text{ or } T^\nu \]
\[(0, 2)\]-Tensor \[ T_{\mu\nu} \]

SR 4-Vector
\[(1, 0)\]-Tensor \[ V^\mu = V = (\vec{v}, v) \]
\[(0, 1)\]-Tensor \[ V_\mu = (v_\mu, \vec{v}) \]

SR 4-Scalar
\[(0, 0)\]-Tensor \[ S \text{ or } S_0 \]
Lorentz Scalar

Trace[\[ T^{\mu\nu} = \eta_{\mu\nu} T^{\mu\nu} = T \]
\[ V \cdot V = V^\mu (\eta_{\mu\nu} V^\nu) = (\vec{v} \cdot \vec{v}) \text{ - } \vec{v} \cdot \vec{v} = (v^2)^2 \]
Lorentz Scalar

4-Gradient
\[ \partial = (\partial/c, -\nabla) \]

ProperTime Derivative
\[ \partial = \gamma (c, u) \rightarrow (\partial/c, -\nabla) = \gamma (\partial + u \cdot \nabla) \]

\[ \gamma d/d\tau = d/d\tau \]
Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)

4 Acceleration
\[ A = \gamma (c, u) \rightarrow (\partial/c, -\nabla) = \gamma (\partial + u \cdot \nabla) \]

4-Vector SRQM Interpretation of QM
SciRealm.org
John B. Wilson
http://scirealm.org/SRQM.pdf

Event WorldLine
\[ \partial = \gamma \partial/c \rightarrow \text{Event is perpendicular to Event WorldLine} \]
\[ U \cdot A = U \cdot U' = 0 \]

Acceleration of Event
\[ \partial = \gamma \partial/c \rightarrow \text{ProperTime Derivative} \]
\[ \partial = \gamma \partial/c \rightarrow \text{ProperTime Derivative} \]

Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)

4-Vector SRQM Interpretation of QM
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Event WorldLine
\[ \partial = \gamma \partial/c \rightarrow \text{Event is perpendicular to Event WorldLine} \]
\[ U \cdot A = U \cdot U' = 0 \]

Acceleration of Event
\[ \partial = \gamma \partial/c \rightarrow \text{ProperTime Derivative} \]
\[ \partial = \gamma \partial/c \rightarrow \text{ProperTime Derivative} \]

Conservation of the 4-Velocity Flow (4-Velocity Flow-Field)
SRQM Diagram:
Local Continuity of 4-Velocity leads to all of the Conservation Laws

Conservation Laws:
All of the Physical Conservation Laws are in the form of a 4-Divergence, which is a Lorentz Invariant Scalar equation.

These are local continuity equations which basically say that the temporal change in a quantity is balanced by the flow of that quantity into or out of a local spatial region.

Conservation of Charge:
\[ \partial \cdot J = \frac{\partial}{\partial t} \rho + \nabla \cdot j = 0 \]

SR 4-Tensor

\[ (2,0) \text{-Tensor } T^{\mu \nu} \]

SR 4-Vector

\[ (1,0) \text{-Tensor } V^\mu = V = (\gamma, \mathbf{v}) \]

SR 4-CoVector: OneForm

\[ (0,1) \text{-Tensor } V_\mu = (\gamma, -\mathbf{v}) \]

SR 4-Scalar

\[ (0,0) \text{-Tensor } S \text{ or } S_0 \text{ Lorentz Scalar} \]
SRQM Motion * Lorentz Scalar
Conservation Laws, Continuity Eqns

SRQM Diagram:

- 4-Displacement \( \Delta R=(c\Delta t, \Delta r) \)
- 4-Position \( R=(ct, r) \)
- 4-Momentum \( p=(mc, \mathbf{u}) \)
- 4-Energy \( E=mc^2 \)
- 4-Mass \( m \)
- 4-Charge \( \phi \)
- 4-WaveVector \( k \)
- 4-Divergence \( \nabla \cdot J \)
- 4-Curl \( \nabla \times A \)

Conservation Laws:
- Conservation of Mass: \( \partial_t \rho = 0 \)
- Conservation of Charge: \( \partial_t \phi = 0 \)
- Conservation of Momentum: \( \partial_t \mathbf{p} = 0 \)
- Conservation of Angular Momentum: \( \partial_t \mathbf{L} = 0 \)

4-Gradient \( \nabla = (\partial_t, \mathbf{\nabla}) \)

These are Fluid or Density-type Conservation/Continuity Laws

These are Individual Particle/Wave/Delta-function Conservation/Continuity Laws

Existing SR Rules

Quantum Principles

SR 4-Tensor
- (2,0)-Tensor \( T^{iv} \)
- (1,1)-Tensor \( T^{iv} \), or \( T^{00} \)
- (0,2)-Tensor \( T^{iv} \)

SR 4-Vector
- (1,0)-Tensor \( V^i = V = (\mathbf{v}, \phi) \)
- SR 4-CoVector: One-Form \( V_i = \mathbf{v}_i \)
- SR 4-Scalar \( (0,0)-Tensor S or S_0 \)

SRQM Motion
- Lorentz Scalar

SRQM Interpretation
- 4-Vector SRQM
- John B. Wilson
- SciRealm.org

Conservation of Charge:
- \( \partial_t \mathbf{J} = (\partial_t \phi + \mathbf{v} \cdot \mathbf{J}) = 0 \)

Conservation of Momentum:
- \( \partial_t \mathbf{P} = 0 \)

Lorentz Gauge
- Conservation of EM Potential: \( \partial_t \mathbf{A} = 0 \)

Einstein de Broglie
- Conservation of 4-WaveVector: \( \partial_t \mathbf{K} = 0 \)

Trace[\( T^{iv} \)] = \( \eta_{iv}T^{iv} = T_{\mathbf{V}} = T \)
- Lorentz Scalar

EM Potential:
- \( \mathbf{A} = \frac{\mathbf{v}}{c} \phi \)

Physicists
- A Tensor Study
- Conservation Laws

SRQM Diagram:
- 4-ProbabilityFlux
- 4-NumberFlux
- 4-Position
- 4-Displacement
- 4-ProbabilityDensity
SRQM: Some Basic 4-Vectors

4-Velocity, 4-Gradient, Time Dilation

The Minkowski Diagram provides a great visual representation of SpaceTime

![Minkowski Diagram](image)

**at-rest worldline** \( U_0 \)

\( u = 0 \)

fully temporal

**const inertial motion worldline** \( U \)

\( 0 < u < c \)

trades some time for space

\[
\begin{align*}
4\text{-Gradient} & = (\partial/c, \nabla) \\
4\text{-Velocity} & = \gamma(c, u) \\
\gamma & = 1/\sqrt{1-(u/c)^2} = 1/\sqrt{1-\beta^2}
\end{align*}
\]

Everything moves into future (+t) at the speed-of-light (c) in its own spatial rest-frame

Since the SpaceTime magnitude of \( U \) is a constant, changes in the components of \( U \) are like "rotating" the 4-Vector without changing its length. However, as \( U \) gains some spatial velocity, it loses some "relative" temporal velocity. Objects that move in some reference frame "age" more slowly relative to those at rest in the same reference frame.

**Time Dilation!**

\[
\Delta t = \gamma \Delta \tau = \gamma \Delta t_o
\]

\[
dt = \gamma d\tau
\]

\[
d/d\tau = \gamma d/t
\]

Each observer will see the other as aging more slowly; similarly to two people moving oppositely along a train track, seeing the other as appearing smaller in the distance.
There are multiple ways of writing out the components of the 4-WaveVector, with each one giving an interesting take on what the 4-WaveVector means.

An SR wave $\Psi$ is actually composed of two tensors:

1. 4-Vector propagation part = $K$ (the engine), in $e^{\lambda(t-k\cdot r)}$, ex. Maxwell Photon Wave
2. Variable amplitude part = $A$ (the load), depends on what is waving...

4-Scalar $A$: $\Psi = A e^{\lambda(t-k\cdot r)}$ ex. KG Quantum Wave
4-Position $R$: $\Psi = R e^{\lambda(t-k\cdot r)}$ ex. Maxwell Photon Wave
4-Tensor $A_{\mu\nu}$: $\Psi^{\mu\nu} = A^{\mu\nu} e^{\lambda(t-k\cdot r)}$ ex. Gravitational Wave Approx.

The $\Psi$ tensor-type will match the $A$ tensor-type, as the propagation part $e^{\lambda(t-k\cdot r)}$ is overall dimensionless.

One comparison I find very interesting is:

- $R \cdot R = (ct)^2 = (ct)^2$
- $K \cdot K = (ct/c\gamma)^2 = (ct/c\gamma)^2$
- $\partial \cdot \partial = (\partial/c\gamma) \cdot (\partial/c\gamma) = (\partial/c\gamma)^2$

I believe the last one is correct: $(\partial \cdot \partial)[R] = 0 = (\partial/c\gamma)^2[R] = A_{\mu\nu} c^2 = 0$: The 4-Acceleration seen in the ProperTime Frame = RestFrame = 0 Normally $(\partial/d\tau)^2[R] = A$, which could be non-zero. But that is for the total derivative, not the partial derivative.

$\Psi(x) = A e^{\lambda(t-k\cdot r)}$: Explicit form of an SR plane wave
$\Psi(x) = \sum \phi [\psi]$ Complete wave is a superposition of multiple plane waves,
$\partial[\psi] = \partial[\phi^{\lambda(i K\cdot r)}] = -i K([\phi^{\lambda(i K\cdot r)}] = -i K[\ hearty\ symbol\ of\ wave\ ]$
$\partial = -i K$ as the condition for a complex-valued plane wave.
$\partial[\psi(x)] = (-i)(-i)(K\cdot R)\[\psi(x)] = -(K\cdot K)[\psi(x)]$
SRQM: Some Basic 4-Vectors

4-Velocity, 4-WaveVector

Wave Properties, Relativistic Doppler Effect

Relativistic SR Doppler Effect

Choose an observer frame for which:
\( \mathbf{K} = (\omega/c, \mathbf{k}) \), with \( k, \hat{n} \) pointing toward observer
\[ \mathbf{K} \cdot \mathbf{U} = \gamma(\omega - k \cdot u) = \omega, \]
\[ \mathbf{K} \cdot \mathbf{U}_{\text{obs}} = (\omega/c, \mathbf{K}) \cdot (\omega/c, \mathbf{n}/v), \rho_{\text{obs}} = \omega, \]
\[ \mathbf{K} \cdot \mathbf{U}_{\text{emit}} = (\omega/c, \mathbf{k}) \gamma(c, u) = \gamma(\omega - k \cdot u) = \omega_{\text{emit}}. \]

For photons, \( \mathbf{K} \) is null \( \rightarrow \mathbf{K} \cdot \mathbf{K} = 0 \rightarrow k = (\omega/c) \hat{n} \)
\[ \omega_{\text{obs}}/\omega_{\text{emit}} = \omega/[\gamma(\omega - (\omega/c) \hat{n} \cdot \mathbf{u})] = 1/[\gamma(1 - \hat{n} \cdot \beta)] = 1/[\gamma(1 - |\beta| \cos[\theta_{\text{obs}}])]. \]

\[ \omega_{\text{obs}}/\omega_{\text{emit}} = \gamma \omega_{\text{obs}}/\omega_{\text{emit}} = \omega_{\text{obs}}/\omega_{\text{emit}}. \]

\[ \omega_{\text{obs}} = \omega_{\text{emit}}/\sqrt{[\gamma(1 - \hat{n} \cdot \beta)]} = \omega_{\text{emit}} \gamma \sqrt{[1 + |\beta|]} \sqrt{[1 + |\beta|]} \]

with \( \gamma = 1/\sqrt{[1 - \beta^2]} = 1/\sqrt{[1 + |\beta|]}. \)

For motion of emitter \( \beta \): (in observer frame of reference)
\( \text{Away from obs, } (\hat{n} \cdot \beta) = -\beta, \omega_{\text{obs}} = \omega_{\text{emit}} \gamma \sqrt{[1 + |\beta|]} \sqrt{[1 + |\beta|]} \) \( \rightarrow \text{Red Shift} \)
\( \text{Toward obs, } (\hat{n} \cdot \beta) = +\beta, \omega_{\text{obs}} = \omega_{\text{emit}} \gamma \sqrt{[1 + |\beta|]} \sqrt{[1 - |\beta|]} \) \( \rightarrow \text{Blue Shift} \)
\( \text{Transverse, } (\hat{n} \cdot \beta) = 0, \omega_{\text{obs}} = \omega_{\text{emit}} \gamma \rightarrow \text{Transverse Doppler Shift} \)

The Phase Velocity of a Photon \( v_{\text{phase}} = c \) equals the Particle Velocity of a Photon \( u = c \)
The Phase Velocity of a Massive Particle \( v_{\text{phase}} > c \) is greater than the Velocity of a Massive Particle \( u < c \)

Trace[Tr^\nu] = \eta_{\nu\mu}T^{\mu\nu} = T_{\nu}^{\nu} = T, \quad V \cdot V = V_{\nu}V^{\nu} = (v_{\nu}v^{\nu}) = (v_{\nu}v^{\nu})^2 = \text{Lorentz Scalar}
**SRQM: Some Basic 4-Vectors**

### 4-Velocity, 4-WaveVector

**Wave Properties, Relativistic Aberration**

- **4-Velocity** $U = \gamma(c, u)$, with $U \cdot U = (c)^2$
  - Rest Angular Frequency $\omega/c^2$
- **4-WaveVector** $K = (\omega/c, k) = (\omega/c, \omega\hat{n}/v_{\text{phase}})$
  - $K \cdot K = (\omega/c)^2$

#### Relativistic SR Aberration Effect

- (\(\hat{n}\)) here is the unit-directional 3-vector of the photon
- \(\omega_{\text{obs}} = \omega_{\text{emit}}/[\gamma(1 - \hat{n} \cdot \beta)] = \omega_{\text{emit}}/[\gamma(1 - |\beta|\cos[\theta_{\text{obs}}])]\)
  - Change reference frames with \{obs→emit\} & \{\beta \rightarrow -\beta\}
  - \(\omega_{\text{emit}} = \omega_{\text{obs}}/[\gamma(1 + \hat{n} \cdot \beta)] = \omega_{\text{obs}}/[\gamma(1 + |\beta|\cos[\theta_{\text{emit}}])]\)
  - \((\omega_{\text{obs}})^*(\omega_{\text{emit}}) = (\omega_{\text{emit}}/[\gamma(1 - |\beta|\cos[\theta_{\text{obs}}])])^*(\omega_{\text{obs}}/[\gamma(1 + |\beta|\cos[\theta_{\text{emit}}])])\)
  - \(1 = (1/\gamma(1 - |\beta|\cos[\theta_{\text{obs}}]))^*(1/\gamma(1 + |\beta|\cos[\theta_{\text{emit}}]))\)
  - \(1 = (\gamma(1 - |\beta|\cos[\theta_{\text{obs}}]))^*(\gamma(1 + |\beta|\cos[\theta_{\text{emit}}]))\)
  - \(1 = \gamma^2(1 - |\beta|\cos[\theta_{\text{obs}}])^*(1 + |\beta|\cos[\theta_{\text{emit}}])\)
  - Solve for $|\beta|\cos[\theta_{\text{obs}}]$ and use \((\gamma^2-1) = \beta^2\gamma^2\)
  - $\cos[\theta_{\text{obs}}] = (\cos[\theta_{\text{emit}}] + |\beta|) / (1 + |\beta|\cos[\theta_{\text{emit}}])$

#### Wave Properties

- Wave Group velocity ($\nu_{\text{group}}$) is mathematically the same as Particle velocity ($u$).
- Wave Phase velocity ($\nu_{\text{phase}}$) is the speed of an individual plane-wave, also the speed of signal synchronicity, the speed of the wave of coordinated flashes.

The Phase Velocity of a Photon \(\nu_{\text{phase}} = c\) equals the Particle Velocity of a Photon \(u = c\)

The Phase Velocity of a Massive Particle \(\nu_{\text{phase}} > c\) is greater than the Velocity of a Massive Particle \(u < c\)
SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

See Hamilton-Jacobi Formulation of Mechanics
for info on the Lorentz Scalar Invariant SR Action.

**Note** This is the Action \( S_{\text{action,free}} \) for a free particle. Generally Action is for the 4-TotalMomentum \( P_t \) of a system.

### SR 4-Vector

- \( (2,0) \)-Tensor \( T_{iv}^{\mu} \)
- \( (1,1) \)-Tensor \( T^v \), or \( T^v \)
- \( (0,2) \)-Tensor \( T_{\mu v} \)

### SR 4-Scalar

- Lorentz Scalar \( S \)

### SR 4-CoVector: OneForm

- \( (1,0) \)-Tensor \( V^i \)
- \( (0,1) \)-Tensor \( V_{\mu} \)

**See SR Wave Definition**
for info on the Lorentz Scalar Invariant SR WavePhase.

### Existing SR Rules

**Quantum Principles**

- 4-Vector SRQM Interpretation of QM
- 4-Vector SRQM
- SciRealm.org
- John B. Wilson

**http://scirealm.org/SCIREALM.pdf**
Some Cool Minkowski Metric Tensor Tricks

4-Gradient, 4-Position, 4-Velocity

SpaceTime is 4D

4-Velocity

\[ U = \gamma (c, u) \]

\[ U \cdot \partial = \gamma (c, u) \cdot (\partial / c - \nabla) = \gamma (\partial + u \cdot \nabla) \]

\[ = d / d \tau = \gamma d / dt \]

4-Position

\[ R = (ct, r) \]

4-Gradient

\[ \partial = (\partial / c, -\nabla) \rightarrow \gamma (\partial / c, -\partial_x, -\partial_y, -\partial_z) \]

\[ \partial [R] = \eta^{\mu \nu} \rightarrow \text{Diag}[1,-1,-1,-1] \]

Minkowski Metric

\[ \partial \cdot R = 4 \]

Space Time Dimension

\[ \eta_{\mu \nu} \rightarrow \text{Diag}[1,-1,-1,-1] \]

Index-Lowered Minkowski Metric

\[ \eta^{\mu \nu} = \text{Diag}[1,1,1,1] \]

Index-Raised Minkowski Metric

\[ \delta^{\mu \nu} = \text{Diag}[1,1,1,1] \]

Kronecker Delta

\[ \text{Trace}[\eta^{\mu \nu}] = \eta^{\mu \mu} = 4 \]

Space Time Dimension

\[ \text{Trace}[\eta_{\mu \nu}] = \eta_{\mu \mu} = 4 \]

Space Time Dimension

Thus

\[ U \cdot \partial [R] = (U^a \partial^a) [R^\alpha] = (U^a \partial_a^{\alpha}) [R^\alpha] = (U^a \partial^b) [R^\alpha] = (U^a) \partial^b [R^\alpha] = (U^a) \eta^{\beta \gamma} = U^\gamma = U = (d / d \tau) [R] \]

Thus

Lorentz Scalar Product \( U \cdot \partial \) = Derivative wrt. ProperTime \((d / d \tau) = \text{Relativistic Factor} \ast \text{Derivative wrt. CoordinateTime} \gamma (d / dt)\):

\[ \Lambda_{\mu \nu} \Lambda^{\mu \nu} = 4 = \Lambda^\mu \Lambda_\mu \]

SR 4-Tensor

\( (2,0) \)-Tensor \( T^{\mu \nu} \)

\( (1,1) \)-Tensor \( T^\mu \), or \( T^\nu \)

\( (0,2) \)-Tensor \( T_{\mu \nu} \)

SR 4-Vector

\( (1,0) \)-Tensor \( V^\mu = V = (v^0, v) \)

\( (0,1) \)-Tensor \( V_\mu \)

SR 4-Vector SRQM Interpretation of QM

Physics

A Tensor Study of Physical 4-Vectors

Interpretation of QM

SciRealm.org

John B. Wilson

SciRealm@aol.com

http://scirealm.org/SRQM.pdf
SRQM+EM Diagram:

4-Vectors

4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
4-Gradient
\[ \partial = (\partial_t/c, -V) \]
4-Position
\[ R = (ct, \vec{r}) \]
4-Velocity
\[ U = (c, \vec{v}) \]

SR 4-Tensor
(2,0)-Tensor \( T^{\mu}_{\nu} \)
(1,1)-Tensor \( T^{\nu} = V = (\vec{v}, v) \)
(0,2)-Tensor \( T^{\nu}_{\mu} \)

SR 4-CoVector: OneForm
(0,1)-Tensor \( V_{\mu} = (v_{\mu} - \vec{v}) \)
(0,0)-Tensor \( S \) or \( S_0 \)
Lorentz Scalar

SR 4-Scalar
(0,0)-Tensor

Complex

SR 4-Vector

4-Acceleration
\[ A = \gamma(cy', y'u + ya) \]
4-Gradient
\[ \partial = (\partial_t/c, -V) \]
4-Acceleration
\[ A = \gamma(cy', y'u + ya) \]

SR 4-NumberFlux
\[ N = (nc, n) = n(c, u) \]
4-Force
\[ F = \gamma(\vec{E}/c, f = \vec{p}) \]
4-Momentum
\[ P = (mc, p) = (E/c, p) \]
4-MassFlux
\[ G = (\rho_c, g) = (\rho / c, g) \]
4-Force
\[ F = \gamma(\vec{E}/c, f = \vec{p}) \]
4-Momentum
\[ P = (mc, p) = (E/c, p) \]
4-Momentum
\[ P_T = (E_T/c, p_T) = (H/c, p) \]

SR 4-Force
\[ \gamma(\vec{E}/c, f = \vec{p}) \]
4-Displacement
\[ \Delta R = (c\Delta t, \Delta r) \]
4-TotalWaveVector
\[ K_T = (\omega/c, k_\nu) \]
4-TotalMomentum
\[ P_T = (E_T/c, p_T) = (H/c, p) \]
4-TotalWaveVector
\[ K_T = (\omega/c, k_\nu) \]
4-TotalMomentum
\[ P_T = (E_T/c, p_T) = (H/c, p) \]

SR 4-CurrentDensity
\[ J = (\rho_c, j) = \rho(c, u) \]
4-Polarization: Spin
\[ \left( \epsilon_0, \epsilon \right) = (\vec{e} \cdot \vec{v}, \vec{e}) \]
4-EMVectorPotential
\[ A = (\phi/c, a) \]
4-ProbabilityFlux
\[ J_{prob} = (\rho_{prob} c, j_{prob}) \]

SR 4-ProbabilityFlux
\[ J_{prob} = (\rho_{prob} c, j_{prob}) \]
4-EMVectorPotential
\[ A = (\phi/c, a) \]
4-ProbabilityFlux
\[ J_{prob} = (\rho_{prob} c, j_{prob}) \]

SR 4-ProbabilityFlux
\[ J_{prob} = (\rho_{prob} c, j_{prob}) \]
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SR 4-ProbabilityFlux
\[ J_{prob} = (\rho_{prob} c, j_{prob}) \]
4-EMVectorPotential
\[ A = (\phi/c, a) \]
4-ProbabilityFlux
\[ J_{prob} = (\rho_{prob} c, j_{prob}) \]
SRQM Diagram: Physical Constants Emphasized

Notice that all the main “Universal” or “Fundamental” Physical Constants are here: G,c,h,ε₀,μ₀.

Some depend on the actual particle type: q,mₒ,ωₒ
Some depend on regional conditions: τ,ρₑo,ρₒ,Φ₂,ψ*ψ
Some depend on interaction: Φ_phase,S_action
Some are mathematical: 0,4,π,i,Diag[1,-1,-1,-1],d/dτ
Conservation Laws are also a type of “zero” constant in this regard.

The majority of the constants are Lorentz Scalars, but some are 4-Vector or 4-Tensor, and all are valid for all inertial observers.

Fundamental Physical Constants are SR Lorentz Scalars

The fact that these “tie together” a network of 4-Vectors is a good argument for why their values are constant. Changing even one would change the relationship properties among all of the 4-Vectors.
SRQM Diagram: Projection Tensors
Temporal, Spatial, Null, SpaceTime

Projection Tensors act as follows:
Generic 4-Vector: 
\[ A' = (a_0', a) = (a_0', a^1, a^2, a^3) \]

Temporal Projection: 
\[ V^{\mu\nu} = \eta^{\mu\nu} V'^{\mu\nu} \rightarrow \text{Diag}[1, 0, 0, 0] \]
\[ V^{\mu\nu}, A' = (a_0', 0, 0, 0) = (a_0', 0) \]

Spatial Projection: 
\[ H^{\mu\nu} = \eta^{\mu\nu} H'^{\mu\nu} \rightarrow \text{Diag}[0, 1, 1, 1] \]
\[ H^{\mu\nu}, A' = (0, a^1, a^2, a^3) = (0, a) \]

SpaceTime Projection: 
\[ V^{\mu\nu} + H^{\mu\nu} = \eta^{\mu\nu}, A' = \delta^{\mu\nu}, A' = A^\mu = (a^0, a) \]

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

Trace\[ T^{\mu\nu} \] = \[ \eta_{\mu\nu} T^{\mu\nu} = T^\nu = T \]
\[ V\cdot V = V^{\mu\nu} \eta_{\mu\nu} V' = (V^0)^2 - V^\mu V^\mu = (V^0)^2 = \text{Lorentz Scalar} \]
SRQM Diagram: Projection Tensors & Perfect-Fluid Stress-Energy Tensor

4-Unit Temporal
\[ T = \gamma(1, \beta) \]

4-Position
\[ \mathbf{R} = (ct, \mathbf{r}) \]

Proper Time
\[ \mathbf{U} \cdot \partial = d\mu / d\tau = \mathbf{c} / \gamma \]

4-Velocity
\[ \mathbf{U} = \gamma(c, \mathbf{u}) \]

4-Gradient
\[ \partial \mathbf{R} = 4 \]

Space-Time Dimension
\[ \mathbf{R} \rightarrow \text{Diag}[1, -1, -1, -1] \]

Minkowski Metric Space-Time Tensor
\[ \mathbf{T}^\mu{}_{\nu} = \mathbf{V}^\mu{}_{\nu} \]

Perfect-Fluid Stress-Energy 4-Tensor:
\[ T^\mu{}_{\nu} \rightarrow \text{Diag}[\rho_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}] \]

Projection Tensors act as follows:
\[ A^\mu{}_{\nu} = (\mathbf{a}^\mu{}_{\nu}, \mathbf{a}^\mu{}_{\nu}, \mathbf{a}^\mu{}_{\nu}, \mathbf{a}^\mu{}_{\nu}) \]

\[ V^\mu{}_{\nu} = \eta_{\mu\nu} \mathbf{V}^\mu{}_{\nu} \rightarrow \text{Diag}[1, 0, 0, 0] \]

\[ V^\mu{}_{\nu} = (\mathbf{a}^\mu{}_{\nu}, 0, 0, 0) = (\mathbf{a}^\mu{}_{\nu}, 0) \]

The rest-energy-density \( (\rho_{\text{eo}}) \) is the Temporal Projection.

The neg rest-pressure \( (-p_{\text{eo}}) \) is the Spatial Projection.

\[ T^\mu{}_{\nu} \rightarrow \text{Diag}[\rho_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}] \]

SR 4-Tensor
\[ (2, 0)-\text{Tensor} \]

SR 4-Vector
\[ (1, 1)-\text{Tensor} \]

SR 4-Scalar
\[ (0, 0)-\text{Tensor} \]

Light Cone
\[ \Delta t \]

\[ \Delta r \]

Lorentz Scalar
\[ \mathbf{V} \cdot \mathbf{V} = \mathbf{V}^\mu{}_{\nu} \mathbf{V}^\mu{}_{\nu} = (\mathbf{V}^\mu{}_{\nu} \mathbf{V}^\mu{}_{\nu}) = (\mathbf{V}^\mu{}_{\nu} \mathbf{V}^\mu{}_{\nu}) = \rho_{\text{eo}} \mathbf{V}^\mu{}_{\nu} \]

Perfect-Fluid Stress-Energy 4-Tensor:
\[ T^\mu{}_{\nu} = (\rho_{\text{eo}}) \mathbf{V}^\mu{}_{\nu} + (-p_{\text{eo}}) H^\mu{}_{\nu} \]

Perfect-Fluid Stress-Energy 4-Tensor:
\[ T^\mu{}_{\nu} \rightarrow \text{Diag}[\rho_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}] \]

Conservation of Stress Energy
\[ \partial \mathbf{T}^\mu{}_{\nu} = 0 \]

Temporal “Vertical” Projection
\[ \mathbf{H}^\mu{}_{\nu} = \eta_{\mu\nu} \mathbf{H}^\mu{}_{\nu} \rightarrow \text{Diag}[0, 1, 1, 1] \]

Spatial “Horizontal” Projection
\[ \mathbf{V}^\mu{}_{\nu} = (\mathbf{a}^\mu{}_{\nu}, 0, 0, 0) = (\mathbf{a}^\mu{}_{\nu}, 0) \]

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

The rest-energy-density \( (\rho_{\text{eo}}) \) is the Temporal Projection.

The neg rest-pressure \( (-p_{\text{eo}}) \) is the Spatial Projection.

\[ T^\mu{}_{\nu} \rightarrow \text{Diag}[\rho_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}] \]

\[ \mathbf{H}^\mu{}_{\nu} = \eta_{\mu\nu} \mathbf{H}^\mu{}_{\nu} \rightarrow \text{Diag}[0, 1, 1, 1] \]

\[ \mathbf{V}^\mu{}_{\nu} = (\mathbf{a}^\mu{}_{\nu}, 0, 0, 0) = (\mathbf{a}^\mu{}_{\nu}, 0) \]

Projection Tensors & Derivation
\[ \partial = (\partial / \partial c, \mathbf{V}) \]

Perfect-Fluid Stress-Energy 4-Tensor:
\[ T^\mu{}_{\nu} = ((\rho_{\text{eo}}) + p_{\text{eo}} / c^2) \mathbf{U}^\mu{}_{\nu} - (p_{\text{eo}}) \eta^\mu{}_{\nu} \]

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

The rest-energy-density \( (\rho_{\text{eo}}) \) is the Temporal Projection.

The neg rest-pressure \( (-p_{\text{eo}}) \) is the Spatial Projection.

\[ T^\mu{}_{\nu} \rightarrow \text{Diag}[\rho_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}] \]

\[ \mathbf{H}^\mu{}_{\nu} = \eta_{\mu\nu} \mathbf{H}^\mu{}_{\nu} \rightarrow \text{Diag}[0, 1, 1, 1] \]

\[ \mathbf{V}^\mu{}_{\nu} = (\mathbf{a}^\mu{}_{\nu}, 0, 0, 0) = (\mathbf{a}^\mu{}_{\nu}, 0) \]

The Minkowski Metric Tensor is the Sum of Temporal & Spatial Projection Tensors, all of which are dimensionless.

The rest-energy-density \( (\rho_{\text{eo}}) \) is the Temporal Projection.

The neg rest-pressure \( (-p_{\text{eo}}) \) is the Spatial Projection.

\[ T^\mu{}_{\nu} \rightarrow \text{Diag}[\rho_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}, p_{\text{eo}}] \]
SRQM Diagram:

4-Tensors and 4-Scalars
generated from 4-Vectors

All SR 4-Tensors can be generated from SR 4-Vectors:

\[ F^{\mu\nu} = \partial^A = \partial^\mu A^\nu - \partial^\nu A^\mu \]
\[ M^{\mu\nu} = X^A P = X^\mu P^\nu - X^\nu P^\mu \]: Faraday EM 4-Tensor (from the 4-Gradient & 4-EMVectorPotential)

\[ \eta^{\mu\nu} = \partial[R] = \partial^\mu [R^\nu] \]
\[ V^{\mu\nu} = T \otimes T = T^\mu T^\nu \]: SR:Minkowski Metric 4-Tensor (from the 4-Gradient & 4-Position)

\[ H^{\mu\nu} = \eta^{\mu\nu} - V^{\mu\nu} \]: (H)orizontal:Spatial Projection 4-Tensor (from previously made 4-Tensors above)

\[ T_{cold\_dust}^{\mu\nu} = P \otimes N = P^\mu N^\nu \]: (Cold)Dust Stress-Energy 4-Tensor (from the 4-Momentum & 4-DustNumberFlux)

\[ \rho_{eo} = T_{\text{Cold\_Dust}}^{\mu\nu} V_{\mu\nu} \]: MCRF EnergyDensity 4-Scalar (from previously made 4-Tensors above)

\[ T_{\text{Lambda\_Vacuum}}^{\mu\nu} = (\rho_{eo}) \eta^{\mu\nu} \]: LambdaVacuum (Dark Energy) Stress-Energy 4-Tensor (from previously made 4-Tensors above)

\[ (p_0) = (k)(1/3)T_{\text{Lambda\_Vacuum}}^{\mu\nu} H_{\mu\nu} \]: MCRF Pressure 4-Scalar (from previously made 4-Tensors above)

with the pressure initially set to the EnergyDensity

\[ T_{\text{Perfect\_Fluid}}^{\mu\nu} = (\rho_{eo}) V^{\mu\nu} + (-p_0) H^{\mu\nu} \]: PerfectFluid Stress-Energy 4-Tensor (from previously made 4-Tensors above)

\[ \text{Equation of State} \]
\[ \text{EoS}[T_{\mu\nu}] = w = p_0 / \rho_{eo} \]

\[ \text{Trace}[T_{\mu\nu}] = \eta_{\mu\nu} T_{\mu\nu} = T \]
\[ V \cdot V = V \eta_{\mu\nu} V^\mu V^\nu = (V^0)^2 - V \cdot V = (V^0)^2 \]
\[ = \text{Lorentz Scalar} \]
Gauss' Theorem in SR:
\[ \int_{\Omega} d^4x \left( \partial_{\mu} V^\mu \right) = \oint_{\partial \Omega} dS \left( V^\mu N_\mu \right) \]

where:
- \( V^\mu = V_\mu \) is a 4-Vector field defined in \( \Omega \)
- \( (\partial \cdot V) = (\partial_\mu V^\mu) \) is the 4-Divergence of \( V \)
- \( (V \cdot N) = (V^\mu N_\mu) \) is the component of \( V \) along the \( N \)-direction
- \( \Omega \) is a 4D simply-connected region of Minkowski SpaceTime
- \( \partial \Omega = S \) is its 3D boundary with its own 3D Volume element \( dS \) and outward pointing normal \( N \).
- \( N_\mu \) is the outward-pointing normal

\[ d^4x = (c \, dt)(d^3x) = (c \, dt)(dx \, dy \, dz) \]

is the 4D differential volume element.

In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface.

Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

In vector calculus, and more generally in differential geometry, the generalized Stokes' theorem is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus.
SRQM Diagram:
Minimal Coupling = (EM)Potential Interaction
Conservation of 4-TotalMomentum

\[ P = (E/c, p): \text{4-Momentum} \]
\[ Q = (V/c, q): \text{4-PotentialMomentum} \]
\[ A = (q/c, a): \text{4-VectorPotential} \]
\[ P_f = (E/c, p_f): \text{4-MomentumIncPotentialField} \]
\[ P_T = (E/c, p_T) = (H/c, p_T): \text{4-TotalMomentum} \]

\[ P = P_f - qA = (E/c-q(\varphi/c), p_f-qA): \text{Minimal Coupling Relation} \]
\[ P_f = P + Q = P + qA: \text{Conservation of 4-MomentumIncPotentialField} \]
\[ P_T = \sum_{n} [P_f]: \text{Conservation of 4-TotalMomentum} \]
4-TotalMomentum is the Sum over all such 4-Momenta

4-MomentumIncField has a contribution from an EM charge (q) interacting with a potential (∂φ/c)
SRQM Study: SRQM Hamiltonian: Lagrangian Connection

\[ H + L = (p_T \cdot u) = \gamma (P_T \cdot U) - (P_T \cdot U)/\gamma \]

4-Momentum \( P = m_o U = (E_o/c^2) U \); 4-Vector Potential \( A = (\phi_o/c^2) U \)

4-Total Momentum \( P_T = (P + qA) = (H/c = E_T/c = (E + q\phi)/c, p_T = p + qa) \)

\[
P \cdot U = \gamma (E - p \cdot u) = E_o = m_o c^2 \; ; \; A \cdot U = \gamma (\phi - a \cdot u) = \phi_o
\]

\[
P_T \cdot U = (P + qA) \cdot U = E_o + q\phi_o = m_o c^2 + q\phi_o
\]

\[
\gamma = 1/\sqrt{1 - \beta \cdot \beta}; \text{ Relativistic Gamma Identity}
\]

\[
(\gamma - 1/\gamma) = (\gamma \beta \cdot \beta); \text{ Manipulate into this form... still an identity}
\]

\[
(\gamma - 1/\gamma) (P_T \cdot U) = (\gamma \beta \cdot \beta) (P_T \cdot U); \text{ Still covariant with Lorentz Scalar}
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma \beta \cdot \beta) (P_T \cdot U)
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma \beta \cdot \beta) (E_o + q\phi_o)
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma \beta \cdot \beta) (E_o + q\phi_o)/c^2
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma \beta \cdot \beta) (E_o/(c^2 + q\phi_o/c^2) \cdot u)
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma \beta \cdot \beta) (E/(c^2 + q\phi_u/c^2) \cdot u)
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (\gamma \beta \cdot \beta) ((p + qa) \cdot u)
\]

\[
\gamma (P_T \cdot U) + -(P_T \cdot U)/\gamma = (p_T \cdot u)
\]

\[
\{ H \} + \{ L \} = (p_T \cdot u); \text{ The Hamiltonian/Lagrangian connection}
\]

H: L Connection in Density Format

\[ H + L = (p_T \cdot u) \]

\[ nH + nL = n(p_T \cdot u), \text{ with number density } n = \gamma n_o \]

\[ H + L = (g_T \cdot u), \text{ with} \]

\[ \text{momentum density } \{ g_r = n p_T \}
\]

Hamiltonian density \( \{ H = nH \}

Lagrangian Density \( \{ L = nL = (\gamma n_o)(L_o/\gamma) = n_o L_o \}

Lagrangian Density is Lorentz Scalar

for an EM field (photonic):

\[ H = (1/2) \{ e \cdot e/\epsilon_o - b \cdot b/\mu_o \}
\]

\[ L = (1/2) \{ e \cdot e/\epsilon_o - b \cdot b/\mu_o \} = (-1/4\mu_o) F_{\mu \nu} F^{\mu \nu}
\]

\[ H + L = \epsilon_o e \cdot e = (g_T \cdot u) \]

\[ |u| = c \]

\[ |g_r| = \epsilon_o e \cdot e / c \]

Poynting Vector \[ |s| = |g| c^2 \rightarrow c \epsilon_o e \cdot e \]

\[ H_o + L_o = 0 \]

Calculating the Rest Values

\[ \gamma = \gamma (P_T \cdot U) = \gamma ((P + qA) \cdot U) = \text{The Hamiltonian with minimal coupling} \]

\[ L = -(P_T \cdot U)/\gamma = -(P + qA) \cdot U)/\gamma \] = The Lagrangian with minimal coupling

\[ H_o = (P_T \cdot U) \]

\[ H = \gamma H_o \]

\[ L_o = -(P_T \cdot U) \]

\[ L = L_o/\gamma \]

4-Vector notation gives a very nice way to find the Hamiltonian/Lagrangian connection:

\[ (H) + (L) = (p_T \cdot u), \text{ where } H = \gamma (P_T \cdot U) \] & \[ L = -(P_T \cdot U)/\gamma \]
SRQM Study: SR Lagrangian, Lagrangian Density, and Relativistic Action (S)

Relativistic Action (S) is Lorentz Scalar Invariant
\[ S = \int L dt = (L_0 - \gamma) (\gamma dt) = (L_0)(dt) \]
\[ S = \int L dt = \int (L/n) dt = \int L(n) dt = \int L(d^3x) dt = \int (L/c)(d^3x)(cdt) = \int (L/c)(d^4x) \]

Explicitly-Covariant Relativistic Action (S)

\[ L = (p\cdot u) - H \]
\[ nL = \int nL \gamma \]
\[ nL = \int nL \gamma = nL \]

Lagrangian \{(L = (p\cdot u) - H)\} is *not* Lorentz Scalar Invariant

Rest Lagrangian \{(L = \gamma L = -(p\cdot u))\} is Lorentz Scalar Invariant

Lagrangian Density \{(L = nL = (\gamma n_o)(L_0/\gamma) = nL_0\} is Lorentz Scalar Invariant

\[ n = \gamma n_o = #/d^4x = #/(dx)(dy)(dz) \text{ number density} \]
\[ cd = (n_o/c)(d^4x) \]

H+L Connection in Density Format for Photonic System (no rest-frame)

H + L = (p\cdot u)
\[ nH + nL = n(p\cdot u), \text{ with number density} \quad n = n_o \]
\[ \mathcal{H} + \mathcal{L} = (g\cdot u), \text{ with} \]
\[ \text{momentum density} \quad \{g_r = np_r\} \]
\[ \text{Hamiltonian density} \quad \{\mathcal{H} = nH\} \]

Lagrangian Density \{(L = nL = (\gamma n_o)(L_0/\gamma) = nL_0\}

Lagrangian Density is Lorentz Scalar

for an EM field (photonic):
\[ \mathcal{H} = (1/2)(\varepsilon_0 e\cdot e + b\cdot b/\mu_o) = n_0 E^2 = \rho_{\varepsilon_0} = \text{EM Field Energy Density} \]
\[ \mathcal{L} = (1/2)(\varepsilon_0 e\cdot e - b\cdot b/\mu_j) = (-1/4\mu_0) F_{\mu\nu} F^{\mu\nu} = (-1/4\mu_0)^* \text{Faraday EM Tensor Inner Product} \]
\[ |u| = c \]
\[ |g| = e\cdot e/c \]
\[ \text{Poynting Vector} \quad |s| = |g|c \rightarrow c\varepsilon_0 e\cdot e \]

\[ \varepsilon_0 \mu_0 = 1/c^2 : \text{Electric:Magnetic Constant Eqn} \]
Lagrangian \( L = (p_T \cdot u) - H \) is \textit{not} a Lorentz Scalar
Rest Lagrangian \( L_o = \gamma L = -(P_T \cdot U) \) is a Lorentz Scalar

Relativistic Action (S) is Lorentz Scalar
\[
S = \int L_o \, dt
\]
\[
S = \int (L_o/\gamma) (\gamma \, dt)
\]
\[
S = \int (L_o) (d\tau)
\]

Explicitly Covariant
Relativistic Action (S)
\[
S = \int L_o \, dt = -\int H_o \, dt
\]
\[
S = -\int (P_T \cdot dR) d\tau
\]
\[
S = -\int (P_T \cdot dR) d\tau
\]
\[
S = \int (P_T \cdot dR) d\tau
\]
\[
S = \int (P_T \cdot dR) d\tau
\]
\[
S = \int (P_T \cdot U + qA \cdot U) d\tau
\]
\[
S = \int (E_o + q \phi_o) d\tau
\]
\[
S = \int (m_o + V) d\tau
\]
\[
S = \int (H_o) d\tau
\]

Hamilton-Jacobi Equation
\[
\partial[-S] = -\partial[S] = P_T
\]
\[
S = \int (E_o + q \phi_o) d\tau
\]
\[
S = -\int (E_o + q \phi_o) d\tau
\]
\[
S = -(E_o + q \phi_o)(\tau + \text{const})
\]

Verified!
\[
R \cdot U = c^2 \tau : \tau = R \cdot U / c^2
\]

The Hamilton-Jacobi Equation is incredibly simple in 4-Vector form
SRQM Diagram:  
Relativistic Hamilton-Jacobi Equation  
$$P_T = -\partial[S]$$  
Differential Format : 4-Vectors
SRQM Diagram: Relativistic Action Equation

\[ (S = -\int (P_T \cdot dR)) \] Integral Format: 4-Scalars

**SRQM Interpretation of QM**

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http://scirealm.org/SRQM.pdf
SRQM Diagram: Relativistic Factors

Hamiltonian & Lagrangian

Relativistic Euler-Lagrange Equation

\[ \delta \text{Lagrangian} = \frac{d}{dt} \frac{\partial \text{Lagrangian}}{\partial \dot{q}} - \frac{\partial \text{Lagrangian}}{\partial q} \]

\[ \delta \text{Hamiltonian} = \frac{d}{dt} \frac{\partial \text{Hamiltonian}}{\partial \dot{p}} - \frac{\partial \text{Hamiltonian}}{\partial p} \]

4-Force \[ F_\mu = \gamma (E/c, \dot{p}_\mu) \]

4-EMPotentialMomentum \[ Q_\mu = \langle U, q \rangle \]

SR Relativistic Scalar (not Lorentz Invariant)

4-Vector SRQM Interpretation of QM

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SRQM Diagram:

Relativistic Euler-Lagrange Equation

The Easy Derivation \( (\mathbf{U} = (d/d\tau)\mathbf{R}) \rightarrow (\partial_{\mathbf{R}} = (d/d\tau)\partial_{\mathbf{U}}) \)

Note Similarity:

4-Velocity is ProperTime Derivative of 4-Position
\( \mathbf{U} = (d/d\tau)\mathbf{R} \) \([\text{m/s}] = [1/s][\text{m}]\)

Relativistic Euler-Lagrange Eqn
\( \partial_{\mathbf{R}} = (d/d\tau)\partial_{\mathbf{U}} \) \([1/m] = [1/s][\text{s/m}]\)

The differential form just inverses the dimensional units, so the placement of the \( \mathbf{R} \) and \( \mathbf{U} \) switch.

That is it: so simple! Much, much easier than how I was taught in Grad School.

To complete the process and create the Equations of Motion, one just applies the base form to a Lagrangian.

This can be:
- a classical Lagrangian
- a relativistic Lagrangian
- a Lorentz scalar
- a Lagrangian scalar

SR 4-Tensor
\((2.0)\)-Tensor \( T^{\mu\nu} \)
\((1.1)\)-Tensor \( T^{\mu} \), or \( T^{\nu}_\lambda \)
\((0.2)\)-Tensor \( T_{\mu\nu} \)

SR 4-Vector
\((1.0)\)-Tensor \( \mathbf{V}^\nu = \mathbf{V} = (V^\nu) \)
\((0.1)\)-Tensor \( \mathbf{V}_\mu = (\mathbf{v}_\mu) \)

SR 4-Scalar
\((0.0)\)-Tensor \( \mathbf{S}_\mu = (S_\mu) \)
Lorentz Scalar

SR 4-CoVector:One-Form
\((0.1)\)-Tensor \( \mathbf{V}^\nu \) or \( \mathbf{V}_\mu \)

SR 4-Vector
\((1.0)\)-Tensor \( \mathbf{V}^\nu = \mathbf{V} = (V^\nu) \)
\((0.1)\)-Tensor \( \mathbf{V}_\mu = (\mathbf{v}_\mu) \)

SR 4-Scalar
\((0.0)\)-Tensor \( \mathbf{S}_\mu = (S_\mu) \)
Lorentz Scalar

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\((0.1)\)-Tensor \( \mathbf{V}_\mu = (\mathbf{v}_\mu) \)

SR 4-Scalar
\((0.0)\)-Tensor \( \mathbf{S}_\mu = (S_\mu) \)
Lorentz Scalar
SRQM Diagram: Relativistic Euler-Lagrange Equation
Alternate Forms: Particle vs. Density

4-Velocity $\mathbf{U}$ is Proper Time Derivative of 4-Position $\mathbf{R}$. The Euler-Lagrange Eqn can be generated by taking the differential form of the same equation.

Relativistic 4-Vector Kinematical Eqn
$\mathbf{U} = (d/d\tau)\mathbf{R}$
$\mathbf{U} \cdot \mathbf{K} = (d/d\tau)\mathbf{R} \cdot \mathbf{K}$

Relativistic Euler-Lagrange Eqns
\{uses gradient-type 4-Vectors\}
$\partial_{R^\alpha} = (d/d\tau) \partial_{U^\alpha}$ \{particle format\}
$\partial_{U^\alpha} = (d/d\tau) \partial_{R^\alpha}$
$\partial_{(\Phi)} = (d/d\tau) \partial_{U^\alpha}$
$\partial_{[\mathbf{R}] K} = (\mathbf{U} \cdot \mathbf{R}) \partial_{[\mathbf{U} \cdot \mathbf{K}]}$
$\partial_{[\mathbf{R}]} = (\mathbf{U} \cdot \mathbf{K}) \partial_{[\mathbf{U} \cdot \mathbf{K}]}$

Relativistic Euler-Lagrange Eqn \{density format\}
$\partial_{[\mathbf{R}]} L = (1/2) \partial_{\mathbf{R}} \mathbf{[}\Phi \mathbf{]} \cdot \partial_{\mathbf{R}} \mathbf{[}\Phi \mathbf{]} - (m_o c^2/h)^2 \Phi^2$ \{KG Lagrangian Density\}

$\partial_{[\mathbf{R}]} L = (\mathbf{U} \cdot \mathbf{R}) \partial_{[\mathbf{U} \cdot \mathbf{R}]} L$: Euler-Lagrange Eqn \{density format\}
$-m_o c^2/h \Phi = (\partial_{\mathbf{R}} \cdot \partial_{[\mathbf{R}]} \Phi)$

Klein-Gordon Relativistic Quantum Wave Eqn

SR 4-Tensor
(2,0)-Tensor $T_{\mu\nu}$
(1,1)-Tensor $T^\alpha_\beta$, or $T^\alpha_\beta$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
$(1,0)$-Tensor $V^\alpha = (v^\alpha, v_\beta)$
SR 4-CoVector: One Form
$(0,1)$-Tensor $V_{\alpha} = (v_\alpha, v_\beta)$

SR 4-Scalar
(0,0)-Tensor $S$ or $S_{\alpha\beta}$
Lorentz Scalar

$\mathbf{V} \cdot \mathbf{V} = V^\alpha V_\alpha = (v^\alpha v_\alpha)^2 = (v^\alpha)^2 - (v_\alpha)^2 = (c^2)^2$ = Lorentz Scalar

$4\text{-Vector SRQM Interpretation of QM}$
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SRQM Diagram:

Relativistic Euler-Lagrange Equation of Motion (EoM) for EM particle

\[ \delta[U] = n^{\alpha} \rightarrow \text{Diag}[1,-1,-1,-1] \]

Minkowski Metric

L_0 = -(P,T,U)
L_\nu = -(P+qA) \cdot U
L_\mu = -(P,qA) \cdot U
L_\nu = -(P+qA) \cdot U
L_\mu = -(P,qA) \cdot U

\[ \delta[U][L_\nu] = \delta[U][A_\nu] = \delta[U][A_\mu] = 0 \]
\[ q\delta[U][A_\nu] = -q\delta[U][A_\mu] = qU, \partial[U][A_\nu] \]
assuming the 4-Gradient \[ \delta[U] \] of the 4-Velocity \[ U \] is zero.

Euler-Lagrange Eqn:

\[ (d/dt)[U][\partial[L][A]] = -(F+qU,\partial[L][A]) \]
\[ F = qU, \partial[L][A] - qU, \partial[L][A] \]
\[ F = qU, \partial[L][A] - qU, \partial[L][A] \]
\[ F_\nu = qU, \partial[L][A] (dP/d\tau) \]
\[ \text{EoM for EM particle} \]

Lorentz Force Equation

\[ \partial[U][\partial[L][A]] = \partial[U][\partial[L][A]] \]

Relativistic Euler-Lagrange Eqn

\[ \partial[U][\partial[L][A]] = \partial[U][\partial[L][A]] \]

Derivative of 4-Position

\[ U = (d/d\tau)[R] \quad [\text{m/s}] = [1/s]*[m] \]

The differential form just inverts the dimensional units

4-Wightman (2,0)-Tensor \[ T^{\nu\tau} i^{\alpha\beta} \]
4-Wightman (1,1)-Tensor \[ T^{\nu\tau} i^{\alpha\beta} \]
4-Wightman (0,2)-Tensor \[ T^{\nu\tau} i^{\alpha\beta} \]
4-Wightman (0,1)-Tensor \[ T^{\nu\tau} i^{\alpha\beta} \]
4-Wightman (0,0)-Tensor \[ T^{\nu\tau} i^{\alpha\beta} \]

SR 4-Tensor

\[ (2,0)-\text{Tensor} \]
\[ (1,1)-\text{Tensor} \]
\[ (0,2)-\text{Tensor} \]

SR 4-Vector

\[ (1,0)-\text{Vector} \]
\[ (1,1)-\text{Vector} \]
\[ (0,1)-\text{Vector} \]

SR 4-Scalar

\[ (0,0)-\text{Scalar} \]
\[ (0,1)-\text{Scalar} \]

4-Force

\[ F = \gamma [a/c, b] \quad \text{EM Faraday} \]
\[ F^{\nu\tau} = \partial[U][\partial[L][A]] \]
\[ \delta[U][\partial[L][A]] = \partial[U][\partial[L][A]] \]

4-Force Equation of motion for charged particle

\[ (dP/d\tau)[F] = qU(F^{\nu\tau}) \]

4-Force

\[ F = \gamma [a/c, b] \quad \text{EM Faraday} \]
\[ F^{\nu\tau} = \partial[U][\partial[L][A]] \]
\[ \delta[U][\partial[L][A]] = \partial[U][\partial[L][A]] \]

4-Force Equation of motion for charged particle

\[ (dP/d\tau)[F] = qU(F^{\nu\tau}) \]

4-Force

\[ F = \gamma [a/c, b] \quad \text{EM Faraday} \]
\[ F^{\nu\tau} = \partial[U][\partial[L][A]] \]
\[ \delta[U][\partial[L][A]] = \partial[U][\partial[L][A]] \]

4-Force Equation of motion for charged particle

\[ (dP/d\tau)[F] = qU(F^{\nu\tau}) \]
SRQM Diagram:
Relativistic Euler-Lagrange Equation
Equation of Motion (EoM) for EM particle

\[
\begin{align*}
\gamma &= 1 / \sqrt{1 - \beta^2}; \text{ Relativistic Gamma Identity} \\
(\gamma - 1 / \gamma) &= (\gamma \beta \cdot \beta); \text{ Manipulate into this form... still an identity} \\
\gamma (\mathbf{P} \cdot \mathbf{U}) + (\mathbf{P} \cdot \mathbf{U}) / \gamma &= ((\gamma \beta \cdot \beta) \mathbf{P} \cdot \mathbf{U}) \\
\gamma (\mathbf{P} \cdot \mathbf{U}) + (\mathbf{P} \cdot \mathbf{U}) / \gamma &= (\mathbf{P} \cdot \mathbf{U}) \\
\{ H \} \{ L \} &= (\mathbf{P} \cdot \mathbf{U}); \text{ The Hamiltonian/Lagrangian connection} \\
H &= \gamma H_0 = \gamma (\mathbf{P} \cdot \mathbf{U}) = \gamma ((\mathbf{P} + \mathbf{q} \mathbf{A}) \cdot \mathbf{U}) = \text{ The Hamiltonian with minimal coupling} \\
L &= L_0 / \gamma = -((\mathbf{P} + \mathbf{q} \mathbf{A}) / \gamma) = \text{ The Lagrangian with minimal coupling} \\
H_0 &= (\mathbf{P} \cdot \mathbf{U}) = -L_0 = (\mathbf{U} \cdot \mathbf{P} \cdot \mathbf{U}) = \text{ Rest Hamiltonian} = \text{ Total RestEnergy} \\
L_0 &= -((\mathbf{P} \cdot \mathbf{U}) = -H_0 \\
(d/dt) \partial_0 [L_0] &= \partial_0 [L_0] \\
4-Velocity is ProperTime \\
4-Derivative of 4-Position \\
\mathbf{U} = (d/dt) \mathbf{R} \quad \text{[m/s]} = [1/s][\text{m}] \\
\text{Relativistic Euler-Lagrange Eqn} \\
\partial_0 = (d/dt) \partial_0 \quad \text{[1/m]} = [1/s][\text{m}] \\
\partial_0 = (d/dt) \partial_0 \quad \mathbf{U} \\
\partial_0 [\mathbf{U}] = (d/dt) [\mathbf{U}] \quad = \mathbf{U} \\
\text{Classical limit, spatial component} \\
\mathbf{a} [\mathbf{U}] = (d/dt) [\mathbf{a}] \quad \mathbf{a} \\
\mathbf{a} [\mathbf{L}] = (d/dt) [\mathbf{L}] \quad \mathbf{a} \\
F_{EM} = \gamma [\mathbf{u} \times \mathbf{E} + \mathbf{c} \cdot \mathbf{U} + \mathbf{u} \times \mathbf{B}] \\
\mathbf{E} = (\nabla \mathbf{u} - \partial_0 \mathbf{u}) \quad \text{a} \text{ and b} = (\mathbf{V} \times \mathbf{a}) \\
\text{If a} = 0, \text{ then } \mathbf{F} = -\mathbf{a} \mathbf{V} = -\mathbf{U}, \text{ the force is neg grad of a potential} \\
\end{align*}
\]
SRQM Diagram: Relativistic Hamilton’s Equations
Equation of Motion (EoM) for EM particle

\[\gamma = 1/\sqrt{1-\beta \cdot \beta} \text{: Relativistic Gamma Identity}\]
\[\gamma = 1/\gamma \text{: Manipulate into this form... still an identity}\]
\[\gamma (P_t \cdot U) + (P_t \cdot U) \gamma = (\gamma \cdot P_t \cdot U)\]
\[\gamma (P_t \cdot U) + (P_t \cdot U)^\gamma = (p_t \cdot U)\]
\{H + L\} = (p_t \cdot U): The Hamiltonian/Lagrangian connection

\[H = \gamma (p_t \cdot U) = \gamma ((P + qA) \cdot U) = \text{The Hamiltonian with minimal coupling}\]
\[L = L_0 = (P_t \cdot U): \text{Rest Hamiltonian} \text{ Total RestEnergy}\]

Thus: \[d\gamma_t[H] = d\gamma_t[U] = \gamma d\delta[U] = \gamma (\partial \delta[P_t][H])\]
\[d\beta_t[H] = \beta d\delta[U] = \partial[U] \partial[U][P_t] = 0 + U \partial[U][P_t] = U = d/d\gamma_t[X]\]
\[d\delta_t[H] = \delta d\delta[U] = \delta[U \cdot P_t + U \cdot \partial[U][P_t] = 0 + U \partial[U][P_t] = d/d\gamma_t[P_t]\]
\[d\delta_t[X] = \delta d\delta[U] = \delta[U \cdot X][U] = \delta[U \cdot X][H]\]

Relativistic Hamilton’s Equations (4-Vector):
\[(d/d\gamma_t)[X] = (\partial \partial[P_t][H])\]
\[(d/d\gamma_t)[P_t] = (\partial \partial[X][X])\]
\[(d/d\gamma_t)[U] = (\partial \partial[P_t][P_t])\]
\[(d/d\gamma_t)[P_t] = (\partial \partial[X][U] = \partial[U \cdot X][H]) = \partial([P_t][U])\]

Taking just the spatial components:
\[\gamma (d/d\gamma_t)[X] = (\partial \partial[P_t][H]) = (\partial \partial[P_t][P_t])\]
\[\gamma (d/d\gamma_t)[P_t] = (\partial \partial[X][X]) = (\partial \partial[X][H])\]
\[\gamma (d/d\gamma_t)[U] = (\partial \partial[U][H])\]
\[\gamma (d/d\gamma_t)[P_t] = (\partial \partial[U][P_t])\]

Take the Classical limit \(\gamma \rightarrow 1\)

Classical Hamilton’s Equations (3-vector):
\[(d/d\gamma_1)[x] = \{+ \partial \partial[p_t][H]\]
\[(d/d\gamma_1)[p_t] = \{- \partial \partial[x][H]\]

Sign-flip difference is interaction of \((- \partial \partial[p_t])\) with \([1/\gamma]\)

4-Velocity
\[U = \gamma(c, u)\]

Rest Hamiltonian: Hamiltonian of Motion
\[H = (P_t \cdot U) = \gamma ((P + qA) \cdot U) = (P + qA) \cdot U = P \cdot U + qA \cdot U\]

4-Position
\[X = (ct, x)\]

4-Velocity
\[\frac{d}{d\tau} [X] = (\partial \partial[P_t][H])\]
\[\frac{d}{d\tau} [P_t] = (\partial \partial[X][X])\]

Relativistic Rest Hamiltonian Hamilton’s Equations of Motion
\[F^\alpha + q(U_\beta \partial^\alpha \Lambda)^\beta = q(\partial^\alpha [A_\beta]U_\beta)\]
\[F^\alpha = q(\partial^\alpha [A_\beta - A_\beta \partial^\alpha \Lambda]U_\beta)\]
\[F^\alpha = q(F^\beta \Lambda U_\beta)\]

Lorentz Force Equation

Trace\[T^\alpha\] = \(n^\alpha_n T^\alpha = T^\alpha\]
\[V \cdot V = V^2 = (V_1^2 + V_2^2)\]

Lorentz Scalar

SR 4-Vector
\[(1,0)-Tensor V^\alpha = (V^\alpha, V^\beta)
SR 4-CoVector:OneForm\[(0,1)-Tensor V_\alpha = (V_\alpha, V_\beta)\]

SR 4-Scalar
\[(0,0)-Tensor S or S_\alpha\]

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SciRealm.org
http://scirealm.org/SRQM.pdf
SRQM Diagram:  
Relativistic Hamilton’s Equations 
Equation of Motion (EoM) for EM particle 

\( \gamma = 1/\sqrt{1 - \beta \cdot \beta} \): Relativistic Gamma Identity

\( \gamma = 1/\sqrt{1 - v^2} \): Manipulate into this form... still an identity

\( \gamma(P_x U) + (\gamma P_x U/\gamma) = (\gamma^2 P_x U) \)

\( \gamma(P_x U) + (\gamma P_x U/\gamma) = (P_x U) \)

\( \{ \gamma H \} + \{ L \} = \{ P_x U \} \): The Hamiltonian/Lagrangian connection

\( H = \gamma P_x U = \gamma((P+qA)U) \): The Hamiltonian with minimal coupling

\( L = L/\gamma = (P_x U)/\gamma = ((P+qA)U)/\gamma \): The Lagrangian with minimal coupling

\( H_0 = (P_x U) \cdot (U \cdot P) \): Rest Hamiltonian = Total RestEnergy

\( L_0 = -(P_x U) \cdot (U \cdot P) \): Relativistic Rest Hamiltonian

Relativistic Hamilton’s Equations (4-Vector):

\[ \frac{d}{dt}[H_0] = \{ \frac{d}{dt}(P+qA) \} \]

\[ \frac{d}{dt}[P_0] = \{ \frac{d}{dt}X \} \]

Classical Hamilton’s Equations (3-vector):

\[ \frac{d}{dt}[\{ H \}] = \{ \frac{d}{dt}(P+qA) \} \]

\[ \frac{d}{dt}[\{ P \}] = \{ \frac{d}{dt}X \} \]

4-Vector SRQM Interpretation of QM

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SRQM Study:

EM Lorentz Force Eqn $\rightarrow$

Classical Force $= - \nabla[\text{Potential}] = -\sqrt{|\text{U}|}$

Lorentz EM Force Equation:

$F^\alpha = q(\mathcal{F}^\beta)U^\beta$

$F^\alpha = q(\partial^\alpha \mathcal{A} - \partial^\beta \mathcal{A}^\beta)U^\beta$

Examine just the spatial components of 4-Force $\mathcal{F}$:

$F^i = q(\mathcal{A}^0 - \mathcal{A}^0)U_0 + q(\mathcal{A}^i - \mathcal{A}^i)U^i$

$\mathcal{F} = q(-\nabla[\phi/c] - (\partial/c)a)(yc) + q(-\nabla[a \cdot u] - u \cdot \nabla[a])y$

$\gamma = q(-\nabla[\phi/c] - (\partial/c)a)(yc) + q(u \cdot \nabla[a] - \nabla[a \cdot u])$

$\gamma = q(-\nabla[\phi] - \partial a + u \cdot \nabla[a] - \nabla[a \cdot u])$

Take the limit of $\{\nabla[\phi] | >> | \partial a - u \times b \}$

$\gamma = q(-\nabla[\phi]) = -\nabla[\varphi] = -\nabla[U] = -\nabla[\text{Potential}]$

The Classical Force $= -\text{Grad}[\text{Potential}]$

when $\{\nabla[\phi] | >> | \partial a - u \times b \}$ or when $\{a = 0\}$

The majority of non-gravitational, non-nuclear potentials dealt with in CM are those mediated by the EM potential.

ex. Spring Potential $\{U = kx^2/2\}$, then $\{f = -\nabla[kx^2/2] = -kx\}$ Hooke’s Law
The Speed-of-Light (c) is THE connection between Time and Space: $d\mathbf{R} = (c dt, dr)$

This physical constant appears in several seemingly unrelated places. You don’t notice these cool relations when you set $c \rightarrow 1$. Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

\[ U \cdot U = \gamma^2 (c^2 - u^2) = c^2 \]

Speed of all things into the Future

\[ (E/m)_\text{rest} = (E/m) = c^2 \text{ Mass is concentrated Energy, } E = mc^2 \]

\[ |u * v| = |v_{\text{group}} * v_{\text{phase}}| = c^2 \]

Particle-Wave “Duality” Correlation

\[ \lambda (\omega_0^2 - 2 \omega_0^2) = \lambda^2 (f^2 - f_0^2) = c^2 \]

Wavelength-Frequency Relation: $\lambda f = c$ for photons

\[ 1/(\epsilon_0 \mu_0) = c^2 \]

Relativistic Quantum Wave Equation

\[ -(h/m)_0^2 (\partial^2 \partial t) = c^2 \]

Klein-Gordon (spin 0), Proca (spin 1), Maxwell (spin 1, m=0)

Factors to Dirac (spin 1/2)

Classical-limit ($|v| < c$) to Schrödinger

\[ (h/\lambda_c m)^2 = c^2 \]

Reduced Compton Wavelength: $\lambda_c = (h/m_c)$

\[ 2GM/R_s = c^2 \]

GR Black Hole Equation

\[ R_s = \text{Schwarzschild Radius} \]

\[ G = \text{GR Gravitational Const, } M = \text{BH Mass} \]

\[ 8\pi G/c^2 = \kappa \]

GR Einstein Curvature Constant: $\kappa = 8\pi G/c^2$

\[ (c^2 \cdot \text{scalar, } 3\text{-vector}) = 4\text{-Vector} \]

Every Physical 4-Vector has a (c) factor to maintain equivalent dimensional units across the whole 4-Vector

\[ 4\text{-Vector SRQM} \]

4-Vector SRQM Interpretation of QM

\[ (\partial^2 \partial t) = (\partial^2 t, \partial^2 r) \]

\[ \partial \mathbf{R} = 4 \text{ SpaceTime Dimension} \]

\[ \partial^n [R^n] = \eta^{uv} \text{ Minkowski Metric} \]

\[ \partial \mathbf{A} = 0 \text{ Lorentz Gauge} \]

\[ \mathbf{F}^{uv} = \partial^a A^a - \partial^a A^a \text{ EM Faraday 4-Tensor} \]

\[ \mathbf{A} = (\mathbf{\phi}/c, \mathbf{a}) \]

\[ 4\text{-EMVector Potential} \]

\[ \mathbf{E} = (E/c^2, E/c^2) \]

\[ \mathbf{B} = (B/c^2, B/c^2) \]

\[ \mathbf{F}^{uv} = \partial^a A^a - \partial^a A^a \text{ EM Faraday 4-Tensor} \]

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\[ \mathbf{A} = (\mathbf{\phi}/c, \mathbf{a}) \]

\[ \mathbf{E} = (E/c^2, E/c^2) \]

\[ \mathbf{B} = (B/c^2, B/c^2) \]

\[ \mathbf{F}^{uv} = \partial^a A^a - \partial^a A^a \text{ EM Faraday 4-Tensor} \]
SRQM: The Speed-of-Light (c) Invariant Relations (part 2)

The Speed-of-Light (c) is THE connection between Time and Space: \( d\mathbf{R} = (cdt, dr) \)

This physical constant appears in several seemingly unrelated places. You don’t notice these cool relations when you set \( c \rightarrow 1 \). Also notice that the set of all these relations definitely rules out a variable speed-of-light. (c) is an Invariant Lorentz Scalar constant.

\[
\begin{align*}
\mathbf{U} \cdot \mathbf{U} &= c^2 \\
\text{Speed of all things into the Future}
\end{align*}
\]

\[
\begin{align*}
(E/m_0) &= (\gamma E/m_0) = (\gamma m_0 c^2)/m_0 \\
\text{Electric: Magnetic} \quad 1/\epsilon_0 \mu_0 &= c^2
\end{align*}
\]

\[
\begin{align*}
\lambda^2(\omega^2 - \omega_0^2) &= \lambda^2(\omega^2 - \omega_0^2) \\
\text{Wavelength-Frequency Relation: } \lambda f &= c \\
\text{Electric (}\epsilon_0\text{) and Magnetic (}\mu_0\text{) EM Field Constants}
\end{align*}
\]

\[
\begin{align*}
(1/\epsilon_0 \mu_0) &= c^2 \\
\text{Relativistic Quantum Wave Equation}
\end{align*}
\]

\[
\begin{align*}
-(h/m_0)^2(\partial \cdot \partial) &= c^2 \\
(h/\lambda_C m_0)^2 &= c^2 \\
\text{Reduced Compton Wavelength: } \lambda_C &= (h/m_0 c)
\end{align*}
\]

\[
\begin{align*}
2GM/R_s &= c^2 \\
\text{GR Black Hole Equation}
\end{align*}
\]

\[
\begin{align*}
8\pi G/\kappa &= c^2 \\
\text{GR Einstein Curvature Constant (mass density form): } \kappa = 8\pi G/c^2
\end{align*}
\]

\[
\begin{align*}
(c^2 \times \text{ scalar, 3-vector}) &= 4\text{-Vector} \\
\text{Every Physical 4-Vector has a (c factor to maintain equivalent dimensional units across the whole 4-Vector}
\end{align*}
\]
Show the damn constants people!

Many authors don’t show constants, which is quite annoying.

It is possible to find this distribution written in multiple ways because they are based on counting arguments.

A covariant way to get this is the Lorentz Scalar Product of the 4-Momentum \( P \) with the 4-ThermalVector \( \Theta \).

\[
F(\text{state}) \sim e^{-\beta E/k_\beta T}, \quad \beta = 1/k_\beta T, \quad \text{(not v/c)}
\]

A covariant way to get this is the Lorentz Scalar Product of the 4-Momentum \( P \) with the 4-ThermalVector \( \Theta \).

\[
F(\text{state}) \sim e^{-\beta E/k_\beta T}, \quad \beta = 1/k_\beta T, \quad \text{(not v/c)}
\]

This also gets Boltzmann’s constant \( (k_\beta) \) out there with the other Lorentz Scalars like \( (c) \) and \( (h) \)

see (Relativistic) Maxwell-Jüttner distribution

\[
f(\text{state}) = N_c/(2c(m_\beta c)^2 K_{(d-1)/2}[m_\beta c/2\pi])^{d-1/2} \cdot e^{-(P \cdot \Theta)}
\]

It is possible to find this distribution written in multiple ways because many authors don’t show constants, which is quite annoying. Show the damn constants people! (\( k_\beta \),c),(h) deserve at least that much respect.
The 4-ThermalVector is used in Relativistic Thermodynamics. It can be used in a partial derivation of Unruh-Hawking Radiation (up to a numerical constant).

Let a “Unruh-DeWitt thermal detector” be in the Momentarily-Comoving-Rest-Frame (MCRF) of a constant spatial acceleration \( \mathbf{a} \), in which \( |u| \to 0 \), \( \gamma \to 1 \), \( \gamma' \to 0 \).

4-Acceleration\(_{\text{MCRF}}\) = \( \mathbf{A}_{\text{MCRF}} = \mathbf{A}_{\text{MCRF}}^\mu = (0, a)_{\text{MCRF}} \)

Take the Lorentz Scalar Product with the 4-ThermalVector
\( \mathbf{A}_{\text{MCRF}} \cdot \Theta = (0, a)_{\text{MCRF}} (c/k_\text{B} T, u/k_\text{B} T) = (-a \cdot u/k_\text{B} T) = \text{Lorentz Scalar Invariant} \)

The \( u \) here is part of the 4-ThermalVector: the 3-velocity of the thermal radiation. (not from \( \mathbf{A}_{\text{MCRF}} \))

Let the thermal radiation be photonic-EM in nature, so \( |u| = c \), and in a direction opposing the acceleration of the ‘thermal detector’, which removes the minus sign.

\( \mathbf{A}_{\text{MCRF}} \cdot \Theta_{\text{radiation}} = (ac/k_\text{B} T) = \text{Invariant Lorentz Scalar} \)

Use Dimensional Analysis to find appropriate Lorentz Scalar Invariant with same units: [Invariant Units] = \([m/s]^2\) /[kg m/s^2] = [kg/m/s] \( c^2 \cdot h = [m/s]^2\) /[kg m/s^2]

\( \mathbf{A}_{\text{MCRF}} \cdot \Theta_{\text{radiation}} = (ac/k_\text{B} T) = \text{Invariant} \sim c^2/h \)

Temperature \( T \sim \hbar/a/k_\text{B} \), (from EM radiation, only from the dir. of acceleration)

Further methods give the constant of proportionality (1/2\( \beta \)):
See (Imaginary Time, Euclideanization, Wick Rotation, Matsubara Frequency)
See (Thermal QFT, Bogoliubov transformation)

\( T_{\text{Unruh}} = \hbar a/2\pi k_\text{B} c \) (due to constant Minkowski-hyperbolic acceleration)

\( T_{\text{Hawking}} = \hbar g/2\pi k_\text{B} c \) (due to gravitational acceleration \( a = g \))

\( T_{\text{Schwarzschild BH}} = \hbar c^2/8\pi G M_\text{K} \) (Temp at BH Event Horizon, \( g = G M_\text{K}/R_\text{s}^2, R_\text{s} = 2GM/c^2 \))

\( \Theta_{\text{SR}} = -\hbar (\mathbf{a} \cdot u)/2\pi k_\text{B} c^2 \) (correct version from 4-Vector derivation \( \mathbf{A}_{\text{MCRF}} \cdot \Theta_{\text{radiation}} = 2\pi c^2/h \))

The 4-InverseTempMomentum
\( \Theta = (\Theta, \Theta) = (c/k_\text{B} T, u/k_\text{B} T) = (\theta/c)(U = 1/k_\text{B} T_0) U \)

\( \mathbf{A}_{\text{MCRF}} \cdot \Theta_{\text{radiation}} = (0, a)_{\text{MCRF}} (c/k_\text{B} T, u/k_\text{B} T) = 0^0 \cdot c/k_\text{B} T \cdot a \cdot u/k_\text{B} T = 0^0 \cdot c/k_\text{B} T \cdot a \cdot u/k_\text{B} T = U = \text{Lorentz Scalar Invariant} \)

\( \Theta = (c/k_\text{B} T)^2 \)

\( \mathbf{P} \cdot \Theta = (E/c, p) \cdot (c/k_\text{B} T, \Theta) = (E/c, p) \cdot (c/k_\text{B} T, \Theta) = (E/c, p) \cdot (c/k_\text{B} T, \Theta) = \text{Just a number} \)

4-Momentum
\( \mathbf{P} = (mc, p) = (E/c, p) = m \cdot \mathbf{U} \)

\( \mathbf{P} \cdot \mathbf{P} = (m c^2)^2 = (E/c)^2 \)

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.
SRQM 4-Vector Study: 4-Thermal Vector
Unruh-Hawking Radiation

Temperature $T \sim \hbar a/k_{\text{B}} c$, (from EM radiation, only from the dir. of acceleration)

Further methods find the constant of proportionality $(1/2\pi)$:
See (Imaginary Time, Euclideanization, Wick Rotation, Matsubara Frequency)
See (Thermal QFT, Bogoliubov transformation)

Alternate forms:
$T = \hbar g/2\pi k_{\text{B}} c$ (due to constant Minkowski-hyperbolic acceleration)
$T_{\text{Hawking}} = h g/2\pi k_{\text{B}} c$ (due to gravitational acceleration $a=\gamma g$)
$T_{\text{Schwarzschild BH Temp at BH Event Horizon, \(g=GM/R^3, R_s=2GM/c^2\)}}$
$T_{\text{SR}} = -h(a/2\pi)/2\pi k_{\text{B}} c^2$ (correct version from 4-Vector derivation $A_{\text{MCRF}} \Theta_{\text{radiation}} = 2\pi c^2/\hbar$)

The $2\pi$ factor is interesting

There are cases when the dimensional units must match. see 4-Momentum related to 4-WaveVector:
$P = \hbar K \rightarrow [J/s/m] = [J/s/\text{rad}]/[\text{rad}/m]$
$h = h/2\pi \rightarrow [J/s/\text{rad}] = [J\cdot\text{s}]/[\pi \text{rad}]$

And other where the $2\pi$ factor doesn’t seem to use [rad] units.
see Circles & Spheres:
$C = 2\pi r \rightarrow [\text{m}] = [2\pi][\text{m}]$
$A = \pi r^2 \rightarrow [\text{m}^2] = [\pi][\text{m}]^2$
$V = (4/3)\pi r^3 \rightarrow [\text{m}^3] = [(4/3)\pi][\text{m}]^3$

Other where $A_{\text{MCRF}} = 2\pi(K)^2$

$A_{\text{MCRF}} = (2\pi^2 c^2)K = (2\pi c^2/\hbar)P$
$dP/dt|_{\text{MCRF}} = \Theta_{\text{radiation}} = 2\pi\omega_o$

$A_{\text{MCRF}} = \Theta_{\text{radiation}} = 2\pi\omega_o$ : { for $m_o = \text{constant} \}$

The 4-Vector SRQM Interpretation of QM

4-Velocity $u = (c, u)$
$u \cdot u = (\gamma c, \gamma u)$
$\gamma d/dt \[\ldots\]$ $d/dt \[\ldots\]$ $\gamma v^2$

4-Acceleration $A = A^\mu = (cy', y'u+y\gamma)$
$= U/d\tau - d^2 R/d\tau^2$

4-Acceleration $A = A_{\text{MCRF}} = (0, a)_{\text{MCRF}}$

$A_{\text{MCRF}} = \Theta_{\text{radiation}} = 2\pi c^2/\hbar$

4-InverseTempMomentum $\Theta = (\theta, \theta) = (c/k_B T, u/k_B T) = (\theta/c) U = (1/k_B T_o) U$

$\Theta \cdot \Theta = (c/k_B T_o)^2$

$P \cdot \Theta = (E/c, p) \cdot (c/k_B T, \theta) = (E/k_B T - p \cdot \theta) = (E/k_B T_o) = \text{Invariant (dimensionless)}$

Just a number

4-Momentum $P = (mc^2, p) = (E/c, p) = m_o U$
$P \cdot P = (mc^2)^2 = (E/c)^2$

Trace[T^\mu_\nu] = n_{\mu} T^\mu_\nu = T^\mu_\nu \rightarrow T$
$V \cdot V = V''_{\mu} V''_\nu = (V_{\gamma})^2 - V \cdot V = (V_{\gamma})^2$

$= \text{Lorentz Scalar}$

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.
SRQM 4-Vector Study:

4-ThermalVector

Wick Rotations, Matsubara Freqs

The QM/QFT←SM Correspondence, via the Wick Rotation

The operator which governs how a quantum system evolves in time, the time evolution operator, and the density operator, a time-independent object which describes the statistical state of a many-particle system in an equilibrium state (with temperature T) can be related via arithmetic substitutions:

Quantum Mechanics (QM)

\[ e^{\frac{-i}{\hbar} (\hat{P} \cdot \Theta)} \]

Statistical Mechanics (SM)

\[ e^{\frac{-H o \tau}{\hbar}} \]

Imaginary Time ↔ Inv Temp

\[ (it/\hbar \leftrightarrow 1/k_b T) \]

where \( \tau \), called Euclidean Time (Imaginary Time) is cyclic with period \( \beta \), (\( 0 \leq \tau \leq +\beta \)).

In Quantum Mechanics (or Quantum Field Theory), the Hamiltonian \( H \) acts as the generator of the Lie group of time translations while in Statistical Mechanics the role of the same Hamiltonian \( H \) is as the Boltzmann weight in an ensemble.

Time Evolution Operator

\[ U(t) = \sum_{n=0}^{\infty} \left[ e^{\frac{-i}{\hbar} (E_n t / \hbar)} \right] | n \rangle \langle n | = e^{\frac{-i}{\hbar} H t / \hbar} \]

Partition Function (time-independent function of state)

\[ Z = \sum_{n=0}^{\infty} \left[ e^{\frac{-E_n}{k_b T}} \right] = \text{Trace} \left[ e^{\frac{-H}{\hbar}} \right] \]

In the Matsubara Formalism, the basic idea (due to Felix Bloch) is that the expectation values of operators in a canonical ensemble:

\[ \langle A \rangle = \frac{\text{Tr} \left[ \exp \left( -\beta H \right) A \right]}{\text{Tr} \left[ \exp \left( -\beta H \right) \right]} \]

may be written as expectation values in ordinary quantum field theory (QFT), where the configuration is evolved by an imaginary time \( \tau = -i t \) (\( 0 \leq \tau \leq \beta \)).

One can therefore switch to a spacetime with Euclidean signature, where the above trace (Tr) leads to the requirement that all bosonic and fermionic fields be periodic and antiperiodic, respectively, with respect to the Euclidean time direction with periodicity \( \beta = \hbar / (k_b T) \).

This allows one to perform calculations with the same tools as in ordinary quantum field theory, such as functional integrals and Feynman diagrams, but with compact Euclidean time.

Note that the definition of normal ordering has to be altered.

In momentum space, this leads to the replacement of continuous frequencies by discrete imaginary (Matsubara) frequencies:

Bosonic \( \omega_n = (n)(2\pi/\beta) \)

Fermionic \( \omega_n = (n+1/2)(2\pi/\beta) \)

and, through the de Broglie relation \( E = \hbar \omega \), to a discretized EM thermal energy spectrum \( E_n = \hbar \omega_n = n(2\pi k_b T) \).
SRQM 4-Vector Study: 4-ThermalVector Covariant Wick Rotation

The operator which governs how a quantum system evolves in time, the time evolution operator, and the density operator, a time-independent object which describes the statistical state of a many-particle system in an equilibrium state (with temperature $T$) can be related via arithmetic substitutions:

$$e^{\frac{-i\{P \cdot R\}}{\hbar}} = e^{\frac{\text{S}_{\text{action}}}{\hbar}}$$

where $\tau$, called Euclidean Time (Imaginary Time) is cyclic with period $\beta$, ($0 \leq \tau \leq +\beta$).

In Quantum Mechanics (or Quantum Field Theory), the Hamiltonian $H$ acts as the generator of the Lie group of time translations while in Statistical Mechanics the role of the same Hamiltonian $H$ is as the Boltzmann weight in an ensemble.

$S_{\text{action}} = -(\frac{P \cdot R}{\hbar}) - \int [\frac{P \cdot dR}{\hbar}] = \int dt \left(-\gamma(H - P \cdot U)\right)$

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.
SRQM 4-Vector Study: Deep Symmetries & Cyclic Imaginary Time ↔ Inv Temp

4-Momentum
\[ P = \begin{pmatrix} p^0 \pm mc \end{pmatrix} = \begin{pmatrix} (mc,p) \end{pmatrix} = \begin{pmatrix} (mc,mp) \end{pmatrix} = m \mathbf{U} \]
\[ (E/c,p) = (E/c^2) \mathbf{U} \]
\[ [\text{kg} \cdot \text{m/s}] = [\text{J} / \text{s/m}] \]

Einstein de Broglie
\[ P = \hbar K \]
\[ h = [\text{m} / \text{J}] \]
\[ \hbar = [\text{m} \cdot \text{s}] / \text{J} \]
\[ 1 / \hbar = [\text{m} / \text{J}] \]

4-WaveVector
\[ K = K^\mu = (\omega/c,k) = (\omega/c^2) \mathbf{U} \]
\[ \omega/c = \omega \mathbf{n} / \sqrt{n_{\text{phase}}} = (1/c \mathbf{T}, \mathbf{n}/\Lambda) \]

Complex Plane-Waves
\[ K = i \mathbf{a} \]
\[ [\text{s/kg} \cdot \text{m}] = [\text{s/kg}] \]

4-Gradient
\[ \partial = \partial_R = \partial / \partial \mathbf{R}_n = \partial^0 = (\partial / c, - \nabla) \]
\[ \rightarrow (\partial / c, - \partial_x, - \partial_y, - \partial_z) \]
\[ = (\partial / c \partial_t, \partial / \partial x, \partial / \partial y, \partial / \partial z) \]

SRQM 4-Vector Study:

- 4-Momentum
- 4-WaveVector
- Complex Plane-Waves
- 4-Gradient

SR 4-Vector
\[ (2,0)-\text{Tensor} \]
\[ T^{\mu \nu} = (\nu^\mu, \nu) \]
\[ P = (E/c^2, p) \]
\[ V = (v^\mu, v) \]
\[ \mathbf{V} = (v^\mu, v) \]
\[ \mathbf{V} \cdot \mathbf{V} = (V_0)^2 \]
\[ \mathbf{V} \cdot \mathbf{V} = (V_0)^2 \]
\[ \mathbf{V} \cdot \mathbf{V} = (V_0)^2 \]

SR 4-Scalar
\[ (0,0)-\text{Tensor} \]
\[ S = (\theta, \theta) \]
\[ \mathbf{S} = (S^\mu, S) \]
\[ \mathbf{S} = (S^\mu, S) \]
\[ \mathbf{S} = (S^\mu, S) \]

Boltzmann Distribution
\[ P \cdot \mathbf{O} = (E/c, p) \cdot (c/k_B T, \theta) \]
\[ = (E/k_B T, r) = (E/k_B T) \]
\[ \mathbf{V} \cdot \mathbf{V} = (v_0)^2 \]
\[ \mathbf{V} \cdot \mathbf{V} = (v_0)^2 \]
\[ \mathbf{V} \cdot \mathbf{V} = (v_0)^2 \]

Trace[\mathbf{T}^\mu \nu] = n_{ij} T^{\mu \nu} = T^\nu = T

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.
The 4-EntropyVector is used in Relativistic Thermodynamics.

Pure Entropy is a Lorentz Scalar in all frames.
Up to this point, we have mostly been exploring the SR aspects of 4-Vectors.

It is now time to show how RQM and QM fit into the works...

This is SRQM, [ SR → QM ]

RQM & QM are derivable from principles of SR

Let that sink in...

Quantum Mechanics is derivable from Special Relativity

GR → SR → RQM → QM → {CM & EM}
The basic idea is to show that Special Relativity plus a few empirical facts lead to Relativistic Wave Equations, and thus RQM, without using any assumptions or axioms from Quantum Mechanics.

Start only with the concepts of SR, no concepts from QM:

1. SR provides the ideas of Invariant Intervals and \( c \) as a Physical Constant, as well as: Poincaré Invariance, Minkowski 4D SpaceTime, ProperTime, and Physical SR 4-Vectors

Note empirical facts which can relate the SR 4-Vectors from the following:

2a) Elementary matter particles each have RestMass, \( m_0 \), a physical constant which can be measured by experiment: eg. collision, cyclotrons, Compton Scattering, etc.

2b) There is a physical constant, \( \hbar \), which can be measured by classical experiment – eg. the Photoelectric Effect, the inverse Photoelectric Effect, LED's=Injection Electroluminescence, Duane-Hunt Law in Bremsstrahlung, the Watt/Kibble-Balance, etc. All known particles obey this constant.

2c) The use of complex numbers \( i \) and differential operators \( \{ \partial_t, \nabla = (\partial_x, \partial_y, \partial_z) \} \) in wave-type equations comes from pure mathematics: not necessary to assume any QM Axioms

These few things are enough to derive the RQM Klein-Gordon equation, the most basic of the relativistic wave equations. Taking the low-velocity limit \( |\mathbf{v}| << c \) (a standard SR technique) leads to the Schrödinger Equation.
Klein-Gordon RWE implies QM

If one has a Relativistic Wave Equation, such as the Klein-Gordon equation, then one has RQM, and thence QM via the low-velocity limit \( |v|<<c \).

The physical and mathematical properties of QM, usually regarded as axiomatic, are inherent in the Klein-Gordon RWE itself.

QM Principles emerge not from \{ QM Axioms + SR \rightarrow RQM \}, but from \{ SR + Empirical Facts \rightarrow RQM \}.

The result is a paradigm shift from the idea of \{ SR and QM as separate theories \} to \{ QM derived from SR \} – leading to a new interpretation of QM: The SRQM or \([\text{SR} \rightarrow \text{QM}]\) Interpretation.

GR \rightarrow (\text{low-mass limit} = \{ \text{curvature } \sim 0 \} \text{ limit}) \rightarrow \text{SR}
SR \rightarrow (+ \text{ a few empirical facts}) \rightarrow \text{RQM}
RQM \rightarrow (\text{low-velocity limit} \{ |v|<<c \}) \rightarrow \text{QM}

The results of this analysis will be facilitated by the use of SR 4-Vectors.
# SRQM 4-Vector Study:
Basic 4-Vectors on the path to QM

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Dimens. Units (SI)</th>
<th>Definition Component Notation</th>
<th>Unites</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>[m]</td>
<td>( R = R^\mu = (r^\mu) = (r^0, r^i) = \langle \text{Event} \rangle = (ct,r) \rightarrow (ct,x,y,z) )</td>
<td>Time, Space ((\text{when, where}) = \text{SR location of } \langle \text{Event} \rangle)</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>[m/s]</td>
<td>( U = U^\mu = (u^\mu) = (u^0, u^i) = \gamma(c,u) )</td>
<td>Temporal velocity, Spatial velocity (\text{nothing faster than } c)</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>[kg\cdot m/s]</td>
<td>( P = P^\mu = (p^\mu) = (p^0, p^i) = (E/c,p) )</td>
<td>Mass:Energy, Momentum used in 4-Momenta Conservation (\Sigma P_{\text{final}} = \Sigma P_{\text{initial}})</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>[{rad}/m]</td>
<td>( K = K^\mu = (k^\mu) = (k^0,k^i) = (\omega/c,k) = (\omega/c,\omega \hat{n}/v_{\text{phase}}) = (1/cT, \hat{n}/\lambda) = 2\pi(1/cT, \hat{n}/\lambda) )</td>
<td>Ang. Frequency, WaveNumber used in Relativistic Doppler Shift (\omega_{\text{obs}}=\omega_{\text{emit}} / [\gamma(1 - \beta \cos[\theta])], k=\omega/c\text{ for photons})</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>[1/m]</td>
<td>( \partial = \partial^\mu = (\partial^\mu) = (\partial^0,\partial^i) = (\partial/c,-\nabla) \rightarrow (\partial/c,-\partial_x,-\partial_y,-\partial_z) )</td>
<td>Temporal Partial, Spatial Partial used in SR Continuity Eqns., ProperTime eg. (\partial \cdot A = 0) means (A) is conserved</td>
</tr>
</tbody>
</table>

All of these are standard SR 4-Vectors, which can be found and used in a totally relativistic context, with no mention or need of QM. I want to emphasize that these objects are ALL relativistic in origin.
SRQM 4-Vector Study:
SR Lorentz Scalar Invariants

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Lorentz Scalar Invariant</th>
<th>What it means in SR...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>( \mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct_o)^2 = (ct)^2 )</td>
<td>SR Invariant Interval</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( \mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2 )</td>
<td>&lt;Event&gt; Motion Invariant Magnitude (c)</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2 )</td>
<td>Einstein Invariant Mass:Energy Relation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} \cdot \mathbf{K} = (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_o/c)^2 )</td>
<td>Wave/Dispersion Invariance Relation</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (\partial_t/c)^2 )</td>
<td>The d'Alembert Invariant Operator</td>
</tr>
</tbody>
</table>

All 4-Vectors have invariant magnitudes, found by taking the scalar product of the 4-Vector with itself. Quite often a simple expression can be found by examining the case when the spatial part is zero. This is usually found when the 3-velocity is zero. The temporal part is then specified by its "rest" value.

For example: \( \mathbf{P} \cdot \mathbf{P} = (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_o/c)^2 = (m_o c)^2 \)
\( E = \text{Sqrt}[ (E_o)^2 + \mathbf{p} \cdot \mathbf{p} c^2 ] \), from above relation
\( E = \gamma E_o \) using \( \{ \gamma = 1/\text{Sqrt}[1-\beta^2] = \text{Sqrt}[1+\gamma^2\beta^2] \} \) and \( \{ \beta = v/c \} \)
meaning the relativistic energy \( E \) is equal to the relative gamma factor \( \gamma \) * the rest energy \( E_o \).
## SR + A Few Empirical Facts: SRQM Overview

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>What it means in SR...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position $\mathbf{R} = (ct, \mathbf{r})$; alt. $\mathbf{X} = (ct, \mathbf{x})$</td>
<td>$\mathbf{R} = \langle \text{Event} \rangle$; alt. $\mathbf{X}$</td>
<td>Location of 4D Spacetime $\langle \text{Event} \rangle$</td>
</tr>
<tr>
<td>4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$</td>
<td>$\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$</td>
<td>Motion of 4D Spacetime $\langle \text{Event} \rangle$</td>
</tr>
<tr>
<td>4-Momentum $\mathbf{P} = (E/c, \mathbf{p}) = (mc, \mathbf{p})$</td>
<td>$\mathbf{P} = m_0\mathbf{U}$</td>
<td>$\langle \text{Events} \rangle$ described as Particles</td>
</tr>
<tr>
<td>4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k})$</td>
<td>$\mathbf{K} = \frac{\mathbf{P}}{\hbar}$</td>
<td>$\langle \text{Events} \rangle$ described as Waves</td>
</tr>
<tr>
<td>4-Gradient $\partial = (\partial_t/c, -\nabla)$</td>
<td>$\partial = -i\mathbf{K}$</td>
<td>Alteration of 4D Spacetime $\langle \text{Event} \rangle$</td>
</tr>
</tbody>
</table>

The Axioms of SR, which is actually a GR limiting-case, lead us to the use of Minkowski SpaceTime and Physical 4-Vectors, which are elements of Minkowski Space (4D SpaceTime).

Empirical Observation leads us to the transformation relations between the components of these SR 4-Vectors, and to the chain of relations between the 4-Vectors themselves. These relations all turn out to be Lorentz Invariant Constants, whose values are measured empirically. They are manifestly invariant relations, true in all reference frames...

The combination of these SR objects and their relations is enough to derive RQM.
SRQM Chart:

**Special Relativity → Quantum Mechanics**

SR→QM Interpretation Simplified

SRQM: The [SR→QM] Interpretation of Quantum Mechanics

Special Relativity (SR) Axioms: Invariant Interval + LightSpeed (c) as Universal Physical Constant lead to SR, although technically SR is itself the Minkowski-SpaceTime low-curvature: "flat" limiting-case of GR.

{c, τ, m₀, ħ, i} = {c: SpeedOfLight, τ: ProperTime, m₀: RestMass, ħ: Dirac/Planck Reduced Constant (ħ = h/2π), i: Imaginary Number √[-1]}: are all Empirically Measured SR Lorentz Invariant Physical Constants and/or Mathematical Constants

**Standard SR 4-Vectors:**

<table>
<thead>
<tr>
<th>Vector Type</th>
<th>Expression</th>
<th>Related by SR Lorentz Invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>( R = (ct, r) )</td>
<td>( (R \cdot R) = (ct)^2 )</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>( U = \gamma (c, u) )</td>
<td>( (U \cdot U) = (c)^2 )</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>( P = (E/c, p) )</td>
<td>( (P \cdot P) = (m_0c)^2 )</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>( K = (\omega/c, k) )</td>
<td>( (K \cdot K) = (m_0c/\hbar)^2 ) KG Equation: (</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>( \partial = (\partial_t/c, \nabla) )</td>
<td>( (\partial \cdot \partial) = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2 = QM Relation \rightarrow RQM \rightarrow QM )</td>
</tr>
</tbody>
</table>

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Quantum Eqn, and thence to QM via the low-velocity limit \{ |v| << c \}, giving the Schrödinger Eqn. This fundamental KG Relation also leads to the other Quantum Wave Equations:

RQM\(_{\text{massless}}\) \{ |v| = c : m₀ = 0 \} → RQM \{ 0 <= |v| < c : m₀ > 0 \} → QM \{ 0 <= |v| << c : m₀ > 0 \}

Spin=0 boson field = 4-Scalar: Free Scalar Wave (Higgs)
Spin=1/2 fermion field = 4-Spinor: Weyl
Spin=1 boson field = 4-Vector: Maxwell (EM photonic)
**SRQM Diagram:**

**RoadMap of SR (4-Vectors)**

- **4-Vector**: \( U = \gamma(c, u) \)
- **4-Momentum**: \( P = (mc, p) = (E/c, p) \)
- **4-Gradient**: \( \partial = (\partial t/c, -\nabla) \)
- **4-Position**: \( R = (ct, r) = \text{<Event>} \)
- **4-WaveVector**: \( K = (\omega/c, k) \)

---

**SR 4-Tensor**

- (2,0)-Tensor \( T^{\mu \nu} \)
- (1,1)-Tensor \( T^\mu \), or \( T_\mu \)
- (0,2)-Tensor \( T_{\mu \nu} \)

**SR 4-Vector**

- (1,0)-Tensor \( V^\mu = V = (\vec{v}, v) \)
- (0,1)-Tensor \( V_\mu = (v_0, -\vec{v}) \)

**SR 4-Scalar**

- (0,0)-Tensor \( S \) or \( S_0 \)

- Lorentz Scalar

---

**Trace**

\[ \text{Trace}[T^{\mu \nu}] = \eta_{\mu \nu} T^{\mu \nu} = T^\mu = T \]

\[ V \cdot V = V^\mu \eta_{\mu \nu} V^\nu = (v^0)^2 - \vec{v} \cdot \vec{v} = (v_0^2) \]

- Lorentz Scalar
SRQM Diagram:
RoadMap of SR (Connections)

4-Gradient
\[ \partial = (\partial/c, -\nabla) \]

Minkowski Metric
\[ \delta^{uv}[R^v] = \eta^{uv} \]

Minkowski Transform
\[ \delta[R^v] = \Lambda^v_v \]

Lorentz Transform

ProperTime
\[ U \cdot \partial = \partial/d\tau = \gamma d/dt \]

Derivative

\[ \partial R = 4 \text{ Dimension} \]

\[ \partial R = 4 \text{ Position} \]
\[ R = (ct, r) = \text{<Event>} \]

4-Position

\[ -\partial[S]_{\text{phase,free}} = K \]

SR Phase

\[ -\partial[S]_{\text{phase}} = K_T \]

Plane-Waves
\[ K_T = -\partial[\Phi] \]

\[ P \cdot R = S_{\text{action,free}} \]

SR Action

\[ P \cdot R = S_{\text{action}} \]

Hamilton-Jacobi
\[ P_T = -\partial[S] \]

4-WaveVector
\[ K = (\omega/c, k) \]

SR 4-Vector
(2,0)-Tensor \( T^{iv} \)
(1,1)-Tensor \( T^v \), or \( T^u \)
(0,2)-Tensor \( T_{iv} \)

SR 4-Vector
(1,0)-Tensor \( V_v = V = (\nabla, v) \)
(0,1)-Tensor \( V_\mu = (v_\nu, -\nabla) \)

SR 4-Scalar
(0,0)-Tensor \( S \) or \( S_0 \)

Lorentz Scalar

\[ \text{Trace}[T^{iv}] = \eta^{iv} T^{iv} = T_v = T \]

\[ V \cdot V = V^\nu \eta_{\nu\lambda} V^\lambda = (\nabla^2 - v \cdot \nabla) = (v_0^2) \]

= Lorentz Scalar

http://scirealm.org/SRQM.pdf
SRQM Diagram: RoadMap of SR (Free Particle)

4-Gradient=Alteration of SR \(<Events>\)
SR SpaceTime Dimension=4
SR SpaceTime 4D Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Proper Time
SR & QM Waves

\[ \delta^\mu[R^\nu] = \eta^{\mu\nu} \text{ Minkowski Metric} \]
\[ \delta[R^\nu] = \Lambda^\nu_\mu \text{ Lorentz Transform} \]

\[ \partial[R^\nu] = 4 \text{ Space Time Dim} \]
\[ \partial[R] = 4 \text{ SpaceTime} \]
\[ \partial[R] = 4 \text{ Proper Time} \]

\[ \text{SR Wave} \ <Events> \text{ have 4-WaveVector=Substantiation oscillations proportional to mass:energy & 3-momentum} \]

\[ \text{SR Particle} \ <Events> \text{ have 4-Momentum=Substantiation mass:energy & 3-momentum} \]

\[ \omega_o/E_o \]

- \[ \gamma \frac{d}{dt} = \frac{d}{d\tau} \]
- \[ \gamma \frac{d}{dt} = \frac{d}{d\tau} \]

\[ \omega_o/c^2 \]
\[ E_o/m_o = \gamma E_o \]

\[ \text{Einstein} \]

\[ \text{Trace}[T^{\mu\nu}] = \eta_{\mu\nu}T^{\mu\nu} = T_{\mu\nu} = T \]
\[ V \cdot V = V^\mu V_\mu = (v_0^2 - v \cdot v) = (v_0^2) \]
\[ = \text{Lorentz Scalar} \]
SRQM Diagram: RoadMap of SR (Free Particle) with Magnitudes

4-Gradient=Alteration of SR <Events>
SR SpaceTime Dimension=4
SR SpaceTime 4D Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time
SR & QM Waves

SR 4-Tensor
(2,0)-Tensor $T^{\mu\nu}$
(1,1)-Tensor $V^\mu = V = (\nabla \cdot, \nabla \times)$
SR 4-Vector
(1,0)-Tensor $V^\mu = V = (\nabla \cdot, \nabla \times)$
SR 4-CoVector:OneForm
(0,1)-Tensor $V_\mu = (\nabla \cdot, \nabla \times)$
SR 4-Scalar
(0,0)-Tensor $S$ or $S_0$

4-Gradient $\partial = (\partial/c, -\nabla)$
\[ \partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla \]
d'Alembertian Free Particle Wave Equation

SR Wave <Events> have 4-WaveVector=Substantiation oscillations proportional to mass:energy & 3-momentum

SR Particle <Events> have 4-Momentum=Substantiation mass:energy & 3-momentum

Trace $[T^\mu] = n_\mu T^{\mu
u} = T^\nu = T$
$V \cdot V = V^2 \Rightarrow V = (V_0^2 - \nabla \cdot, \nabla \times)$

SR SpaceTime 4D Metric
$\partial^\mu [R^\nu] = \eta^{\mu\nu}$

Lorentz Transform
$\partial \cdot R = 4$

SpaceTime Dim

SR Proper Time
SR & QM Waves

4-Position $R = (ct, r)$

$\partial \cdot U = \gamma(c^2 u \cdot u)$

4-Velocity $U = (c, u)$

4-Momentum $P = (mc, p)$

$P \cdot P = (E/c)^2 - p \cdot p$

$E_0/c^2$

Hamilton-Jacobi
$P = -\partial S$

Wave Velocity
$V = c$

Einstein
$E = mc^2$

SR 4-Vector
$V_\mu = (\nabla \cdot, \nabla \times)$

SR 4-CoVector:OneForm
$V_\mu = (\nabla \cdot, \nabla \times)$

SR 4-Scalar
$S_0$

Lorentz Scalar
SRQM Diagram: RoadMap of SR (EM Potential)

4-Vector SRQM Interpretation of QM

SR SpaceTime Dimension=4
SR SpaceTime 4D Metric
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Action → 4-Momentum
SR Lorentz Transforms
SR Proper Time
SR & QM Waves

4-Gradient=

\[ \partial = (\partial /c - \nabla) \]

SR Proper Time
SR Phase → 4-WaveVector
SR Action → 4-Momentum
SR Lorentz Transforms
SR SpaceTime Dimension=4

\[ 4-Gradient = \nabla \]
# SRQM Study: The Empirical 4-Vector Facts

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>Discoverer</th>
<th>Physics</th>
</tr>
</thead>
</table>
| 4-Position  | \( \mathbf{R} = \langle \text{Event} \rangle \) | Newton+ Einstein | \[
\begin{align*}
\text{[ t & r]} & \quad \text{Time & Space Dimensions} \\
\text{[ R=(ct,r)]} & \quad \text{SpaceTime as 4D=(1+3)D}
\end{align*}
\] |
| 4-Velocity  | \( \mathbf{U} = \frac{d\mathbf{R}}{d\tau} \) | Newton Einstein | \[
\begin{align*}
\text{[ v=\dot{r}=dr/dt]} & \quad \text{Calculus of motion} \\
\text{[ U=\gamma(c,u)=d\mathbf{R}/d\tau]} & \quad \text{Gamma & Proper Time}
\end{align*}
\] |
| 4-Momentum  | \( \mathbf{P} = m_0 \mathbf{U} \) | Newton Einstein | \[
\begin{align*}
\text{[ p=mv]} & \quad \text{Classical Mechanics} \\
\text{[ P=(E/c,p)=m_0 \mathbf{U}] } & \quad \text{SR Mechanics}
\end{align*}
\] |
| 4-WaveVector| \( \mathbf{K} = \frac{\mathbf{P}}{\hbar} \) | Planck Einstein de Broglie | \[
\begin{align*}
\text{[ h]} & \quad \text{Photon Thermal Distribution} \\
\text{[ E=\hbar \nu=\hbar \omega]} & \quad \text{Photoelectric Effect (h=h/2\pi)} \\
\text{[ p=\hbar \mathbf{k}]} & \quad \text{Matter Waves} \\
\text{[ P=(E/c,p)=\hbar \mathbf{K}=\hbar(\omega/c,\mathbf{k})]} & \quad \text{as 4-Vector Math}
\end{align*}
\] |
| 4-Gradient  | \( \partial = -i \mathbf{K} \) | Schrödinger | \[
\begin{align*}
\text{[ \omega=i\partial, \mathbf{k}=-i\nabla]} & \quad \text{(SR) Wave Mechanics} \\
\text{[ P=(E/c,p)=i\hbar \partial=i\hbar(\partial/c,-\nabla)]} & \quad \text{(QM) 4-Vector}
\end{align*}
\] |

1. The SR 4-Vectors and their components are related to each other via constants.
2. We have not taken any 4-vector relation as axiomatic, the constants come from experiment.
3. \( c, \tau, m_0, \hbar \) come from physical experiments, (-i) comes from the general mathematics of waves.
**SRQM Study: 4-Vector Relations Explained**

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>Empirical Fact</th>
<th>What it means in SRQM...</th>
<th>Lorentz Invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position $\mathbf{R} = (ct, \mathbf{r})$</td>
<td>$\mathbf{R} = \langle \text{Event} \rangle$</td>
<td>Space-Time as Unified Concept</td>
<td>$c = \text{LightSpeed}$</td>
</tr>
<tr>
<td>4-Velocity $\mathbf{U} = \gamma(c, \mathbf{u})$</td>
<td>$\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$</td>
<td>Velocity is ProperTime Derivative</td>
<td>$\tau = t_o = \text{ProperTime}$</td>
</tr>
<tr>
<td>4-Momentum $\mathbf{P} = (E/c, \mathbf{p})$</td>
<td>$\mathbf{P} = m_o \mathbf{U}$</td>
<td>Mass:Energy-Momentum Equivalence</td>
<td>$m_o = \text{RestMass}$</td>
</tr>
<tr>
<td>4-WaveVector $\mathbf{K} = (\omega/c, \mathbf{k})$</td>
<td>$\mathbf{K} = \frac{\mathbf{P}}{\hbar}$</td>
<td>Wave-Particle Duality</td>
<td>$\hbar = \text{UniversalAction}$</td>
</tr>
<tr>
<td>4-Gradient $\partial = (\partial_t/c, -\nabla)$</td>
<td>$\partial = -i\mathbf{K}$</td>
<td>Unitary Evolution, Operator Formalism</td>
<td>$i = \text{ComplexSpace}$</td>
</tr>
</tbody>
</table>

Three old-paradigm QM Axioms: Particle-Wave Duality $[(\mathbf{P})=\hbar(\mathbf{K})]$, Unitary Evolution $[\partial=(-i)\mathbf{K}]$, Operator Formalism $[(\partial)=(-i)\mathbf{K}]$ are actually just empirically-found constant relations between known SR 4-Vectors. Note that these constants are in fact all Lorentz Scalar Invariants.

Minkowski Space and 4-Vectors also lead to idea of Lorentz Invariance. A Lorentz Invariant is a quantity that always has the same value, independent of the motion of inertial observers. Lorentz Invariants can typically be derived using the scalar product relation.

$$\mathbf{U} \cdot \mathbf{U} = c^2, \quad \mathbf{U} \cdot \partial = \frac{d}{d\tau}, \quad \mathbf{P} \cdot \mathbf{U} = m_o c^2, \text{ etc.}$$

A very important Lorentz invariant is the Proper Time $\tau$, which is defined as the time displacement between two points on a worldline that is at rest wrt. an observer. It is used in the relations between 4-Position $\mathbf{R}$, 4-Velocity $\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$, and 4-Acceleration $\mathbf{A} = \frac{d\mathbf{U}}{d\tau}$. 
SRQM: The SR Path to RQM
Follow the Invariants...

<table>
<thead>
<tr>
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<th>Lorentz Invariant</th>
<th>What it means in SRQM...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-Position</td>
<td>$\mathbf{R} \cdot \mathbf{R} = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (ct)^2$</td>
<td>SR Invariant Interval</td>
</tr>
<tr>
<td>4-Velocity</td>
<td>$\mathbf{U} \cdot \mathbf{U} = \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = c^2$</td>
<td>Events move into future at magnitude c</td>
</tr>
<tr>
<td>4-Momentum</td>
<td>$\mathbf{P} \cdot \mathbf{P} = (m_o c)^2$</td>
<td>Einstein Mass:Energy Relation</td>
</tr>
<tr>
<td>4-WaveVector</td>
<td>$\mathbf{K} \cdot \mathbf{K} = (m_o c/\hbar)^2 = (\omega_o/c)^2$</td>
<td>Matter-Wave Dispersion Relation</td>
</tr>
<tr>
<td>4-Gradient</td>
<td>$\partial \cdot \partial = (-im_o c/\hbar)^2 = -(m_o c/\hbar)^2$</td>
<td>The Klein-Gordon Equation $\rightarrow$ RQM!</td>
</tr>
</tbody>
</table>

$\mathbf{U} = \frac{d\mathbf{R}}{d\tau}$
Remember, everything after 4-Velocity was just a constant times the last 4-vector, and the Invariant Magnitude of the 4-Velocity is itself a constant $\mathbf{P} = m_o \mathbf{U}$, $\mathbf{K} = \mathbf{P}/\hbar$, $\partial = -i\mathbf{K}$, so e.g. $\mathbf{P} \cdot \mathbf{P} = m_o \mathbf{U} \cdot m_o \mathbf{U} = m_o^2 \mathbf{U} \cdot \mathbf{U} = (m_o c)^2$

The last equation is the Klein-Gordon RQM Equation, which we have just derived without invoking any QM axioms, only SR plus a few empirical facts
SRQM: Some Basic 4-Vectors

4-Momentum, 4-WaveVector, 4-Position, 4-Velocity, 4-Gradient, Wave-Particle

4-Momentum: $P = (mc, p) = (E/c, p)$

4-Gradient: $\partial \Phi = (\partial/c, -\nabla)$

4-WaveVector: $K = (\omega/c, k) = (\omega/c, \omega\hat{n}/c^2)$

Treating motion like a particle:
Moving particles have a 4-Velocity
4-Momentum is the negative 4-Gradient of the SR Action (S)

Treating motion like a wave:
Moving waves have a 4-Velocity
4-WaveVector is the negative 4-Gradient of the SR Phase (Φ)


See SR Wave Definition for info on the Lorentz Scalar Invariant SR WavePhase.

SR 4-Vector

1. $T^{\mu\nu}$ Tensor
2. $V^\mu = V$ (Spatial Velocity)
3. $V^\mu = V$ (Temporal Velocity)
4. $T^{\mu\nu}$ Tensor
5. $V^\mu = V$ (Spatial Velocity)
6. $V^\mu = V$ (Temporal Velocity)

Existing SR Rules

Quantum Principles

SRQM Interpreation of QM

SciRealm.org
John B. Wilson
SciRealm.com
http://scirealm.org/SRQM.pdf
SRQM: Wave-Particle Diffraction/Interference Types

The 4-Vector Wave-Particle relation is inherent in all particle types: Einstein-de Broglie \( \mathbf{P} = \left( \frac{E}{c}, \mathbf{p} \right) = \hbar \mathbf{K} = \hbar \left( \frac{\omega}{c}, \mathbf{k} \right) \).

All waves can superpose, interfere, diffract: Water waves, gravitational waves, photonic waves of all frequencies, etc. In all cases: experiments using single particles build the diffraction/interference pattern over the course many iterations.

**Photon/light Diffraction:** Photonic particles diffracted by matter particles.
Photons of any frequency encounter a translucent “solid=matter” object, grating, or edge. Most often encountered are diffraction gratings and the famous double-slit experiment.

**Matter Diffraction:** Matter particles diffracted by matter particles.
Electrons, neutrons, atoms, small molecules, buckyballs (fullerenes), macromolecules, etc. have been shown to diffract through crystals.
Crystals may be solid single pieces or in powder form.

**Kapitsa-Dirac Diffraction:** Matter particles diffracted by photonic standing waves.
Electrons, atoms, super-sonic atom beams have been diffracted from resonant standing waves of light.

**Photonic-Photonic Diffraction?:** Delbruck scattering & Light-by-light scattering
Normally, photons do not interact, but at high enough relative energy, virtual particles can form which allow interaction.
SRQM: Hold on, aren't you getting the “ℏ” from a QM Axiom?

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>SR Empirical Fact</th>
<th>What it means...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-WaveVector</td>
<td>$K = (\omega/c, k) = (\omega/c, \omega \hat{n}/v_{\text{phase}}) = (\omega_c/c^2)U$</td>
<td>Wave-Particle Duality</td>
</tr>
</tbody>
</table>

ℏ is actually an empirically measurable quantity, just like e or c. It can be measured classically from the photoelectric effect, the inverse photoelectric effect, from LED’s (injection electroluminescence), from the Duane-Hunt Law in Bremsstrahlung, Electron Diffraction in crystals, the Watt/Kibble-Balance, etc.

For the LED experiment, one uses several different LED’s, each with its own characteristic wavelength. One then makes a chart of wavelength ($\lambda$) vs threshold voltage ($V$) needed to make each individual LED emit. One finds that: $\lambda = h*c/(eV)$, where $e=$ElectronCharge and $c=$LightSpeed. ℏ is found by measuring the slope.

Applying our 4-Vector knowledge, we recognize this as the temporal components of a 4-Vector relation. $(E/c, ...) = ℏ(\omega/c, k) = ℏ*4-$WaveVector $K$.

The spatial component (due to De Broglie) follows naturally from the temporal component (due to Einstein) via to the nature of 4-Vector (tensor) mathematics.

This is also derivable from pure SR 4-Vector (Tensor) arguments: $P = m_0U = (E_0/c^2)U$ and $K = (\omega_0/c^2)U$

Since $P$ and $K$ are both Lorentz Scalar proportional to $U$, then by the rules of tensor mathematics, $P$ must also be Lorentz Scalar proportional to $K$

i.e. Tensors obey certain mathematical structures:

Transitivity (if $a\sim b$ and $b\sim c$, then $a\sim c$) & Euclideaness: {if $a\sim c$ and $b\sim c$, then $a\sim b$} **Not to be confused with the Euclidean Metric**

This invariant proportional constant is empirically measured to be (ℏ) for each known particle type, massive ($m_o>0$) or massless ($m_o=0$): $P = m_0U = (E_0/c^2)U = (E_0/c^2)(\omega_0/c^2)K = (E_0/c^2)\gamma(E_0/c^2)K = (E_0/c^2)\gamma\gamma K = (E_0/c^2)K = (ℏ)K$

also from standard SR Lorentz 4-Vector Scalar Products: $P\cdot U = E_0: K\cdot U = \omega_0: P\cdot K = m_0\omega_0: P\cdot P = (m_0c)^2: K\cdot K = (\omega_0/c)^2$

$P\cdot U = E_0: K\cdot U = \omega_0: P\cdot K = m_0\omega_0: P\cdot P = (m_0c)^2: K\cdot K = (\omega_0/c)^2$

$P\cdot K = m_0\omega_0(\omega_0/c^2): P\cdot P = (m_0c)^2: P\cdot P = (m_0c)^2$

$P\cdot U = E_0: K\cdot U = \omega_0: P\cdot K = m_0\omega_0: P\cdot P = (m_0c)^2: K\cdot K = (\omega_0/c)^2$

$P\cdot R = (-S_{\text{action,free}}/(-\Phi_{\text{phase,plane}})) → |P||K| = (ℏ) = E_0/\omega_0$

The spatial component (due to De Broglie) follows naturally from the temporal component (due to Einstein) via to the nature of 4-Vector (tensor) mathematics.

SciRealm.org
John B. Wilson
SciRealm@aol.com
http://scirealm.org/SRQM.pdf
SRQM: Hold on, aren't you getting the “K” from a QM Axiom?

<table>
<thead>
<tr>
<th>SR 4-Vector</th>
<th>SR Empirical Fact</th>
<th>What it means...</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-WaveVector</td>
<td>( \mathbf{K} = (\omega/c, \mathbf{k}) = (\omega/c, \omega \hat{n}/v_{\text{phase}}) = (\omega/c^2)\mathbf{U} )</td>
<td>Wave-Particle Duality</td>
</tr>
</tbody>
</table>

\( \mathbf{K} \) is a standard SR 4-Vector, used in generating the SR formulae:

**Relativistic Doppler Effect:**
\[
\omega_{\text{obs}} = \omega_{\text{emit}} / [\gamma (1 - \beta \cos[\theta])], \quad k = \omega/c \text{ for photons}
\]

**Relativistic Aberration Effect:**
\[
\cos[\theta_{\text{obs}}] = (\cos[\theta_{\text{emit}}] + |\beta|) / (1 + |\beta|\cos[\theta_{\text{emit}}])
\]

The 4-WaveVector \( \mathbf{K} \) can be derived in terms of periodic motion, where families of surfaces move through space as time increases, or alternately, as families of hypersurfaces in SpaceTime, formed by all events passed by the wave surface. The 4-WaveVector is everywhere in the direction of propagation of the wave surfaces.

\[
\mathbf{K} = -\partial[\Phi_{\text{phase}}]
\]

From this structure, one obtains relativistic/wave optics without ever mentioning QM.
SRQM:
Hold on, aren't you getting the “-i” from a QM Axiom?

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<td>4-Gradient</td>
<td>( \partial = (\partial_t/c, -\nabla) = -iK )</td>
<td>Unitary Evolution of States Operator Formalism</td>
</tr>
</tbody>
</table>

\[ \partial = -iK \] gives the sub-equations \[ \partial_t = -i\omega \] and \[ \nabla = iK \], and is certainly the main equation that relates QM and SR by allowing Operator Formalism. But, this is a basic equation regarding the general mathematics of plane-waves; not just quantum-waves, but anything that can be mathematically described by plane-waves and superpositions of plane-waves…

This includes purely SR waves, an example of which would be EM plane-waves (i.e. photons)...

\[ \psi(t, r) = ae^{i(k \cdot r - \omega t)} \]: Standard mathematical plane-wave equation

\[
\begin{align*}
\partial_\tau [\psi(t, r)] &= \partial_\tau [ae^{i(k \cdot r - \omega t)}] = (-i\omega)ae^{i(k \cdot r - \omega t)} = (-i\omega)\psi(t, r), \text{ or } [\partial_t = -i\omega] \\
\nabla [\psi(t, r)] &= \nabla [ae^{i(k \cdot r - \omega t)}] = (ik)ae^{i(k \cdot r - \omega t)} = (ik)\psi(t, r), \text{ or } [\nabla = ik]
\end{align*}
\]

In the more economical SR notation:

\[ \partial [\psi(R)] = \partial [ae^{-iK \cdot R}] = (-iK)ae^{-iK \cdot R}] = (-iK)\psi(R), \text{ or in 4-Vector shorthand } [\partial = -iK] \]

This one is more of a mathematical empirical fact, but regardless, it is not axiomatic. It can describe purely SR waves, again without any mention of QM.
SRQM:
Hold on, aren't you getting the “∂” from a QM Axiom?

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<td>4-Gradient</td>
<td>( \partial = (\partial_t/c,-\nabla) = -iK )</td>
<td>4D Gradient Operator</td>
</tr>
</tbody>
</table>

\( [\partial = (\partial_t/c,-\nabla)] \) is the SR 4-Vector Gradient Operator. It occurs in a purely relativistic context without ever mentioning QM.

\[
\partial \cdot X = (\partial_t/c,-\nabla) \cdot (ct,x) = (\partial_t/c[ct] - (-\nabla \cdot x)) = (\partial_t[t] + \nabla \cdot x) (1)+(3) = 4
\]
The 4-Divergence of the 4-Position \( (\partial \cdot X = \partial^\mu \eta_{\mu\nu} X^\nu) \) gives the dimensionality of SpaceTime.

\[
\partial[X] = (\partial_t/c,-\nabla)[(ct,x)] = (\partial_t/c[ct],-\nabla[x]) = \text{Diag}[1,-I_{(3)}] = \eta^{\mu\nu}
\]
The 4-Gradient acting on the 4-Position \( (\partial[X] = \partial^\mu[X^\nu]) \) gives the Minkowski Metric Tensor.

\[
\partial \cdot J = (\partial_t/c,-\nabla) \cdot (pc,j) = (\partial_t/c[pc] - (-\nabla \cdot j)) = (\partial_t[p] + \nabla \cdot j) = 0
\]
The 4-Divergence of the 4-CurrentDensity is equal to 0 for a conserved current. It can be rewritten as \( (\partial_t[p] = -\nabla \cdot j) \), which means that the time change of ChargeDensity is balanced by the space change or divergence of CurrentDensity. It is a Continuity Equation, giving local conservation of ChargeDensity. It is related to Noether’s Theorem.
SRQM:

Hold on, doesn’t using “∂” in an Equation of Motion presume a QM Axiom?

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<tr>
<td>4-(Position)Gradient</td>
<td>∂ₚ = ∂ = (∂/c,-∇) = -iK</td>
<td>4D Gradient Operator</td>
</tr>
</tbody>
</table>

Klein-Gordon Relativistic Quantum Wave Equation
∂·∂[Ψ] = -(mₒc/ℏ)²[Ψ] = -(ωₒ/c)²[Ψ]

Relativistic Euler-Lagrange Equations
∂ₚ[L] = (d/dτ)∂ₓ[L]: {particle format}
∂ₓ[Φ[L] = (∂ₚ) ∂ₓ(∂ₚ)[L]: {density format}

[∂ = (∂/c,-∇)] is the SR 4-Vector (Position)Gradient Operator. It occurs in a purely relativistic context without ever mentioning QM. There is a long history of using the gradient operator on classical physics functions, in this case the Lagrangian. And, in fact, it is another area where the same mathematics is used in both classical and quantum contexts.
The QM Schrödinger Relation

\[ P = \imath \hbar \frac{\partial}{\partial t} \]

This is derived from the combination of:

1. The Einstein-de Broglie Relation
2. Complex Plane-Waves
   \[ K = \imath \frac{\partial}{\partial t} \]

These are the standard QM Schrödinger Relations.

It is this Lorentz Scalar Invariant relation \( \imath \hbar \) which connects the 4-Momentum to the 4-Gradient, making it into a QM operator.

Note that these 4-Vectors are already connected in multiple ways in standard SR.

**SRQM Diagram:**

RoadMap of SR → QM

QM Schrödinger Relation

The QM Schrödinger Relation

\[ P = \imath \hbar \frac{\partial}{\partial t} \]

This is derived from the combination of:

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   \[ K = \imath \frac{\partial}{\partial t} \]

These are the standard QM Schrödinger Relations.

It is this Lorentz Scalar Invariant relation \( \imath \hbar \) which connects the 4-Momentum to the 4-Gradient, making it into a QM operator.

Note that these 4-Vectors are already connected in multiple ways in standard SR.
SRQM:
Review of SR 4-Vector Mathematics

- 4-Gradient: \( \partial = (\partial/c, \nabla) \)
- 4-Position: \( X = (ct, \mathbf{x}) \)
- 4-Velocity: \( U = \gamma(c, \mathbf{u}) \)
- 4-Momentum: \( P = (E/c, \mathbf{p}) = (E_0/c^2)U \)
- 4-WaveVector: \( K = (\omega/c, \mathbf{k}) = (\omega_0/c^2)U \)

\[
\begin{align*}
\partial \cdot X &= (\partial/c, -\nabla)(ct, \mathbf{x}) = (\partial/c[ct](-\nabla \cdot \mathbf{x})) = 1 - (-3) = 4: \\
U \cdot \partial &= \gamma(c, \mathbf{u}) \cdot (\partial/c, -\nabla) = \gamma(\partial + \mathbf{u} \cdot \nabla) = \gamma(d/d\tau) = d/d\tau: \\
\partial[X] &= (\partial/c, -\nabla)(ct, \mathbf{x}) = (\partial/c[ct], -\nabla \cdot \mathbf{x}) = \text{Diag}[1, -1] = \eta^{\mu
u}: \\
\partial[K] &= (\partial/c, -\nabla)(\omega/c, \mathbf{k}) = (\partial/c[\omega/c], -\nabla \cdot \mathbf{k}) = [0] \\
K \cdot X &= (\omega/c, \mathbf{k}) \cdot (ct, \mathbf{x}) = (\omega t - \mathbf{k} \cdot \mathbf{x}) = \phi: \\
\partial[K \cdot X] &= \partial[K] \cdot X + K \cdot \partial[X] = K = -\partial[\phi]: \\
(\partial \cdot \partial)[K \cdot X] &= ((\partial/c)^2 - \nabla \cdot \nabla)(\omega t - \mathbf{k} \cdot \mathbf{x}) = 0 \\
(\partial \cdot \partial)[K \cdot X] &= \partial \cdot (\partial[K \cdot X]) = \partial \cdot K = 0:
\end{align*}
\]

Dimensionality of SpaceTime
Derivative wrt. ProperTime is Lorentz Scalar
The Minkowski Metric
Phase of SR Wave
Neg 4-Gradient of Phase gives 4-WaveVector
Wave Continuity Equation, No sources or sinks
Standard mathematical plane-waves if \( b = -i \)
Unitary Evolution, Operator Formalism
The Klein-Gordon Equation \( \rightarrow \) RQM

\[
\begin{align*}
\partial \cdot X &= (\partial/c, -\nabla)(ct, \mathbf{x}) = (\partial/c[ct](-\nabla \cdot \mathbf{x})) = 1 - (-3) = 4: \\
X \cdot X &= ((ct)^2 - \mathbf{x} \cdot \mathbf{x}) = (ct_0)^2 = (ct)^2: \text{Invariant Interval Measure} \\
U \cdot U &= \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = (c)^2 \\
P \cdot P &= (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 \\
K \cdot K &= (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2 \\
\partial \cdot X &= (\partial/c, -\nabla)(ct, \mathbf{x}) = (\partial/c[ct](-\nabla \cdot \mathbf{x})) = 1 - (-3) = 4: \\
U \cdot U &= \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = (c)^2 \\
P \cdot P &= (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 \\
K \cdot K &= (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2:
\end{align*}
\]

\[
\begin{align*}
\partial \cdot P &= \gamma(c, \mathbf{u}) \cdot (\partial/c, -\nabla) = \gamma(\partial + \mathbf{u} \cdot \nabla) = \gamma(d/d\tau) = d/d\tau: \\
\partial \cdot K &= \gamma(c, \mathbf{u}) \cdot (\partial/c, -\nabla)(\omega/c, \mathbf{k}) = \gamma(\partial/c[\omega/c], -\nabla \cdot \mathbf{k}) = [0] \\
P \cdot X &= (E/c, \mathbf{p}) \cdot (ct, \mathbf{x}) = (E_0/c^2)(ct_0) = (ct): \\
P \cdot P &= (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 \\
K \cdot K &= (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2:
\end{align*}
\]

\[
\begin{align*}
\partial \cdot X &= (\partial/c, -\nabla)(ct, \mathbf{x}) = (\partial/c[ct](-\nabla \cdot \mathbf{x})) = 1 - (-3) = 4: \\
U \cdot U &= \gamma^2(c^2 - \mathbf{u} \cdot \mathbf{u}) = (c)^2 \\
P \cdot P &= (E/c)^2 - \mathbf{p} \cdot \mathbf{p} = (E_0/c)^2 \\
K \cdot K &= (\omega/c)^2 - \mathbf{k} \cdot \mathbf{k} = (\omega_0/c)^2:
\end{align*}
\]

Note that no QM Axioms are assumed: This is all just pure SR 4-vector (tensor) manipulation
Klein-Gordon Equation:

\[ \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -(m_o c / \hbar)^2 = -(\omega_o / c)^2 = -(1/\lambda C)^2 \]

Let \( \mathbf{X}_T = (ct + c \Delta t, \mathbf{x}) \), then \( \partial [\mathbf{X}_T] = (\partial / c, -\nabla)(ct + c \Delta t, \mathbf{x}) = \text{Diag}[1, -I_{(3)}] = \partial [\mathbf{X}] = \eta^{\mu \nu} \)

so \( \partial [\mathbf{X}_T] = \partial [\mathbf{X}] \) and \( \partial [\mathbf{K}] = \text{[[0]]} \)

let \( f = ae^{-i(\mathbf{K} \cdot \mathbf{X}_T)} \), the time translated version

\[ (\partial \cdot \partial)[f] \]
\[ \partial \cdot (\partial [f]) \]
\[ \partial \cdot (\partial [e^{-i(\mathbf{K} \cdot \mathbf{X}_T)}]) \]
\[ \partial \cdot (e^{-i(\mathbf{K} \cdot \mathbf{X}_T)} \partial [-i(\mathbf{K} \cdot \mathbf{X}_T)]) \]
\[ -i \partial \cdot (f \partial [\mathbf{K} \cdot \mathbf{X}_T]) \]
\[ -i \partial [f] \partial [\mathbf{K} \cdot \mathbf{X}_T]) + \Psi (\partial \cdot \partial)[\mathbf{K} \cdot \mathbf{X}_T]) \]
\[ (-i)^2 f (\partial [\mathbf{K} \cdot \mathbf{X}_T])^2 + 0 \]
\[ (-i)^2 f (\partial [\mathbf{K}] \cdot \mathbf{X}_T + \mathbf{K} \cdot \partial [\mathbf{X}_T])^2 \]
\[ (-i)^2 f (0 + \mathbf{K} \cdot \partial [\mathbf{X}])^2 \]
\[ (-i)^2 f (\mathbf{K})^2 \]
\[ -(\mathbf{K} \cdot \mathbf{K})f \]
\[ -(\omega_o / c)^2 f \]
SRQM:
What does the Klein-Gordon Equation give us?
A lot of RQM!

Relativistic Quantum Wave Equation: \[ \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -\left(\frac{m_o c}{\hbar}\right)^2 = \left(\frac{im_o c}{\hbar}\right)^2 = -\left(\frac{\omega_o}{c}\right)^2 \]

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles (4-Scalars)
Factoring the KG Eqn leads to the RQM Dirac Equation for spin=1/2 particles (4-Spinors)
Applying the KG Eqn to a 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Taking the low-velocity-limit of the KG leads to the standard QM non-relativistic Schrödinger Eqn, for spin=0
Taking the low-velocity-limit of the Dirac leads to the standard QM non-relativistic Pauli Eqn, for spin=1/2

Setting RestMass \( m_o \rightarrow 0 \) leads to the RQM Free Wave Eqn., Weyl Eqn., and Free Maxwell (Standard EM) Eqn.

In all of these cases, the equations can be modified to work with various potentials by using more SR 4-Vectors, and more empirically found relations between them, e.g. the Minimal Coupling Relations:
4-TotalMomentum \( P_{\text{tot}} = P + qA \), where \( P \) is the particle 4-Momentum, \( q \) is a charge, and \( A \) is a 4-VectorPotential, typically the 4-EMVectorPotential.

Also note that generating QM from RQM (via a low-energy limit) is much more natural than attempting to “relativize or generalize” a given NRQM equation. Facts assumed from a non-relativistic equation may or may not be applicable to a relativistic one, whereas the relativistic facts are still true in the low-velocity limiting-cases. This leads to the idea that QM is an approximation only of a more general RQM, just as SR is an approximation only of GR.
### SRQM: Relativistic Quantum Wave Eqns.

| Spin-(Statistics) | Relativistic Light-like Mass = 0 | Relativistic Matter-like Mass > 0 | Non-Relativistic Limit (|v|<<c) Mass >0 | Field Representation |
|-------------------|---------------------------------|-----------------------------------|------------------------------------------|----------------------|
| 0-(Boson)         | Free Wave N-G Bosons            | Klein-Gordon Higgs Bosons, maybe Axions | Schrödinger Common NRQM Systems         | Scalar (0-Tensor)    |
|                   | (\(\partial \cdot \partial\))\(\Psi = 0\) | \((\partial \cdot \partial + (m_o c/\hbar)^2 - c'(-i\hbar \nabla - qa)^2)\)(\(\Psi = 0\) | with minimal coupling \((i\hbar \partial_i - q\phi) -(p-q\alpha)^2)2m_o\(\Psi = 0\) | \(\Psi = \Psi[K_o X]\) |
|                   |                                 |                                   |                                          | \(\Psi[\Phi]\) |
| 1/2-(Fermion)     | Weyl Idealized Matter Neutinos  | Dirac Matter Leptons/Quarks       | Pauli Common NRQM Systems w Spin        | Spino\(r\)         |
|                   | \((\sigma \cdot \partial)\)(\(\Psi = 0\) | \((\partial \cdot \partial - m_o c/\hbar)\)(\(\Psi = 0\) | \((i\hbar \partial_i - [\(\sigma \cdot p\)^2]/2m_o)\)(\(\Psi = 0\) | \(\Psi = \Psi[K_o X]\) |
|                   | factored to Right & Left Spinors | \((\partial \cdot \partial - m_o c/\hbar)\)(\(\Psi = 0\) | with minimal coupling \((i\hbar \partial_i - q\phi - [(\sigma \cdot p \alpha)^2]/2m_o)\)(\(\Psi = 0\) | \(\Psi[\Phi]\) |
|                   | \(L = i\Psi^\dagger \sigma^\mu \partial^\mu \Psi\) | \(L = i\Psi^\dagger \sigma^\alpha \partial^\alpha \Psi\) |                                          |                      |
|                   |                                 | \(L = i\Psi^\dagger \sigma^\mu \partial^\mu \Psi\) |                                          |                      |
| 1-(Boson)         | Maxwell Photons/Gluons          | Proca Force Bosons                | 4-Vector SRQM Interpretation of QM      | 4-Vector (1-Tensor)  |
|                   | \((\partial \cdot \partial)A = 0\) free | \((\partial \partial)^2 - (m_o c/\hbar)^2\)\(A = 0\) |                                          | \(A = A'[K_o X]\)  |
|                   | \((\partial \partial)A = \mu_o J\) w current src \(\partial \cdot \partial A = 0\) | \(\partial \partial)^2 - (m_o c/\hbar)^2\)\(A' = 0\) |                                          | \(A'[\Phi]\)   |
|                   | \((\partial \partial)A = \mu_e e^\gamma \gamma \Psi\) QED | \(\partial \partial)^2 - (m_o c/\hbar)^2\)\(A' = 0\) |                                          |                      |
SRQM: Factoring the KG Equation → Dirac Eqn

Klein-Gordon Equation: $\partial^2 - \nabla^2 = -(m_0c/\hbar)^2$

Since the 4-vectors are related by constants, we can go back to the 4-Momentum description/representation:

$\left(\frac{\partial^2}{c^2} - \nabla^2\right) = -(m_0c/\hbar)^2$

$E^2 - c^2 p \cdot p - (m_0 c^2)^2 = 0$

Factoring: $[E - c \alpha \cdot p - \beta (m_0 c^2)] [E + c \alpha \cdot p + \beta (m_0 c^2)] = 0$

$E$ & $p$ are quantum operators, $\alpha$ & $\beta$ are matrices which must obey $\alpha \beta = -\beta \alpha, \alpha \alpha = -\alpha \alpha, \alpha^2 = \beta^2 = I$

The left hand term can be set to 0 by itself, giving...

$[E - c \alpha \cdot p - \beta (m_0 c^2)] = 0$, which is the momentum-representation form of the Dirac equation

Remember: $P^\mu = (p^0, p)$ and $\alpha^\mu = (\alpha^0, \alpha)$ where $\alpha^0 = I_{(2)}$

$[E - c \alpha \cdot p - \beta (m_0 c^2)] = [c \alpha^0 p^0 - c \alpha \cdot p - \beta (m_0 c^2)] = [c \alpha^0 P^\mu - \beta (m_0 c^2)] = 0$

$\alpha^\mu \partial_\mu = -\beta (im_0 c/\hbar)$

Transforming from Pauli Spinor (2 component) to Dirac Spinor (4 component) form:

Dirac Equation: $(\gamma^\mu \partial_\mu)\psi = -(im_0 c/\hbar)\psi$

Thus, the Dirac Eqn is guaranteed by construction to be one solution of the KG Eqn

The KG Equation is at the heart of all the various relativistic wave equations, which differ based on mass and spin values, but all of them respect $E^2 - c^2 p \cdot p - (m_0 c^2)^2 = 0$
SRQM Study:
Lots of Relativistic Quantum Wave Equations
A lot of RQM!

Relativistic Quantum Wave Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = -(m_0 c/\hbar)^2 = (i m_0 c/\hbar)^2 = -(\omega_0 / c)^2 \)

The Klein-Gordon Eqn is itself the Relativistic Quantum Equation for spin=0 particles \{Higgs\} (4-Scalars)
Factoring the KG Eqn ("square root method") leads to the RQM Dirac Equation for spin=1/2 particles (4-Spinors)
Applying the KG Eqn to a SR 4-Vector field leads to the RQM Proca Equation for spin=1 particles (4-Vectors)

Setting RestMass \( m_0 \rightarrow 0 \) leads to the:
RQM Free Wave (4-Scalar massless)
RQM Weyl (4-Spinor massless)
Free Maxwell Eqns (4-Vector massless) = Standard EM

So, the same Relativistic Quantum Wave Equation is simply applied to different SR Tensorial Quantum Fields
See Mathematical_formulation_of_the_Standard_Model at Wikipedia:

- **4-Scalar (massive)**: Higgs Field \( \phi \)
  - Free Field Eqn→Klein-Gordon Eqn
  - \( \partial \cdot [\partial \phi] = -(m_0 c/\hbar)^2 \phi \)

- **4-Vector (massive)**: Weak Field \( Z^\mu, W^\mu \)
  - Free Field Eqn→Proca Eqn
  - \( \partial \cdot [\partial Z^\mu] = -(m_0 c/\hbar)^2 Z^\mu \)

- **4-Vector (massless \( m_0 = 0 \))**: Photon Field \( A^\mu \)
  - Free Field Eqn→EM Wave Eqn
  - \( \partial \cdot [\partial A^\mu] = 0 \)

- **4-Spinor (massive)**: Fermion Field \( \psi \)
  - Free Field Eqn→Dirac Eqn
  - \( \gamma \cdot [\partial \psi] = -(i m_0 c/\hbar) \psi \)

*The Fermion Field is a special case, the Dirac Gamma Matrices \( \gamma^\mu \) and 4-Spinor field \( \Psi \) work together to preserve Lorentz Invariance.
Relativistic Quantum Wave Equation: \( \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = -(m_0c/\hbar)^2 = (im_0c/\hbar)^2 = -\left(\omega_0/c\right)^2 \)

\( \partial \cdot \partial = -(m_0c/\hbar)^2 \)

\[ \partial \cdot \partial = 0 \] \[ \nu \]: The Free Classical Maxwell EM Equation \{no source, no spin effects\}

\[ \partial \cdot \partial = \mu_0 J_\nu \]: The Classical Maxwell EM Equation \{with 4-Current \( J \) source, no spin effects\}

\[ \partial \cdot \partial = q(\bar{\psi} \gamma^\nu \psi) \]: The QED Maxwell EM Spin-1 Equation \{with QED source, including spin effects\}

So, the same Relativistic Quantum Wave Equation is simply applied to different SR Tensorial Quantum Fields

See Mathematical_formulation_of_the_Standard_Model at Wikipedia:

4-Scalar (massive) \( \) Higgs Field \( \varphi \) \[ [\partial \cdot \partial = -(m_0c/\hbar)^2] \varphi \] Free Field Eqn→Klein-Gordon Eqn \[ \partial \cdot \partial [\varphi] = -(m_0c/\hbar)^2 \varphi \]

4-Vector (massive) \( \) Weak Field \( Z^\mu, W^\mu \) \[ [\partial \cdot \partial = -(m_0c/\hbar)^2] Z^\mu \] Free Field Eqn→Proca Eqn \[ \partial \cdot \partial [Z^\mu] = -(m_0c/\hbar)^2 Z^\mu \]

4-Vector (massless \( m_0 = 0 \)) \( \) Photon Field \( A^\mu \) \[ [\partial \cdot \partial = 0] A^\mu \] Free Field Eqn→EM Wave Eqn \[ \partial \cdot \partial [A^\mu] = 0^\mu \]

4-Spinor (massive) \( \) Fermion Field \( \psi \) \[ [\partial \cdot \partial = -im_0c/\hbar] \psi \] Free Field Eqn→Dirac Eqn \[ \gamma \cdot \partial [\psi] = -(im_0c/\hbar) \psi \]

*The Fermion Field is a special case, the Dirac Gamma Matrices \( \gamma^\mu \) and 4-Spinor field \( \psi \) work together to preserve Lorentz Invariance.*

We can also do the same physics using Lagrangian Densities.

Proca Lagrangian Density \( L = -(1/2)(\partial_\mu B^\nu - \partial_\nu B^\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) + (m_0c/\hbar)^2 B^\mu B^\nu \) with \( B^\mu = (\varphi/c, a)((ct, r)) \) is a generalized complex 4-(Vector)Potential

KG Lagrangian Density \( L = -\eta^{\mu\nu}(\partial_\mu \psi^* \partial_\nu \psi - (m_0c/\hbar)^2 \psi^* \psi) \) with \( \psi = \psi((ct, r)) \)

Dirac Lagrangian Density \( L = \bar{\psi}(\gamma^\mu \partial_\mu - m_0c/\hbar) \psi \) with \( \psi = \text{a spinor } \psi((ct, r)) \)

QED Lagrangian Density \( L = \bar{\psi}(i\gamma^\mu D^\mu - m_0c)\psi - (1/4)F_{\mu\nu}F^{\mu\nu} \) with \( D^\mu = \partial^\mu + iqA^\mu + iqB^\mu \) and \( A^\mu \)=EM field of the e^−, \( B^\mu \)= external source EM field

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
In relativistic quantum mechanics and quantum field theory, the Bargmann–Wigner equations describe free particles of arbitrary spin \( j \), an integer for bosons (\( j = 1, 2, 3 \ldots \)) or half-integer for fermions (\( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots \)). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields.

Bargmann–Wigner equations: 
\[
(-\gamma^\mu P_\mu + mc)_{\alpha r \alpha' r'} \psi_{\alpha_1 \ldots \alpha_2 \ldots \alpha_{2j}} = 0
\]

In relativistic quantum mechanics and quantum field theory, the Joos–Weinberg equation is a relativistic wave equations applicable to free particles of arbitrary spin \( j \), an integer for bosons (\( j = 1, 2, 3 \ldots \)) or half-integer for fermions (\( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots \)). The solutions to the equations are wavefunctions, mathematically in the form of multi-component spinor fields. The spin quantum number is usually denoted by \( s \) in quantum mechanics, however in this context \( j \) is more typical in the literature.

Joos–Weinberg equation: 
\[
\gamma^{\mu_1 \mu_2 \ldots \mu_{2j}} P_{\mu_1} P_{\mu_2} \ldots P_{\mu_{2j}} + (mc)^{2j} \Psi = 0
\]

The primary difference appears to be the expansion in either the wavefunctions for (BW) or the Dirac Gamma’s for (JW)

For both of these: A state or quantum field in such a representation would satisfy no field equation except the Klein-Gordon equation.

Yet another form is the Duffin-Kemmer-Petiau Equation vs Dirac Equation

DKP Eqn {spin \( 0 \) or \( 1 \)}: \((i\hbar \partial_a - m_c)\Psi = 0\), with \( \beta^a \) as the DKP matrices

Dirac Eqn (spin \( \frac{1}{2} \)): \((i\gamma^a \partial_a - m_c)\Psi = 0\), with \( \gamma^a \) as the Dirac Gamma matrices
# SRQM: A few more SR 4-Vectors

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**SRQM:** A treatise of SR → QM by John B. Wilson ([SciRealm@aol.com](mailto:SciRealm@aol.com))
### SR 4-Vector

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Minimal Coupling = Potential Interaction

Klein-Gordon Eqn → Schrödinger Eqn

\[ P_T = P + Q = P + qA \]
\[ K = i\hbar \]
\[ P = \hbar K \]
\[ P = i\hbar \partial \]

\[ P = (E/c, p) = P_T - qA = (E_T - q\phi/c, p_T - qa) \]
\[ \partial = (\partial/c, \nabla) = \partial_T + (i\hbar c/\hbar)A = (\partial_T/c + (i\hbar c/\hbar)\phi/c, -\nabla_T + (i\hbar c/\hbar)a) \]
\[ \partial \cdot \partial = (\partial/c)^2 - \nabla^2 = -(m_o c/\hbar)^2 \]
\[ P \cdot P = (E/c)^2 - p^2 = (m_o c)^2 \]

\[ E^2 = (m_o c^2)^2 + c^2 p^2 \]
\[ E \sim [ (m_o c^2) + p^2/2m_o ] \]
\[ (E_T - q\phi)^2 = (m_o c)^2 + c^2 (p_T - qa)^2 \]
\[ (E_T - q\phi) \sim [ (m_o c)^2 + (p_T - qa)^2/2m_o ] \]
\[ (i\hbar \partial_T - q\phi)^2 = (m_o c)^2 + c^2 (-i\hbar \nabla_T - qa)^2 \]
\[ (i\hbar \partial_T - q\phi) \sim [ (m_o c)^2 + (-i\hbar \nabla_T - qa)^2/2m_o ] \]
\[ (i\hbar \partial_T) \sim [ q\phi + (m_o c^2) + (i\hbar \nabla_T + qa)^2/2m_o ] \]
\[ (i\hbar \partial_T) \sim [ V + (i\hbar \nabla_T + qa)^2/2m_o ] \]
\[ (i\hbar \partial_T) \sim [ V - (i\hbar \nabla_T)^2/2m_o ] \]

\[ K = i\hbar \]
\[ P = \hbar K \]

The better statement is that the Schrödinger Eqn is the limiting low-velocity case of the more general KG Eqn, not that the KG Eqn is the relativistic generalization of the Schrödinger Eqn.

The Schrödinger NRQM Wave Equation (non-relativistic QM)

\[ (i\hbar \partial_T)\psi > \sim [ V - (i\hbar \nabla_T)^2/2m_o ]\psi > \]
SRQM: Once one has a Relativistic Wave Eqn...

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2 = -(\omega_0/c)^2 \)

Once we have derived a RWE, what does it imply?

The KG Eqn. was derived from the physics of SR plus a few empirical facts. It is a 2\textsuperscript{nd} order, linear, wave PDE that pertains to physical objects of reality from SR.

Just being a linear wave PDE implies all the mathematical techniques that have been discovered to solve such equations generally: Hilbert Space, Superpositions, \( \langle \text{Bra}|\text{Ket}\rangle \) notation, wavevectors, wavefunctions, etc. These things are from mathematics in general, not only and specifically from an Axiom of QM.

Therefore, if one has a physical RWE, it implies the mathematics of waves, the formalism of the mathematics, and thus the mathematical Principles and Formalism of QM. Again, QM Axioms are not required – they emerge from the physics and math...
Once one has a Relativistic Wave Eqn...

Examine Photon Polarization

From the Wikipedia page on [Photon Polarization]

Photon polarization is the quantum mechanical description of the classical polarized sinusoidal plane electromagnetic wave. An individual photon can be described as having right or left circular polarization, or a superposition of the two. Equivalently, a photon can be described as having horizontal or vertical linear polarization, or a superposition of the two.

The description of photon polarization contains many of the physical concepts and much of the mathematical machinery of more involved quantum descriptions and forms a fundamental basis for an understanding of more complicated quantum phenomena. Much of the mathematical machinery of quantum mechanics, such as state vectors, probability amplitudes, unitary operators, and Hermitian operators, emerge naturally from the classical Maxwell's equations in the description. The quantum polarization state vector for the photon, for instance, is identical with the Jones vector, usually used to describe the polarization of a classical wave. Unitary operators emerge from the classical requirement of the conservation of energy of a classical wave propagating through lossless media that alter the polarization state of the wave. Hermitian operators then follow for infinitesimal transformations of a classical polarization state.

Many of the implications of the mathematical machinery are easily verified experimentally. In fact, many of the experiments can be performed with two pairs (or one broken pair) of polaroid sunglasses.

The connection with quantum mechanics is made through the identification of a minimum packet size, called a photon, for energy in the electromagnetic field. The identification is based on the theories of Planck and the interpretation of those theories by Einstein. The correspondence principle then allows the identification of momentum and angular momentum (called spin), as well as energy, with the photon.
SRQM:

Principle of Superposition:
From the mathematics of waves

Klein-Gordon Equation: $\partial \cdot \partial = (\partial / c)^2 - \nabla \cdot \nabla = (-im_0 c / \hbar)^2 = -(m_0 c / \hbar)^2 = -(\omega_0 / c)^2$

The Extended Superposition Principle for Linear Equations

Suppose that the non-homogeneous equation, where $L$ is linear, is solved by some particular $u_p$
Suppose that the associated homogeneous problem is solved by a sequence of $u_i$
$L(u_p) = C ; \ L(u_0) = 0 , \ L(u_1) = 0, \ L(u_2) = 0 ...$
Then $u_p$ plus any linear combination of the $u_n$ satisfies the original non-homogeneous equation:
$L(u_p + \sum a_n u_n) = C,$
where $a_n$ is a sequence of (possibly complex) constants and the sum is arbitrary.

Note that there is no mention of partial differentiation. Indeed, it's true for any linear equation, algebraic or integro-partial differential-whatever.

QM superposition is not axiomatic, it emerges from the mathematics of the Linear PDE
The Klein-Gordon Equation is a 2nd-order LINEAR Equation.
This is the origin of superposition in QM.
SRQM:

Klein-Gordon obeys Principle of Superposition

Klein-Gordon Equation: \( \partial \cdot \partial = (\partial/c)^2 - \nabla \cdot \nabla = (-im_0c/\hbar)^2 = -(m_0c/\hbar)^2 = -(\omega_0/c)^2 \)

\( K \cdot K = (\omega/c)^2 - k \cdot k = (\omega_0/c)^2 \): The particular solution (w rest mass)

\( K_n \cdot K_n = (\omega_n/c)^2 - k_n \cdot k_n = 0 \): The homogenous solution for a (virtual photon?) microstate n

Note that \( K_n \cdot K_n = 0 \) is a null 4-vector (photonic)

Let \( \Psi_p = Ae^{-i(K \cdot X)} \), then \( \partial \cdot \partial[\Psi_p] = (-i)^2(K \cdot K)\Psi_p = -(\omega_0/c)^2\Psi_p \)

which is the Klein-Gordon Equation, the particular solution...

Let \( \Psi_n = A_ne^{-i(K_n \cdot X)} \), then \( \partial \cdot \partial[\Psi_n] = (-i)^2(K_n \cdot K_n)\Psi_n = (0)\Psi_n \)

which is the Klein-Gordon Equation homogeneous solution for a microstate n

We may take \( \Psi = \Psi_p + \sum_n \Psi_n \)

Hence, the Principle of Superposition is not required as an QM Axiom, it follows from SR and our empirical facts which lead to the Klein-Gordon Equation. The Klein-Gordon equation is a linear wave PDE, which has overall solutions which can be the complex linear sums of individual solutions – i.e. it obeys the Principle of Superposition. This is not an axiom – it is a general mathematical property of linear PDE’s. This property continues over as well to the limiting case \( \{ |v| \ll c \} \) of the Schrödinger Equation.
SRQM:

QM Hilbert Space: From the mathematics of waves

Klein-Gordon Equation: \[ \partial \cdot \partial = (\partial_t/c)^2 - \nabla \cdot \nabla = (-im_0c/\hbar)^2 = -(m_o c/\hbar)^2 = -(\omega_0/c)^2 \]

Hilbert Space (HS) representation:
if \(|\Psi\rangle \in \text{HS}, \text{then } c|\Psi\rangle \in \text{HS}, \text{where } c \text{ is complex number}
if \(|\Psi_1\rangle \text{ and } |\Psi_2\rangle \in \text{HS}, \text{then } |\Psi_1\rangle + |\Psi_2\rangle \in \text{HS}
if \(|\Psi\rangle = c_1|\Psi_1\rangle + c_2|\Psi_2\rangle, \text{then } \langle \Phi|\Psi\rangle = c_1\langle \Phi|\Psi_1\rangle + c_2\langle \Phi|\Psi_2\rangle \text{ and } \langle \Psi| = c_1^*\langle \Psi_1| + c_2^*\langle \Psi_2| \]
\[ \langle \Phi|\Psi\rangle = \langle \Psi|\Phi\rangle \]
\[ \langle \Psi|\Psi\rangle \geq 0 \]
if \(\langle \Psi|\Psi\rangle = 0, \text{then } |\Psi\rangle = 0 \]

etc.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. They are indispensable tools in the theories of partial differential equations, Fourier analysis, signal processing, heat transfer, ergodic theory, and Quantum Mechanics.

The QM Hilbert Space emerges from the fact that the KG Equation is a linear wave PDE – Hilbert spaces as solutions to PDE's are a purely mathematical phenomenon – no QM Axiom is required.

Likewise, this introduces the <bra|,|ket> notation, wavevectors, wavefunctions, etc.

Note:

One can use Hilbert Space descriptions of Classical Mechanics using the Koopman-von Neumann formulation. One cannot use Hilbert Space descriptions of Quantum Mechanics by using the Phase Space formulation of QM.
The Standard QM Canonical Commutation Relation is simply an axiom in standard QM. It is just given, with no explanation. You just had to accept it.

I always found that unsatisfactory.

There are at least 4 parts to it:

Where does the commutation ([ , ]) come from?
Where does the imaginary constant (i) come from?
Where does the Dirac:reduced-Planck constant (ℏ) come from?
Where does the Kronecker Delta (δ_{jk}) come from?

See the next page for SR enlightenment...
The SR Metric is the source of “quantization”.

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
Let \( (f) \) be an arbitrary SR function
\[ X[t] = X_f, \quad \partial[t] = \partial \]
\( X \), function or not, has no effect on \( (f) \)
\[ \partial = \partial \] is definitely an SR function:operator

\[ X[a[t]] = X[a[t]] \]
\[ \partial[a[t]] = a[\partial[a[t]] + X[a[t]]] \]
\[ \partial[a[t]] - X[a[\partial]] = a[X][f] \]

Recognize this as a commutation relation
\[ [\partial, X]\] is definitely an SR function:operator

\[ [\partial, X] = a[X] \]
\[ = \partial[a[X]] \]
\[ = \partial[a[c, T]][(ct, x)] \]
\[ = \partial[a[c, -\partial, \partial_y, \partial_x][(ct, x, y, z)] \]
\[ = \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\delta^k] \]
\[ = \eta^{ij} = \text{Minkowski Metric} \]

\[ [\partial_i, X^j] = \eta^{ij} \]
\[ \partial_i^j = \text{Tensor form: true for all observers} \]

\[ \partial_i^j = i\hbar \delta_{ij} \]
\[ = \text{Independently true from empirical constants} \]
\[ = (i, \hbar) \]
\[ = \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -\delta^k] \]
\[ = \eta^{ij} = \text{Minkowski Metric} \]

\[ [x_i, p_i] = i\hbar \delta_{ij} \]

Position: Momentum
\[ X^j = -i\hbar(1) \]

Time: Energy
\[ X^i = \gamma(x, t) = -\partial[S] \]

4-Velocity
\[ U = \gamma(c, v) \]

4-Displacement
\[ \Delta X = (\Delta t, \Delta x) \]

4-Position
\[ X = (ct, x) \]

4-Gradient
\[ \partial = (\partial/c, -\nabla) \]

4-WaveVector
\[ K = (\omega/c, k) \]

Wave Velocity
\[ v_{\text{group}} = \frac{c}{v_{\text{phase}}} = c^2 \]

Einstein de Broglie
\[ P = \hbar K \]

4-Momentum
\[ P = (mc, p) = (E/c, p) \]

Non-Zero Commutation Relation via SR 4-Momentum
\[ \{P = \hbar K\} \]

Non-Zero Commutation Relation via SR 4-WaveVector
\[ \{i[\partial, X] = [i\partial, X] = [K, X] = i\eta^{ij} \} \]

Lorentz Transform
\[ \delta_i^j = \delta_i^j(X^i) = \Lambda_i^j \]

Minkowski Metric
\[ \delta_i^j = \delta_i^j = \eta^{ij} \]

Complex Plane-waves
\[ K = i\delta \]

Non-Zero Commutation Relation via natural SR 4-Gradient
\[ \{[K, X] = [\Phi_{\text{phase}}], \ldots \} \]

Electric Dipole Moment
\[ E = mc^2 \]

E = mc^2
\[ m_0 \]

\[ E/c^2 \]

\[ E/\omega_0 \]

\[ (\hbar) \]

\[ i[K, X] = i[\partial, X] = [K, X] = i\eta^{ij} \]

Trace[T^\mu_\nu] = \eta_\mu^\nu T^\mu_\nu = T^\nu_\nu = T

\[ V \cdot V = V'' \cdot V' = (\omega^2 V' - V' \omega) = (\omega^2 V')^2 = \text{Lorentz Scalar} \]
As we have seen, this relation is generated from simple SR math.

\[
[\partial_\mu, X^\nu] = \partial_\mu [X^\nu] = \partial^\mu [X^\nu] = (\partial_\mu/c, -\nabla_\mu)[(ct, x, y, z)] = \text{Diag}(1, -1, -1, -1) = \text{Diag}[1, -\delta^\mu\nu] = \eta^{\mu\nu} = \text{Minkowski Metric}
\]

\[
[\partial_\mu, X^\nu] = \eta^{\mu\nu}
\]

\[
[P_\mu, X^\nu] = i\hbar \eta^{\mu\nu} : \text{This is the more general 4D version, with the Standard QM version being just the spatial part.}
\]

One of the great misconceptions on modern physics is that since QM is about “tiny” things, that ALL things should be built up from it. That paradigm of course works well for many things:

- Compounds are built-up from smaller molecules.
- Molecules are built-up from smaller elements.
- Elements are built-up from smaller atoms.
- Atoms are built-up from smaller protons, neutrons, and electrons.
- Protons and neutrons are built-up from smaller quarks.

And all experiments to-date show that electrons and quarks appear to be point-like, with wave-type properties giving extent.

So, one can mistakenly think that “SpaceTime” must be made up of smaller “quantum” stuff as well. However, that is not what the math says. The “quantization” paradigm doesn’t apply to SpaceTime itself, just to <events>. All of the “quantum”-sized things above, electrons and quarks, are material things, <events>, which move around “within” SpaceTime. Their “quantization” comes about from the properties of the math and metric of SR.

The math does *NOT* say that SpaceTime itself is “quantized”. It says that SR Minkowski SpaceTime is the source of “quantization.”
SRQM Study: 4-Position and 4-Gradient

4-Displacement
\[ \Delta R = (c \Delta t, \Delta r) \]
dR = (cdt, d\mathbf{r})

4-Position
\[ R = R^\mu = (ct, \mathbf{r}) \]

SR: Minkowski Metric
\[ \partial[R] = \partial^\mu R^\nu = \eta^{\mu \nu} \]
\[ \rightarrow \text{Diag}[1, -1, -1, -1] \]
\[ = \text{Diag}[1, -1_3] = \text{Diag}[1, -\delta^3] \]
\[ \text{(in Cartesian form) "Particle Physics" Convention} \]
\[ \{\eta_{\mu \nu}\} = 1/\{\eta^{\mu \nu}\} \]
\[ \text{Tr}[\eta^{\mu \nu}] = 4 \]
\[ \eta^\mu \propto \delta^\mu \]

SR: Lorentz Transform
\[ \partial[R'] = \partial^\mu R'^\nu = \Lambda^\nu_\mu \]
\[ \Lambda^\mu_\nu \Lambda^\alpha_\beta = \eta^\mu \delta^\nu_\beta \]
\[ \eta_{\mu \nu} \Lambda^\alpha_\nu \Lambda^\nu_\beta = \eta_{\alpha \beta} \]
\[ (\text{Det}[\Lambda])^2 = 1 \]
\[ \text{Det}[\Lambda] = \pm 1 \]
\[ \Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu \]
\[ \Lambda_{\mu \nu} \Lambda^{\mu \nu} = 4 \]

Rotations
Boosts
CPT

SpaceTime
\[ \partial \cdot R = \partial_\mu R^\mu = 4 \]

Dimension

SRQM: Tensor Zero Exterior Product
\[ \partial^\mu R = \partial^\mu R^\nu - \partial^\nu R^\mu = 4 \]

SRQM: Non-Zero Commutation
\[ [\partial, R] = [\partial^\mu, R^\nu] = \partial^\mu R^\nu - \partial^\nu R^\mu = 4 \]

Total Derivative Chain Rule
\[ dR \cdot \partial = (cdt \partial t + d\mathbf{r} \cdot \nabla) \]
\[ = dt(\partial / \partial t) + dx(\partial / \partial x) + dy(\partial / \partial y) + dz(\partial / \partial z) \]

Invariant Interval
\[ R \cdot R = (ct)^2 - \mathbf{r} \cdot \mathbf{r} = (c \tau)^2 \]

Invariant d'Alembertian Wave Equation
\[ \partial \cdot \partial = (\partial / c)^2 \nabla \cdot \nabla = (\partial / c)^2 \]

Invariant Calculus
\[ dR \cdot \partial \cdot \partial = (cdt, d\mathbf{r}) \cdot (\partial / c, -\nabla) \]
\[ = dt(\partial / \partial t) + dx(\partial / \partial x) + dy(\partial / \partial y) + dz(\partial / \partial z) \]

4-Gradient
\[ \partial \cdot \partial = (\partial / c, -\nabla) \]
\[ = \partial / \partial R^\mu \]

SR 4-Tensor
(2,0)-Tensor \( T^{\mu \nu} \)
(1,1)-Tensor \( T^\nu_\mu \), or \( T^1_2 \)
(0,2)-Tensor \( T^\mu_\nu \)

SR 4-Vector
(1,0)-Tensor \( V^\mu = V = (v^0, \mathbf{v}) \)
(0,1)-Tensor \( V_\nu = (v_0, -\mathbf{v}) \)

SR 4-CoVector: OneForm
(0,1)-Tensor \( V_\nu = (v_0, -\mathbf{v}) \)

SR 4-Scalar
(0,0)-Tensor \( S \) or \( S_0 \)
Lorentz Scalar

Trace[\( T^{\mu \nu} \)] = \( \eta_{\mu \nu} T^{\mu \nu} = T^\nu_\nu = T \]
\[ V \cdot V = V^\nu \eta_{\nu \nu} V^\nu = (v^0)^2 - \mathbf{v} \cdot \mathbf{v} = (v^0)^2 \]
\[ = \text{Lorentz Scalar} \]
SRQM: Heisenberg Uncertainty Principle: Viewed from SRQM

Heisenberg Uncertainty \( \{ \sigma^2_\text{A}, \sigma^2_\text{B} \} \geq (1/2)|\langle [A,B]\rangle| \)
arises from the non-commuting nature of certain operators.

The commutator is \([A,B] = AB-BA\), where A & B are functional “measurement” operators. The Operator Formalism arose naturally from our SR → QM path: \([\partial = -i\mathbf{K}]\).

The Generalized Uncertainty Relation: \( \sigma^2_\text{f} \sigma^2_\text{g} \geq (\Delta F) \cdot (\Delta G) \geq (1/2)|\langle i [F,G] \rangle| \)

The uncertainty relation is a very general mathematical property, which applies to both classical or quantum systems. From Wikipedia: Photon Polarization: “This is a purely mathematical result. No reference to a physical quantity or principle is required.”

The Cauchy–Schwarz inequality asserts that (for all vectors \( f \) and \( g \) of an inner product space, with either real or complex numbers):
\[ \sigma^2_\text{f} \sigma^2_\text{g} = \langle f | f \cdot g | g \rangle \geq |\langle f | g \rangle|^2 \]

But first, let's back up a bit; Using standard complex number math, we have:
\[ z = a + ib \]
\[ z^* = a - ib \]
\[ \text{Re}(z) = a = (z + z^*)/(2) \]
\[ \text{Im}(z) = b = (z - z^*)/(2i) \]
\[ |z|^2 = [(z + z^*)/(2)]^2 + [(z - z^*)/(2i)]^2 \]

Now, generically, based on the rules of a complex inner product space we can arbitrarily assign:
\[ z = \langle f | g \rangle, \quad z^* = \langle g | f \rangle \]

Which allows us to write:
\[ |\langle f | g \rangle|^2 = \langle f | f \rangle \langle g | g \rangle + \langle f | g \rangle \langle g | f \rangle \geq (1/2)|\langle f | g \rangle|^2 \]

We can also note that:
\[ |\langle f | = F| \Psi \rangle \text{ and } |\langle g | = G| \Psi \rangle \]

Thus,
\[ |\langle f | g \rangle|^2 = (|\langle \Psi | F^* G| \Psi \rangle \rangle + \langle \Psi | G^* F| \Psi \rangle \rangle)/(2) |^{2} + (|\langle \Psi | F^* G| \Psi \rangle \rangle - \langle \Psi | G^* F| \Psi \rangle \rangle)/(2i) |^{2} \]

For Hermetian Operators...
\[ F^* = +F, \quad G^* = +G \]

For Anti-Hermetian (Skew-Hermetian) Operators...
\[ F^* = -F, \quad G^* = -G \]

Assuming that F and G are either both Hermetian, or both anti-Hermetian...
\[ |\langle f | g \rangle|^2 = (|\langle \Psi | (\pm)FG| \Psi \rangle \rangle + \langle \Psi | (\pm)GF| \Psi \rangle \rangle)/(2) |^{2} + (|\langle \Psi | (\pm)FG| \Psi \rangle \rangle - \langle \Psi | (\pm)GF| \Psi \rangle \rangle)/(2i) |^{2} \]

We can write this in commutator and anti-commutator notation...
\[ |\langle f | g \rangle|^2 = (|\langle \Psi | (\pm)FG| \Psi \rangle \rangle/(2) |^{2} + (|\langle \Psi | (\pm)GF| \Psi \rangle \rangle/(2i) |^{2} \]

Due to the squares, the \((\pm)\)'s go away, and we can also multiply the commutator by an \(( i^2)\)
\[ |\langle f | g \rangle|^2 = (|\langle \Psi | F| G\rangle \Psi \rangle \rangle/(2) |^{2} + (|\langle \Psi | G| F\rangle \Psi \rangle \rangle/(2i) |^{2} \]

The Cauchy–Schwarz inequality again...
\[ \sigma^2_\text{f} \sigma^2_\text{g} = \langle f | f \rangle \langle g | g \rangle \geq (|\langle f | g \rangle|^2) + (\langle f | g \rangle)^2 \]

Taking the root:
\[ \sigma^2_\text{f} \sigma^2_\text{g} \geq (1/2)|\langle i | F| G \rangle | \]

Which is what we had for the generalized Uncertainty Relation.

*Note* This is not a QM axiom - This is just pure math. At this stage we already see the hints of commutation and anti-commutation. It is true generally, whether applying to a physical or purely mathematical situation.
Heisenberg Uncertainty Principle: Simultaneous vs Sequential

Heisenberg Uncertainty \( \{ \sigma_A^2 \sigma_B^2 \geq (1/2)|<[A,B]>| \} \) arises from the non-commuting nature of certain operators.

\[
[\partial^\mu, X^\nu] = \partial[X] = \eta^{\mu \nu} = \text{Minkowski Metric}
\]

\[
[P^\mu, X^\nu] = [i\hbar \partial^\mu, X^\nu] = i\hbar[\partial^\mu, X^\nu] = i\hbar \eta^{\mu \nu}
\]

Consider the following:
Operator A acts on System \(|\Psi\rangle\) at SR Event A: \(A|\Psi\rangle \rightarrow |\Psi'\rangle\)
Operator B acts on System \(|\Psi'\rangle\) at SR Event B: \(B|\Psi'\rangle \rightarrow |\Psi''\rangle\)
or \(BA|\Psi\rangle = B|\Psi'\rangle = |\Psi''\rangle\)

If measurement Events A & B are space-like separated, then there are observers who can see \{A before B, A simultaneous with B, A after B\}, which of course does not match the quantum description of how Operators act on Kets.

If Events A & B are time-like separated, then all observers will always see A before B. This does match how the operators act on Kets, and also matches how \(|\Psi\rangle\) would be evolving along its worldline, starting out as \(|\Psi\rangle\), getting hit with operator A at Event A to become \(|\Psi'\rangle\), then getting hit with operator B at Event B to become \(|\Psi''\rangle\).

The Uncertainty Relation here does NOT refer to simultaneous (space-like separated) measurements, it refers to sequential (time-like separated) measurements. This removes the need for ideas about the particles not having simultaneous properties. There are simply no “simultaneous measurements” of non-zero commuting properties on an individual system, a single worldline – they are sequential, and the first measurement places the system in such a state that the outcome of the second measurement will be altered wrt. if the order of the operations had been reversed.
The Pauli Exclusion Principle is a result of the empirical fact that nature uses identical (indistinguishable) particles, and this combined with the Spin-Statistics theorem from SR, leads to an exclusion principle for fermions (anti-symmetric, Fermi-Dirac statistics) and an aggregation principle for bosons (symmetric, Bose-Einstein statistics). The Spin-Statistics Theorem is related as well to the CPT Theorem.

For large numbers and/or mixed states these both tend to the Maxwell-Boltzmann statistics. In the \(kT \gg (\epsilon_i - \mu)\) limit, Bose-Einstein reduces to Rayleigh-Jeans. The commutation relations here are based on space-like separation particle exchanges. Exchange operator \(P\), \(P^2 = +1\), Since two exchanges bring one back to the original state. \(P\) thus has two eigenvalues \((\pm 1)\) and two eigenvectors \(|\text{Symm}\rangle, |\text{AntiSymm}\rangle\)

\[
P|\text{Symm}\rangle = +|\text{Symm}\rangle \\
P|\text{AntiSymm}\rangle = -|\text{AntiSymm}\rangle
\]

<table>
<thead>
<tr>
<th>Spin-Symmetry</th>
<th>Particle Type</th>
<th>Quantum Statistics</th>
<th>Classical (kT \gg (\epsilon_i - \mu))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{spin:}(0,1,...,N)) bosons symmetric</td>
<td>Indistinguishable, Commutation relation ([a,b] = ab-ba = [b,a] = \text{constant} \ (ab = ba) ) if commutes</td>
<td>Bose-Einstein: (n_i = g_i / \left[ e^{(\epsilon_i - \mu)/kT} - 1 \right]) aggregation principle</td>
<td>Rayleigh-Jeans: from (e^x \sim (1 + x + ...)) (n_i = g_i / \left[ (\epsilon_i - \mu)/kT \right])</td>
</tr>
<tr>
<td>Multi-particle Mixed</td>
<td>Distinguishable, or high temp, or low density</td>
<td>Maxwell-Boltzmann: (n_i = g_i / \left[ e^{(\epsilon_i - \mu)/kT} + 0 \right])</td>
<td>Maxwell-Boltzmann: (n_i = g_i / \left[ e^{(\epsilon_i - \mu)/kT} \right])</td>
</tr>
<tr>
<td>(\text{spin:}(1/2,3/2,...,N/2)) fermions anti-symmetric</td>
<td>Indistinguishable, Anti-commutation relation ([a,b] = ab+ba = +[b,a] = \text{constant} \ (ab = - ba) ) if anti-commutes</td>
<td>Fermi-Dirac: (n_i = g_i / \left[ e^{(\epsilon_i - \mu)/kT} + 1 \right]) exclusion principle</td>
<td></td>
</tr>
</tbody>
</table>
Complex 4-vectors are simply 4-Vectors where the components may be complex-valued

\[ A = A^\mu = (a^0, a) = (a^0, a^1, a^2, a^3) \rightarrow (a^t, a^x, a^y, a^z) \]

\[ B = B^\mu = (b^0, b) = (b^0, b^1, b^2, b^3) \rightarrow (b^t, b^x, b^y, b^z) \]

Examples of 4-Vectors with complex components are the 4-Polarization and the 4-ProbabilityCurrentDensity

Minkowski Metric \( g^{\mu\nu} \rightarrow \eta^{\mu\nu} = \eta_{\mu\nu} \rightarrow \text{Diag}[1, -1, -1, -1] = \text{Diag}[1, -I(3)] \), which is the \{curvature\~0 limit = low-mass limit\} of the GR metric \( g^{\mu\nu} \).

Applying the Metric to raise or lower an index also applies a complex-conjugation *

Scalar Product = Lorentz Invariant \( \rightarrow \) Same value for all inertial observers

\[ A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^x B^y = A^\mu B^\mu \ast = (a^0 b^0 - a^\ast \cdot b) \text{ using the Einstein summation convention} \]

This reverts to the usual rules for real components
However, it does imply that \( A \cdot B = B \cdot A \)

**SRQM: A treatise of SR→QM by John B. Wilson**
The Phase is a Lorentz Scalar Invariant – all observers must agree on its value.

\[ K \cdot X = (\omega/c) \cdot (\text{cl},x) = (\omega - k \cdot x) = -0: \text{Phase of SR Wave} \]

We take the point of view of an observer operating on a particle at 4-Position \( X \), which has an initial 4-Vector Wave \( K \). The 4-Position \( X \) of the particle, the operation's event, will not change: we are applying the various operations only to the particle's 4-Momentum \( K \).

Note that for matter particles \( K = (\omega/c)T \), where \( T \) is the Unit-Temporal 4-Vector \( T = \gamma(1,\beta) \), which defines the particle's worldline at each point. The gamma factor \( \gamma \) will be unaffected in the following operations, since it uses the square of \( \beta \): \( \gamma = 1/\sqrt{1 - \beta^2} \).

For photonic particles, \( K = (\omega/c)N \), where \( N \) is the "Unit"-Null 4-Vector \( N = (1,n) \) and \( n \) is a unit-spatial 3-vector. All operations listed below work similarly on the Null 4-Vector.

Do a Time Reversal Operation: \( T \)
The particle's temporal direction is reversed & complex-conjugated:

\[ T = -T = \gamma(-1,\beta)* \]

Do a Parity Operation (Space Reflection): \( P \)
Only the spatial directions are reversed:

\[ T = \gamma(1,-\beta) \]

Do a Charge Conjugation Operation: \( C \)
Charge Conjugation actually changes all internal quantum #s:
charge, lepton #, etc.

Feynman showed this is the equivalent of a world-line reversal & complex-conjugation:

\[ T_C = \gamma(-1,-\beta)* \]

Pairwise combinations:

- \( T_T = T_P = T_C = \gamma(1,-\beta)* \)
- \( T_T = T_P = T_C = \gamma(-1,\beta)* \), a CP event is mathematically the same as a \( T \) event

They all remain temporal 4-vectors

\[ T_{TP} = T_{PT} = T_{CP} = \gamma(1,\beta) \quad T_{CC} = \gamma(1,\beta) \quad T_{TP} = T_{CP} = \gamma(1,\beta) \quad T_{TT} = T = \gamma(1,\beta) \]

It is only the combination of all three ops: \{C,P,T\}, or pairs of singles: \{CC,PP,TT\} that leave the Unit-Temporal 4-Vector, and thus the Phase, Invariant.
SRQM: CPT Theorem (Charge) vs (Parity) vs (Time)

Identity and Space-Parity are Unitary Time-Reversal and Charge-Conjugation are Anti-Unitary.

Original 4-Vector
A=A'=\left(\begin{array}{c}a^0, a^1, a^2, a^3\end{array}\right)

identity

SR 4-Tensor
\left(\begin{array}{c}2,0\end{array}\right)\text{Tensor } T_{\mu\nu}

SR 4-Vector
\left(\begin{array}{c}1,0\end{array}\right)\text{Tensor } V^\mu = (\vec{v}^\mu, v^0)

SR 4-CoVector: OneForm
\left(\begin{array}{c}0,1\end{array}\right)\text{Tensor } V_\nu = (v_0, \vec{v})

SR 4-Scalar
\left(\begin{array}{c}0,0\end{array}\right)\text{Tensor } S \text{ or } S_0 \text{ Lorentz Scalar}
SRQM Transforms: Venn Diagram

Poincaré = Lorentz + Translations

(10) (6) (4)

Transformations

(# of independent parameters = # continuous symmetries = # Lie Dimensions)

Poincaré Transformation Group aka. Inhomogeneous Lorentz Transformation
Lie group of all affine isometries of SR:Minkowski TimeSpace (preserve quadratic form $\eta_{\mu\nu}$)
General Linear,Affine Transform $X' = \Lambda^\mu_\nu X^\nu + \Delta X^\mu$ with $\text{Det}[\Lambda_{\mu\nu}] = \pm 1$

$(6+4=10)$

4-Tensor {mixed type-(1,1)}

4-Vector {Charge}

4-Vector {Partiy}

4-Vector {Time}

Discrete

Continuous

Lorentz Transform

$\Lambda^\mu_\nu$

$\Lambda^\mu_\nu \rightarrow T^\mu_\nu$

$\Lambda^\mu_\nu \rightarrow P^\mu_\nu$

$\Lambda^\mu_\nu \rightarrow C^\mu_\nu$

Discrete

Continuous

SpatialFlipCombos

$\Lambda^\mu_\nu \rightarrow F^\mu_\nu$

$\Lambda^\mu_\nu \rightarrow P^\mu_\nu$

$\Lambda^\mu_\nu \rightarrow C^\mu_\nu$

Identity $I_{(4)}$

$x,y,z \rightarrow x,y,z$

Rotation $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$

$x,y,z \rightarrow x,y,z$

Boost $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$

$x,y,z \rightarrow x,y,z$

Charge-Conjugation $\Lambda^\mu_\nu \rightarrow C^\mu_\nu$

$R \rightarrow -R$, $q \rightarrow -q$

CPT Symmetry

{Charge}

{Partiy}

{Time}

Isotropy

{same all directions}

Homogeneity

{same all points}

Temporal

$\Delta X^\mu \rightarrow (c\Delta t,0)$

$\Delta t$

Spatial

$\Delta X^\mu \rightarrow (0,\Delta x)$

$\Delta x$ | $\Delta y$ | $\Delta z$

4-Zero

$\Delta X^\mu \rightarrow (0,0)$

$\Delta t$

4-AngularMomentum $M^\mu_\nu = X^\mu \wedge P^\nu = X^\mu P^\nu - X^\nu P^\mu$

= Generator of Lorentz Transformations (6)

= { $\Lambda^\mu_\nu \rightarrow R^\mu_\nu$, Rotations (3) + $\Lambda^\mu_\nu \rightarrow B^\mu_\nu$, Boosts (3) }

4-LinearMomentum $P^\mu$

= Generator of Translation Transformations (4)

= { $\Delta X^\mu \rightarrow (c\Delta t,0)$ Time (1) + $\Delta X^\mu \rightarrow (0,\Delta x)$ Space (3) }

$\text{Det}[\Lambda_{\mu\nu}] = +1$ for Proper Lorentz Transforms

$\text{Det}[\Lambda_{\mu\nu}] = -1$ for Improper Lorentz Transforms

Lorentz Matrices can be generated by a matrix $M$ with $\text{Tr}[M]=0$ which gives:

{ $\Lambda = e^M = e^+(\theta \mathbf{J} - \xi \mathbf{K})$ }

{ $\Lambda^T = (e^M)^T = e^+(\theta \mathbf{J} - \xi \mathbf{K})$ }

{ $\Lambda^{-1} = (e^M)^{-1} = e^-(\theta \mathbf{J} - \xi \mathbf{K})$ }

$\Lambda = e^M M = e^+(\theta \mathbf{J} - \xi \mathbf{K})$

$\text{Det}[\Lambda_{\mu\nu}] = \pm 1$

$\Lambda_{\mu\nu} \Lambda^\nu_\mu = 4$

SR: Lorentz Transform

$\partial_i [R^\mu] = \delta R^\mu / \delta R^\nu = \Lambda^\mu_\nu$

$\Lambda^\mu_\nu = (\Lambda^{-1})^\mu_\nu : \Lambda^\alpha_\nu \Lambda^\nu_\alpha = \eta_{\alpha\beta} \Lambda^\mu_\nu \Lambda^\nu_\mu = \eta_{\alpha\beta}$

Rotations $J_i = -\varepsilon_{mn} M_m^i$, Boosts $K_i = M_0^i$

[ $(R \rightarrow -R^*)$ or $(t \rightarrow -t^*)$ & $(r \rightarrow -r)$ ] imply $q \rightarrow -q$

Feynman-Stueckelberg Interpretation

Amusingly, Inhomogeneous Lorentz adds homogeneity.

SR $\rightarrow$ QM Physics

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John B. Wilson

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http://scirealm.org/SRQM.pdf
Hermitian Generators
Noether's Theorem - Continuity

The Hermitian Generators that lead to translations and rotations via unitary operators in QM...

These all ultimately come from the Poincaré Invariance → Lorentz Invariance that is at the heart of SR and Minkowski Space.

Infintesimal Unitary Transformation
\( \hat{U}_t(\hat{G}) = 1 + i\varepsilon \hat{G} \)

Finite Unitary Transformation
\( \hat{U}_\alpha(\hat{G}) = e^{i\alpha \hat{G}} \)

let \( \hat{G} = P/\hbar = K \)
let \( \alpha = \Delta x \)

\( \hat{U}_{\Delta x}(P/\hbar)\Psi(X) = e^{i\Delta x \cdot P/\hbar}\Psi(X) = e^{i\Delta x \cdot \hat{\partial}}\Psi(X) = \Psi(X - \Delta x) \)

Time component: \( \hat{U}_{\Delta t}(P/\hbar)\Psi(ct) = e^{i(\Delta tE/\hbar)\hat{\partial}}\Psi(ct) = e^{i\Delta t\hat{\partial}}\Psi(ct) = \Psi(ct - c\Delta t) = c\Psi(t - \Delta t) \)
Space component: \( \hat{U}_{\Delta x}(p/\hbar)\Psi(x) = e^{i\Delta x \cdot p/\hbar}\Psi(x) = e^{i\Delta x \cdot \nabla}\Psi(x) = \Psi(x + \Delta x) \)

By Noether's Theorem, this leads to \( \hat{\partial} \cdot K = 0 \)

We had already calculated
(\( \hat{\partial} \cdot \hat{\partial} \))\( [K \cdot X] \) = (\( \hat{\partial} \cdot \hat{\partial} \))(\( \omega t - k \cdot x \)) = 0
(\( \hat{\partial} \cdot \hat{\partial} \))\( [K \cdot X] \) = \( \hat{\partial} \cdot (\hat{\partial}[K \cdot X]) = \hat{\partial} \cdot K = 0 \)

Poincaré Invariance also gives the Casimir invariants of mass and spin, and ultimately leads to the spin-statistics theorem of RQM.
SRQM:
QM Correspondence Principle:
Analogous to the GR and SR limits

Basically, the old school QM Correspondence Principle says that QM should give the same results as classical physics in the realm of large quantum systems, i.e. where macroscopic behavior overwhelms quantum effects. Perhaps a better way to state it is when the change of system by a single quantum has a negligible effect on the overall state.

There is a way to derive this limit, by using Hamilton-Jacobi Theory:

\[(\imath\hbar \partial_t)\Psi \sim \left[ V - \frac{\hbar}{2m} \nabla^2 \right]\Psi : \text{The Schrödinger NRQM Equation for a point particle (non-relativistic QM)}\]

Examine solutions of form \(\Psi = \Psi_0 e^{\imath \Phi} = \Psi_0 e^{\imath S/\hbar} \) where \(S \) is the QM Action

\[
\partial_t[S] = (i/\hbar)\Psi \partial_t[S] \text{ and } \nabla^2[\Psi] = (i/\hbar)\Psi \nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2 \\
(i\hbar)(i/\hbar)\Psi \partial_t[S] = V\Psi - (\hbar^2/2m_o)((i/\hbar)\Psi \nabla^2[S] - (\Psi/\hbar^2)(\nabla[S])^2) \\
\partial_t[S] = V + (i\hbar/2m_o)\nabla^2[S] - (1/2m_o)(\nabla[S])^2 \\
\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2] = (i\hbar/2m_o)\nabla^2[S] : \text{Quantum Single Particle Hamilton-Jacobi} \\
\partial_t[S] + [V+(1/2m_o)(\nabla[S])^2] = 0 : \text{Classical Single Particle Hamilton-Jacobi}
\]

Thus, the classical limiting case is:

\[
\nabla^2[\Phi] \ll (\nabla[\Phi])^2 \\
\hbar \nabla^2[S] \ll (\nabla[S])^2 \\
\hbar \nabla \cdot p \ll (p \cdot p) \\
(p \lambda) \nabla \cdot p \ll (p \cdot p)
\]
QM Correspondence Principle:
Analogous to the GR and SR limits

\[ \partial_t [S] + [V + (1/2m_0)(\nabla[S])^2] = (i\hbar/2m_0)\nabla^2[S] \] : Quantum Single Particle Hamilton-Jacobi
\[ \partial_t [S] + [V + (1/2m_0)(\nabla[S])^2] = 0 \] : Classical Single Particle Hamilton-Jacobi

Thus, the quantum→classical limiting-case is:
\[ \hbar^2[\nabla^2[S]] \ll (\nabla^2[S])^2 \]
\[ \nabla^2[\phi] \ll (\nabla^2[\phi])^2 \]
\[ \nabla \cdot \nabla [S] \ll (\nabla [S])^2 \]
\[ \nabla \cdot [\Phi] \ll (\nabla [\Phi])^2 \]
\[ \hbar \cdot p \ll (p \cdot p) \]
\[ (p \lambda) \cdot p \ll (p \cdot p) \]

with
\[ P = (E/c, p) = (-\partial [S], -\nabla [S]) = (-\partial [\phi], -\nabla [\phi]) \]
\[ K = (\omega/c, k) = (-\partial [\phi], -\nabla [\phi]) \]

It is analogous to GR→SR in limit of low curvature (low mass), or SR→CM in limit of low velocity \( |v| \ll c \).
It still applies, but is now understood as the same type of limiting-case as these others.

*Note* The commonly seen form of \( (c \to \infty, \hbar \to 0) \) as limits are incorrect!
c and \( \hbar \) are universal constants – they never change.
If \( c \to \infty \), then photons (light-waves) would have infinite energy \( E = pc \). This is not true classically.
If \( \hbar \to 0 \), then photons (light-waves) would have zero energy \( E = h\omega \). This is not true classically.
Always better to write the SR Classical limit as \( |v| \ll c \), the QM Classical limit as \( \nabla^2[\phi] \ll (\nabla[\phi])^2 \)

Again, it is more natural to find a limiting-case of a more general system than to try to unite two separate theories which may or may not ultimately be compatible. From logic, there is always the possibility to have a paradox result from combination of arbitrary axioms, whereas deductions from a single true axiom will always give true results.

This page needs some work. Source was from Goldstein.
Conservation of Probability : Probability Current : Charge Current
Consider the following purely mathematical argument (based on Green's Vector Identity):
\[ \partial \cdot ( f \partial [g] - \partial [f] g ) = f \partial \cdot \partial [g] - \partial \cdot \partial [f] g \]
with (f) and (g) as SR Lorentz Scalar functions

Proof:
\[ \partial \cdot ( f \partial [g] - \partial [f] g ) = \partial \cdot ( f \partial [g] ) - \partial \cdot ( \partial [f] g ) = ( f \partial \cdot \partial [g] + \partial [f] \cdot \partial [g] ) - ( \partial \cdot \partial [f] g ) \]

We can also multiply this by a Lorentz Invariant Scalar Constant \( s \)
\[ s ( f \partial \cdot \partial [g] - \partial \cdot \partial [f] g ) = s \partial \cdot ( f \partial [g] - \partial [f] g ) = \partial \cdot s ( f \partial [g] - \partial [f] g ) \]

Ok, so we have the math that we need…

Now, on to the physics… Start with the Klein-Gordon Eqn.
\[ \partial \cdot \partial [g] + (m_o c/\hbar)^2 g = 0 \]

Let it act on SR Lorentz Invariant function \( g \)
\[ \partial \cdot \partial [g] + (m_o c/\hbar)^2 [g] = 0 \]
Then pre-multiply by \( f \)
\[ [f] \partial \cdot \partial [g] + [f] (m_o c/\hbar)^2 [g] = [f] 0 \]
\[ [f] \partial \cdot \partial [g] + (m_o c/\hbar)^2 [f][g] = 0 \]
Now, subtract the two equations
\[ \{[f] \partial \cdot \partial [g] + (m_o c/\hbar)^2 [f][g] = 0\} - \{ \partial \cdot \partial [f][g] + (m_o c/\hbar)^2 [f][g] = 0 \} \]
\[ [f] \partial \cdot \partial [g] - \partial \cdot \partial [f][g] = 0 \]

And as we noted from the mathematical Green's Vector identity at the start…
\[ [f] \partial \cdot \partial [g] - \partial \cdot \partial [f][g] = \partial \cdot ( f \partial [g] - \partial [f] g ) = 0 \]

Therefore,
\[ s \partial \cdot ( f \partial [g] - \partial [f] g ) = 0 \]
\[ \partial \cdot s ( f \partial [g] - \partial [f] g ) = 0 \]

Thus, there is a conserved current 4-Vector, \( J_{prob} = s ( f \partial [g] - \partial [f] g ) \), for which \( \partial \cdot J_{prob} = 0 \), and which also solves the Klein-Gordon equation.

Let's choose as before \( (\partial = -iK) \) with a plane wave function \( f = ae^{-i(K\cdotX)} = \psi \), and choose \( g = f^* = ae^{+i(K\cdotX)} = \psi^* \) as its complex conjugate.

At this point, I am going to choose \( s = (\hbar/2m_o) \), which is Lorentz Scalar Invariant, in order to make the probability have dimensionless units and be normalized to unity in the rest case.
The 4-Divergence of Probability \( \{ \partial \cdot J_{\text{prob}} = 0 \} \) by construction via Green's Vector Identity and the Klein-Gordon RQM Eqn.

The reason for \( s = (ih/2m_c) \) becomes more clear by examining our diagram: Start at the 4-Gradient and follow the arrows toward the 4-ProbabilityFlux You immediately see where the \((ih/m_c)\) factor comes from. The \( p_{\text{prob} \cdot \omega} \) is then a function of the \( \psi \)'s divided by 2.

Examine the temporal component, the Relativistic Probability Density
\[
\rho_{\text{prob}} = (ih/2m_c^2)(\psi^* \partial \psi) = (ih/2m_c^2)(\psi^* \partial \psi)
\]
Assume wave solution in following general form:
\[
\{ \psi = A f[k] e^{i(\omega t)} \}
\]
\[
\{ \psi^* = A^* f[k]^* e^{-i(\omega t)} \}
\]
then
\[
\{ \partial \psi = (i\omega)A f[k] e^{-i(\omega t)} = (i\omega)\psi \}
\]
\[
\{ \partial \psi^* = (i\omega)A^* f[k]^* e^{i(\omega t)} = (i\omega)\psi^* \}
\]
then
\[
\rho_{\text{prob}} = (ih/2m_c^2)(\psi^* \partial \psi - \partial \psi^* \psi)
\]
\[
\rho_{\text{prob}} = (ih/2m_c^2)((-i\omega)\psi^* \psi + (i\omega)\psi^* \psi)
\]
\[
\rho_{\text{prob}} = (ih/2m_c^2)(-2\iota \omega)\psi^* \psi
\]
\[
\rho_{\text{prob}} = (h\omega/m_c^2)(\psi^* \psi)
\]
\[
\rho_{\text{prob}} = (h\omega/m_c^2)(\psi^* \psi)
\]
Finally, multiply by charge \( q \) to get standard SR EM
4-CurrentDensity = 4-ChargeFlux = \( J = (cp) \psi = qJ_{\text{prob}} = q(c \rho_{\text{prob}} \cdot j_{\text{prob}}) \)

\[
\text{SR 4-Scalar} \quad (0,0)-\text{Tensor} \quad T_{\text{scalar}} = \text{Lorentz Scalar}
\]

\[
\text{SR 4-Vector} \quad (1,0)-\text{Tensor} \quad V^\mu = \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Tensor} \quad (2,0)-\text{Tensor} \quad T_{\mu\nu} = \text{T = (T^\mu\nu)}
\]

\[
\text{SR 4-Vector} \quad (0,1)-\text{Tensor} \quad \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Vector} \quad (1,0)-\text{Tensor} \quad \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Vector} \quad (0,1)-\text{Tensor} \quad \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Scalar} \quad (0,0)-\text{Tensor} \quad \text{T = (T^\mu )}
\]

\[
\text{SR 4-Vector} \quad (1,0)-\text{Tensor} \quad \text{V^\mu = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Vector} \quad (0,1)-\text{Tensor} \quad \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Vector} \quad (1,0)-\text{Tensor} \quad \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Scalar} \quad (0,0)-\text{Tensor} \quad \text{T = (T^\mu )}
\]

\[
\text{SR 4-Vector} \quad (1,0)-\text{Tensor} \quad \text{V^\mu = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Vector} \quad (0,1)-\text{Tensor} \quad \text{V = (V^\mu, V_\mu)}
\]

\[
\text{SR 4-Scalar} \quad (0,0)-\text{Tensor} \quad \text{T = (T^\mu )}
\]

\[
\text{SR 4-Vector} \quad (1,0)-\text{Tensor} \quad \text{V^\mu = (V^\mu, V_\mu)}
\]
4-Vector Quantum Probability

4-ProbabilityFlux, Klein-Gordon RQM Eqn
with Minimal Coupling

4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux
\[ J_{\text{prob}} = \left( \frac{1}{2m_c} \right) (\psi^* \partial[\psi] - \partial[\psi^*] \psi) \]
follow back past \( 1/\gamma \)

Start at \( J \)

If we include minimal coupling:
\[ \text{4-Divergence of Probability} \{ \partial \cdot J_{\text{prob}} = 0 \} \]

by construction via Green’s Vector Identity and the Klein-Gordon RQM Eqn.

Follow past Born Rule

\( \partial \cdot J_{\text{prob}} = (i\hbar/2m_c)(\psi^* \partial[\psi] - \partial[\psi^*] \psi) + (q/m_c)(\psi^* \psi)A \)

Now have the additional factor:
\[ + (q/m_c)(\psi^* \psi)A \]

Follow back past Born Rule \((\psi^* \psi)\)

An alternate way would be to take \( A \) to \( U \) via the direct route:
\[ + (c^2/q_m)(\psi^* \psi)A \]

Subtracting the Born term leads to a term like:
\[ \rho_{\text{min}} \rightarrow (\gamma)(\psi^* \psi) + (\gamma)(\phi_p/\phi_m)(\psi^* \psi) = (\gamma)(1 + \phi_p/\phi_m)(\psi^* \psi) \]

with potential due to particle \((\phi_p)\) typically much less than the potential due to the whole field \((\phi_m)\)

\((\phi_p) << (\phi_m)\)

Minimal Coupling \( \partial \equiv (\partial/c, -\partial) \)

4-Gradient

\[ \dot{J} = (\partial/c, -\partial) \]

J-Klein-Gordon

\[ \dot{J} \]

4-ProbabilityCurrentDensity, a.k.a. 4-ProbabilityFlux
\[ J_{\text{prob}} = \left( \frac{1}{2m_c} \right) (\psi^* \partial[\psi] - \partial[\psi^*] \psi) = (\rho_{\text{prob}}) \gamma (c, u) = (\gamma p_{\text{prob}})(c, u) = (\rho_{\text{prob}})(c, u) \]

with 4-Divergence of Probability \{ \partial \cdot J_{\text{prob}} = 0 \} by construction via Green’s Vector Identity and the Klein-Gordon RQM Eqn.
4-Vector Quantum Probability

**Newtonian Limit**

4-ProbabilityDensityCurrent $J_{\text{prob}} = (c \rho_{\text{prob}} \cdot j_{\text{prob}}) = (i \hbar/2m_0)(\psi^* \partial[\psi] - \partial[\psi^*] \psi) + (q/m_0)(\psi^* \psi)A$

Examine the temporal component:

$\rho_{\text{prob}} = (i \hbar/2m_0c^2)(\psi^* \partial[\psi] - \partial[\psi^*] \psi) + (q/m_0)(\psi^* \psi)(\phi/c^2)$

$\rho_{\text{prob}} \rightarrow (\gamma)(\psi^* \psi) + (\gamma)(q \phi_0/m_0c^2)(\psi^* \psi) = (\gamma)[1 + q \phi_0/E_0]\psi^* \psi$

Typically, the particle EM potential energy $(q\phi_0)$ is much less than the particle rest energy $(E_0)$, else it could generate new particles. So, take $(q\phi_0 << E_0)$, which gives the EM factor $(q\phi_0/E_0) \sim 0$

Now, taking the low-velocity limit $(\gamma \rightarrow 1)$, $\rho_{\text{prob}} = \gamma[1 + ~0](\psi^* \psi)$, $\rho_{\text{prob}} \rightarrow (\psi^* \psi) = (\rho_{\text{prob}} o)$ for $|v|<<c$

The Standard Born Probability Interpretation, $(\psi^* \psi) = (\rho_{\text{prob}} o)$, only applies in the low-potential-energy & low-velocity limit

This is why the {non-positive-definite} probabilities and {probabilities > 1} in the RQM Klein-Gordon equation gave physicists fits, and is the reason why one must regard the probabilities as charge conservation instead.

The original definition from SR is Continuity of Worldlines, $\partial \cdot J_{\text{prob}} = 0$, for which all is good and well in the RQM version.

The definition says there are no external sources or sinks of probability = conservation of probability.

The Born idea that $(\rho_{\text{prob}} o) \rightarrow \text{Sum}[(\psi^* \psi)] = 1$ is just the Low-Velocity QM limit.

Only the non-EM rest version $(\rho_{\text{prob}} o) = \text{Sum}[(\psi^* \psi)] = 1$ is true.

It is not a fundamental axiom, it is an emergent property which is valid only in the NRQM limit

We now multiply by charge $(q)$ to instead get a

4-"Charge"DensityCurrent $J = (c p_{\text{prob}} \cdot j_{\text{prob}}) = q J_{\text{prob}} = q(c p_{\text{prob}} \cdot j_{\text{prob}})$, which is the standard SR EM 4-CurrentDensity

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SR 4-Vector Study:
The QM Compton Effect

Compton Scattering Derivation: Compton Effect

\[ \mathbf{P}_p \cdot \mathbf{P} = (m_c)^2 \]

\[ \mathbf{P}_p \cdot \mathbf{P}_{photon} = h \mathbf{k}_p \mathbf{k} = h \mathbf{\hat{n}}_p \mathbf{k} = (h \omega_c c^2) (1 - \mathbf{\hat{n}}_p \cdot \mathbf{\hat{n}}) \]

\[ \mathbf{P}_p + \mathbf{P}_{mass} = \mathbf{P'}_{photon} + \mathbf{P'}_{mass} : 4\text{- Momentum Conservation in Photon-Mass Interaction} \]

\[ \mathbf{P}_p + \mathbf{P}_{mass} = \mathbf{P'}_{photon} + \mathbf{P'}_{mass} : \text{rearrange} \]

\[ (0 + 2 \mathbf{P}_{mass} \mathbf{P'}_{photon} - 2 \mathbf{P}\mathbf{P'}_{mass} - 2 \mathbf{P}\mathbf{P'}_{mass} + 2 \mathbf{P}\mathbf{P'}_{mass} + 2 \mathbf{P}_{mass} \mathbf{P'}_{photon} + 0) = (m_c c^2)^2 \]

\[ \mathbf{K} = \omega_c c^2 = (m_c c^2)^2 \]

\[ \Delta \lambda = (\lambda' - \lambda) = (h/m_c)(1 - \cos[\theta]) = \lambda_c (1 - \cos[\theta]) \]

The Compton Effect: Compton Scattering

with

\[ \lambda_c = \lambda_c / 2 \pi = (h/m_c) = \text{Reduced Compton Wavelength} \]

\[ \lambda_c = (h/m_c c) = \text{Compton Wavelength (not a rest-wavelength, but the wavelength of a photon with the energy equivalent to a massive particle of rest-mass m_c)} \]

Calculates the wavelength shift of a photon scattering from an electron (ignoring spin)

Proves that light does not have a “wave-only” description, photon 4-Momentum required

\[ E/\omega = \gamma_E/\gamma_c \]

\[ K_{\text{photon}} = (\omega_c)(1, \mathbf{\hat{n}}) = \text{null} \]

\[ \{\omega = 0\} \leftrightarrow \{\mathbf{K} \cdot U = 0\} \leftrightarrow \{\mathbf{K} \text{ is null}\} \]

Einstein de Broglie

\[ \mathbf{P} = m_c \mathbf{p} = (m_c, \mathbf{p}) = (h \mathbf{\hat{n}}) \]

\[ \mathbf{E} \cdot \omega \]

\[ \sum \mathbf{E} \]

\[ \text{4-Velocity U} = \gamma (c, \mathbf{u}) \]

\[ \mathbf{P'} = (m_c, \mathbf{p'}) = (E/c, \mathbf{p'}) \]

\[ \sum \mathbf{P} \]

\[ \text{4-TotalMomentum} \]

\[ \delta[\mathbf{R}'] = 2 \mathbf{C} \]

\[ \text{Minkowski Metric} \]

\[ \mathbf{V} \cdot \mathbf{V} = 4 \mathbf{V}_0^2 \]

\[ \text{Lorentz Scalar} \]

\[ \mathbf{V}_0 = \mathbf{V}_0^\perp = (\mathbf{V}_0^\perp)^2 = (\mathbf{v}_0^\perp)^2 \]

\[ \mathbf{V} = \mathbf{V}_0^\perp \]

\[ \mathbf{v}_0 = \mathbf{v}_0^\perp \]

\[ \mathbf{V}_0 = \mathbf{V}_0^\perp \]

\[ \mathbf{v}_0 = \mathbf{v}_0^\perp \]

\[ \mathbf{V} = \mathbf{V}_0^\perp \]

\[ \text{Lorentz Scalar} \]

\[ \mathbf{V}_0 = \mathbf{V}_0^\perp \]

\[ \mathbf{v}_0 = \mathbf{v}_0^\perp \]

\[ \mathbf{V} = \mathbf{V}_0^\perp \]
Aharonov-Bohm Effect

The EM 4-Vector Potential gives the Aharonov-Bohm Effect.

\[ \Phi_{\text{pot}} = -\frac{q}{\hbar} A \cdot X = -K \cdot X \]

or taking the differential...

\[ d\Phi_{\text{pot}} = -(q/\hbar)A \cdot dX \]

over a path...

\[ \Delta \Phi = \int_{\text{path}} d\Phi_{\text{pot}} \]

\[ \Delta \Phi = -(q/\hbar) \int_{\text{path}} A \cdot dX \]

\[ \Delta \Phi = -(q/\hbar) \int_{\text{path}} [(\phi/c)(cdt) - a \cdot dX] \]

\[ \Delta \Phi = -(q/\hbar) \int_{\text{path}} (\phi dt - a \cdot dX) \]

Note that both the Electric and Magnetic effects come out by using the 4-Vector notation.

Electric AB effect: \[ \Delta \Phi_{\text{Elec}} = -(q/\hbar) \int_{\text{path}} (\phi dt) \]

Magnetic AB effect: \[ \Delta \Phi_{\text{Mag}} = +(q/\hbar) \int_{\text{path}} (a \cdot dX) \]

Proves that the 4-Vector Potential \( A \) is more fundamental than \( e \) and \( b \) fields, which are just components of the Faraday EM Tensor.

\[ A \cdot dX = (q\phi dt - a \cdot dX) \]
SRQM 4-Vector Study:

The QM Josephson Junction Effect = SuperCurrent

EM 4-Vector Potential \( A = -(\hbar/q)\partial[\Phi_{pot}] \)

Josephson Effect

The EM 4-Vector Potential gives the Aharonov-Bohm Effect.

Phase \( \Phi_{pot} = -(q/\hbar)A \cdot X = -K_{pot} \cdot X \)

Rearrange the equation a bit:

\( -\hbar/q\Delta \Phi_{pot} = A \cdot \Delta X \)

\( d/dt[A] \cdot \Delta X = d/dt[-(\hbar/q)\Delta \Phi_{pot}] = d/dt[A] \cdot \Delta X + A \cdot d/dt[\Delta X] = d/dt[A] \cdot \Delta X + A \cdot U \)

Assume that \( d/dt[A] \cdot \Delta X \sim 0 \)

\( A = -(\hbar/q)K; |K| = (q/\hbar)A = (q/\hbar)(q/c,a) \)

Which explains Josephson Effect criteria:

\( \Delta X \sim 0: \text{small gap} \)

\( d/dt[A] \sim 0; \text{"critical current" & no voltage} \)

\( d/dt[A] \cdot \Delta X \sim \text{orthogonal}; ?? \)

Take the temporal part:

EM Scalar Potential \( \phi = -(\hbar/q)(\delta/\partial \Delta \Phi_{pot}) \); \( \omega = (q/\hbar)\phi \)

If the charge \((q)\) is a Cooper-electron-pair: \( q = -2e \)

Voltage \( V(t) = \phi(t) = (\hbar/2e)(\delta/\delta t)[\Delta \Phi_{pot}] \); AngFreq \( \omega = -2eV/\hbar \)

This is the superconducting phase evolution equation of the Josephson Effect

\( (h/2e) \) is defined to be the Magnetic Flux Quantum \( \Phi_o \)
SRQM Symmetries:
Hamilton-Jacobi vs Relativistic Action
Josephson vs Aharonov-Bohm
Differential (4-Vector) vs Integral (4-Scalar)

**SR Hamilton-Jacobi Equation**

\[ P = P + qA = P + Q = -\frac{\partial}{\partial t}[\Delta S_{\text{action}}] = -\frac{\partial}{\partial t}[\hbar \Delta \Phi_{\text{phase}}] \]
\[ = -\frac{\partial}{\partial t}[\hbar (\Delta \Phi_{\text{phase,dyn}} + \Delta \Phi_{\text{phase,potential}})] \]

**Integral Formats : 4-Scalars : Action**

\[ \Delta S_{\text{action}} = -\int_{\text{path}} P_T \cdot dX = -\int_{\text{path}} (P + qA) \cdot dX = -\int_{\text{path}} (P + Q) \cdot dX \]
\[ = \hbar \Delta \Phi_{\text{phase}} = \hbar (\Delta \Phi_{\text{phase,dyn}} + \Delta \Phi_{\text{phase,potential}}) \]

**SR Action Equation**

**Potential Part**

\[ \Delta S_{\text{act,pot}} = \hbar \Delta \Phi_{\text{phase,potential}} = -\int_{\text{path}} (qA) \cdot dX = -\int_{\text{path}} (Q) \cdot dX \]

**Potential Part**

\[ \Delta \Phi_{\text{phase,potential}} = -\frac{q}{\hbar} \int_{\text{path}} A \cdot dX \]

**Action**

\[ \text{Action} \]

**Minimal Coupling**

\[ P = (P + qA) = (P_T + qA) \]

**Inverse**

\[ \text{A Tensor Study of Physical 4-Vectors} \]

**Technically, the standard Josephson Junction uses just the temporal part \{ A = (\varphi/c, a) \} & Cooper-pair-electrons \{ q = -2e \} giving V(t) = \varphi = (h/2e)\partial/\partial t[\Delta \Phi_{\text{pot}}]. There should be a spatial part as well.**
SRQM Symmetries:
Schrödinger Relations ↔ Inv Temp

Cyclic Imaginary Time

SRQM Interpretation of QM

SciRealm.org
John B. Wilson
SciRealm@aol.com
http://scirealm.org/SRQM.pdf

4-Vector SRQM Interpretation of QM

Note that the temperature here is relativistically direction-specific, unlike in the classical use of temperature.

Trace[T^α] = n_μ T^α μ = T^α α = T

V·V = V_u V^μ = [(v_μ)^2 - v·v] = (v_0)^2

Lorentz Scalar
SRQM Symmetries: Wave-Particle

\( P = (E/c, p) = -\partial[S] = (\partial/c\partial[S], V[S]) \) 
{temporal component} \( E = -\partial/c\partial[S] = -\partial[S] \) 
{spatial component} \( p = V[S] \) 

**Note** This is the Action \( S_{\text{action,free}} \) for a free particle. Generally Action is for the 4-TotalMomentum \( P_t \) of a system.

**SR 4-Tensor**
(2,0)-Tensor \( T_{\mu
u} \)
(1,1)-Tensor \( T_{\mu} \), or \( T_{\nu} \)
(0,2)-Tensor \( T_{\nu
\mu} \)

**SR 4-Vector**
(1,0)-Tensor \( V^\mu = V = (\omega, v) \)
4-CoVector: One Form
(0,1)-Tensor \( V_\nu = (\omega, v) \)

**SR 4-Scalar**
(0,0)-Tensor \( S \) or \( S_\eta \) Lorentz Scalar

**See SR Wave Definition** for info on the Lorentz Scalar Invariant SR WavePhase. 
\( \{ K = (\omega/c, k) = (\omega/c, \omega \eta/v) \) 
{temporal component} \( \omega = -\partial/\partial[\phi] = -\partial[\phi] \) 
{spatial component} \( k = V[\phi] \) 

**Note** This is the Phase \( \phi \) for a single free plane-wave. Generally WavePhase is for the 4-TotalWaveVector \( K_\tau \) of a system.

**4-Position** \( R = (ct, r) \)

**4-Momentum** \( P = (mc, p) = (E/c, p) \)
\( P = -\partial[S] \)

**4-Gradient** \( \partial = (\partial/c, -\nabla) \rightarrow (\partial/c, -\partial_x, -\partial_y, -\partial_z) \)

**Treating motion like a particle** Moving particles have a 4-Velocity
**Treating motion like a wave** Moving waves have a 4-WaveVector

**SRQM Interpreation of QM**

**4-Vector SRQM Interpretation** of QM

**SR Physics**

**SR** (1,1)-Tensor \( T_{\nu} \), or \( T_{\nu} \)

**SR Action** for info on the Lorentz Scalar Invariant SR Action. 
See Hamilton-Jacobi Formulation of Mechanics.
SRQM Symmetries:
Relativistic Euler-Lagrange Equation

The Easy Derivation \((U=(d/d\tau)R) \rightarrow (\partial R=(d/d\tau)\partial U)\)

Note Similarity:
4-Velocity is Proper Time Derivative of 4-Position
\(U = (d/d\tau)R \quad [m/s] = [1/s][m]\)

Relativistic Euler-Lagrange Eqn
\(\partial_\alpha R = (d/d\tau)\partial_\alpha U \quad [1/m] = [1/s][m]\)

The differential form just inverses the dimensional units, so the placement of the \(R\) and \(U\) switch.

That is it: so simple! Much, much easier than how I was taught in Grad School.

To complete the process and create the Equations of Motion, one just applies the base form to a Lagrangian.

This can be:
a classical Lagrangian
a relativistic Lagrangian
a Lorentz scalar
a Lagrangian
a quantum Lagrangian

SR 4-Tensor
(2,0)-Tensor \(T^{\mu\nu}\)

SR 4-Vector
(1,0)-Tensor \(\mathbf{V}^\mu = \mathbf{V} = (v^0, \mathbf{v})\)

SR 4-CoVector: One-Form
(0,1)-Tensor \(V_\mu = (v_0, -\mathbf{v})\)

SR 4-Scalar
(0,0)-Tensor \(S = S_0\) Lorentz Scalar

SR 4-Vector
(1,1)-Tensor \(\mathbf{V}^{\mu\nu}\) or \(T^{\mu\nu}\)

SR 4-CoVector: One-Form
(0,1)-Tensor \(V_\mu\) or \(S_0\)

SR 4-Scalar
(0,0)-Tensor \(S = S_0\) Lorentz Scalar

\[\mathbf{V} \cdot \mathbf{V} = V_\mu V^\mu = (v^0)^2 - \mathbf{v} \cdot \mathbf{v} = (v^0)^2 = \text{Lorentz Scalar}\]
SRQM Symmetries:

Lorentz Transform Connection Map – Trace Identification
CPT, Big-Bang, (Matter-AntiMatter), Arrow(s)-of-Time

All Lorentz Transforms have Tensor Invariants: Determinant = ±1 and InnerProduct = 4. However, one can use the Tensor Invariant Trace to Identify CPT Symmetry & AntiMatter

Tr[ NM-Rotate ] = {0...+4} \( Tr[\text{NM-Identity}] = +4 \) \( Tr[\text{NM-Boost}] = +4...+\infty \)

Tr[ AM-Rotate ] = {0...-4} \( Tr[\text{AM-Identity}] = -4 \) \( Tr[\text{AM-Boost}] = -4...-\infty \)

Trace : Determinant
\( Tr = +4 \) : Det = +1 Proper
\( Tr = +2 \) : Det = -1 Improper
\( Tr = 0 \) : Det = +1 Proper
\( Tr = -2 \) : Det = -1 Improper
\( Tr = -4 \) : Det = +1 Proper

Trace : Determinant
\( Tr = +4 \) : Det = +1 Proper
\( Tr = +2 \) : Det = -1 Improper
\( Tr = 0 \) : Det = +1 Proper
\( Tr = -2 \) : Det = -1 Improper
\( Tr = -4 \) : Det = +1 Proper

Two interesting properties of (1,1)-Tensors, of which the Lorentz Transform is an example:

\( \text{Trace} = \sum \text{EigenValues} \) : Determinant = Product (\( \text{EV}_i \)) of EigenValues

As 4D Tensors, each Lorentz Transform has 4 EigenValues (EV’s). Create an Anti-Transform which has all EigenValue Tensor Invariants negated.

\( \sum [-\text{EV}_i] = -\sum \text{EV}_i \) : The Anti-Transform has negative Trace of the Transform.

\( \prod [-\text{EV}_i] = (-1)^4 \prod \text{EV}_i \) : The Anti-Transform has equal Determinant.

The Trace Invariant identifies a “Dual” Negative-Side for all Lorentz Transforms.
The $\hbar$ Connection

$P = hK$: Basic Einstein-de Broglie

$P+Q = P + Q$
$P+Q = hK_{dyn} + hK_{pot}$
$P+Q = h(K_{dyn} + K_{pot})$

Sum over $n$ particles: $P_T = \Sigma_n (P+Q), K_T = \Sigma_n (K_{dyn} + K_{pot})$

$P_T = hK_T$

$\{SR\text{ Hamilton-Jacobi}\} = h\{QM\text{ Complex Plane-Waves}\}$

The SR Hamilton-Jacobi Equation, and the QM idea of Complex Plane-Waves, are related by a simple constant ($\hbar$) relation.
Dimensionless Physical Objects

There are a number of dimensionless physical objects in SR that can be constructed from Physical 4-Vectors. Most are 4-Scalars, but there are few 4-Vector and 4-Tensors.

\[ \delta X = 4: \text{SpaceTime Dimension} \]
\[ \delta P = \eta_{x y}^{x y}: \text{The SR Minkowski Metric} \]
\[ T \times T = 1: \text{Lorentz Scalar Magnitude}^2 \]
\[ T \times S = 0: \text{Lorentz Scalar of 4-UnitTemporal with 4-UnitSpatial} \]
\[ S \times S = -1: \text{Lorentz Scalar Magnitude}^2 \]
\[ K \times X = \omega t - k \cdot x = -\phi_{\text{phase dyn}}: \text{Phase of an SR Wave used in SRQM wave functions} \]
\[ \theta = (E/c)/(k_{B}T_{o}): \text{4-Momentum with 4-InverseTemporalMomentum used in statistical mechanics particle distributions} \]
\[ F(x) = e^{-\left(\frac{E_{x}}{k_{B}T_{o}}\right)} \]
\[ \alpha = \left(\frac{1}{4\pi\epsilon_{0}}\right)(e^{2}/\hbar c) = (\mu_{B}/4\pi)(e^{2}/\hbar c): \text{Fine Structure Constant constructed from Lorentz 4-Scalars, which are themselves constructed from 4-Vectors via the Lorentz Scalar Product.} \]
\[ \text{ex. h} = (P \times X)(X \times X) ; \theta = (\eta_{x y}^{x y})(A \times X) \rightarrow e \text{ for electron}; c=(T \cdot U) \]
\[ \mu = (\partial \phi)(A \times X)/(J \cdot X) \]
\[ \{y^{2}\}: \text{Dirac Gamma Matrix ("4-Vector") \{4 component\}} \]
\[ \{\sigma^{n}\}: \text{Pauli Spin Matrix ("4-Vector") \{2 component\}} \]

\[ \text{Components are matrices of numbers, not just numbers} \]

---

**SRQM 4-Vector Study: Dimensionless Physical Objects**
SRQM: QM Axioms Unnecessary

QM Principles emerge from SR

QM is derivable from SR plus a few empirical facts – the “QM Axioms” aren’t necessary. These properties are either empirically measured or are emergent from SR properties...

3 “QM Axioms” are really just empirical constant relations between purely SR 4-Vectors:
- Particle-Wave Duality \[(P) = \hbar(K)\]
- Unitary Evolution \[[\partial] = (-i)K\]
- Operator Formalism \[(\partial) = -iK\]

2 “QM Axioms” are just the result of the Klein-Gordon Equation being a linear wave PDE:
- Hilbert Space Representation (<bra|,|ket>, wavefunctions, etc.) & The Principle of Superposition

3 “QM Axioms” are a property of the Minkowski Metric and the empirical fact of Operator Formalism
- The Canonical Commutation Relation
- The Heisenberg Uncertainty Principle (time-like-separated measurement exchange)
- The Pauli Exclusion Principle (space-like-separated particle exchange)

1 “QM Axiom” only holds in the NRQM case
- The Born QM Probability Interpretation – Not applicable to RQM, use Conservation of Worldlines instead

1 “QM Axiom” is really just another level of limiting cases, just like SR \(\rightarrow\) CM in limit of low velocity
- TheQM Correspondence Principle (QM \(\rightarrow\) CM in limit of \(\Delta^2[\phi] \ll (\nabla[\phi])^2\))
The SRQM interpretation fits fairly well with Carlo Rovelli's Relational QM interpretation:

Relational QM treats the state of a quantum system as being observer-dependent, that is, the QM State is the relation between the observer and the system. This is inspired by the key idea behind Special Relativity, that the details of an observation depend on the reference frame of the observer.

All systems are quantum systems: no artificial Copenhagen dichotomy between classical/macroscopic/conscious objects and quantum objects.

The QM States reflect the observers' information about a quantum system. Wave function "collapse" is informational – not physical. A particle always knows its complete properties. An observer has at best only partial information about the particle's properties.

No Spooky Action at a Distance. When a measurement is done locally on an entangled system, it is only the partial information about the distant entangled state that "changes/becomes-available-instantaneously". There is no superluminal signal. Measuring/physically-changing the local particle does not physically change the distant particle.

ex. Place two identical-except-for-color marbles into a box, close lid, and shake. Without looking, pick one marble at random and place it into another box. Send that box very far away. After receiving signal of the far box arrival at a distant point, open the near box and look at the marble. You now instantaneously know the far marble's color as well. The information did not come by signal. You already had the possibilities (partial knowledge). Looking at the near marble color simply reduced the partial knowledge of both marble's color to complete knowledge of both marbles' color. No signal was required, superluminal or otherwise.

ex. The quantum version of the same experiment uses the spin of entangled particles. When measured on the same axis, one will always be spin-up, the other will be spin-down. It is conceptually analogous. Entanglement is only about correlations of system that interacted in the past and are determined by conservation laws.
Einstein and Bohr can both be “right” about EPR:
Per Einstein: The QM State measured is not a “complete” description, just one observer’s point-of-view.
Per Bohr: The QM State measured is a “complete” description, it’s all that a single observer can get.

The point is that many observers can all see the “same” system, but see different facets of it. But a single measurement is the maximal information that a single observer can get without re-interacting with the system, which of course changes the system in general. Remember, the Heisenberg Uncertainty comes from non-zero commutation properties which require separate measurement arrangements. The properties of a particle are always there. Properties define particles. We as observers simply have only partial information about them.

Relativistic QM, being derived from SR, should be local – The low-velocity limit to QM may give unexpected anomalous results if taken out of context, or out of the applicable validity range, such as with velocity addition $v_{12} = v_1 + v_2$, where the correct formula should be the relativistic velocity composition $v_{12} = (v_1 + v_2)/(1 + v_1 v_2/c^2)$

These ideas lead to the conclusion that the wavefunction is just one observer’s state of information about a physical system, not the state of the physical system itself. The “collapse” of the wavefunction is simply the change in an observer’s information about a system brought about by a measurement or, in the case of EPR, an inference about the physical state.

EPR doesn’t break Heisenberg because measurements are made on different particles. The happy fact is that those particles interacted and became correlated in the causal past. The EPR-Bell experiments prove that it is possible to maintain those correlations over long distances. It does not prove superluminal (FTL) signaling.
SRQM Interpretation:
Range-of-Validity Facts & Fallacies

We should not be surprised by the “quantum” probabilities being correct instead of “classical” in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Examples

*Classical Physics as the limit of \( \hbar \to 0 \) {Fallacy}:
\( \hbar \) is a Lorentz Scalar Invariant and Fundamental Physical Constant. It never becomes 0. {Fact}

*The classical commutator being zero \( [p^k, x^j] = 0 \) {Fallacy}:
\( [P^\mu, X^\nu] = i\hbar \eta^{\mu \nu}; [p^k, x^j] = [E/c, ct] = [E, t] = i\hbar; \) Again, it never becomes 0 {Fact}

*Using Maxwell-Boltzmann (distinguishable) statistics for counting probabilities of (indistinguishable) quantum states {Fallacy}:
Must use Fermi-Dirac statistics for Fermions: Spin=(n+1/2); Bose-Einstein statistics for Bosons: Spin=(n) {Fact}

*Using sums of classical probabilities on quantum states {Fallacy}:
Must use sums of quantum probability-amplitudes {Fact}

*Ignoring phase cross-terms and interference effects in calculations {Fallacy}:
Quantum systems and entanglement require phase cross-terms {Fact}

*Assuming that one can simultaneously “measure” non-commuting properties at a single spacetime event {Fallacy}:
Particle properties always exist. However, non-commuting ones require separate measurement arrangements to get information about the properties. The required measurement arrangements on a single particle/worldline are at best sequential events, where the temporal order plays a role; {Fact}
However, EPR allows one to “infer (not measure)” the other property of a particle by the separate measurement of an entangled partner. {Fact}
This does not break Heisenberg Uncertainty, which is about the order of operations (measurement events) on a single worldline. {Fact}
In the entangled case, both/all of the entangled partners share common past-causal entanglement events, typically due to a conservation law. {Fact}
Information is not transmitted at FTL. The particles simply carried their normal respective “correlated” properties (no hidden variables) with them. {Fact}

*Assuming that QM is a generalization of CM, or that classical probabilities apply to QM {Fallacy}:
CM is a limiting-case of QM for when changes in a system by a few quanta have a negligible effect on the whole/overall system. {Fact}
SRQM Interpretation:
Quantum Information

We should not be surprised by the "quantum" probabilities being correct instead of "classical" in the EPR and Bell Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

{from Wikipedia}

No-Communication Theorem/No-Signaling:
A no-go theorem from quantum information theory which states that, during measurement of an entangled quantum state, it is not possible for one observer, by making a measurement of a subsystem of the total state, to communicate information to another observer. The theorem shows that quantum correlations do not lead to what could be referred to as "spooky communication at a distance". SRQM: There is no FTL signaling/communication.

No-Teleportation Theorem:
The no-teleportation theorem stems from the Heisenberg uncertainty principle and the EPR paradox: although a qubit $|\psi\rangle$ can be imagined to be a specific direction on the Bloch sphere, that direction cannot be measured precisely, for the general case $|\psi\rangle$. The no-teleportation theorem is implied by the no-cloning theorem.

SRQM: Ket states are informational, not physical.

No-Cloning Theorem:
In physics, the no-cloning theorem states that it is impossible to create an identical copy of an arbitrary unknown quantum state. This no-go theorem of quantum mechanics proves the impossibility of a simple perfect non-disturbing measurement scheme. The no-cloning theorem is normally stated and proven for pure states; the no-broadcast theorem generalizes this result to mixed states. SRQM: Measurements are arrangements of particles that interact with a subject particle.

No-Broadcast Theorem:
Since quantum states cannot be copied in general, they cannot be broadcast. Here, the word "broadcast" is used in the sense of conveying the state to two or more recipients. For multiple recipients to each receive the state, there must be, in some sense, a way of duplicating the state. The no-broadcast theorem generalizes the no-cloning theorem for mixed states. The no-cloning theorem says that it is impossible to create two copies of an unknown state given a single copy of the state.

SRQM: Conservation of worldlines.

No-Deleting Theorem:
In physics, the no-deleting theorem of quantum information theory is a no-go theorem which states that, in general, given two copies of some arbitrary quantum state, it is impossible to delete one of the copies. It is a time-reversed dual to the no-cloning theorem, which states that arbitrary states cannot be copied.

SRQM: Conservation of worldlines.

No-Hiding Theorem:
The no-hiding theorem is the ultimate proof of the conservation of quantum information. The importance of the no-hiding theorem is that it proves the conservation of wave function in quantum theory.

SRQM: Conservation of worldlines. RQM wavefunctions are Lorentz 4-Scalars (spin=0), 4-Spinors (spin=1/2), 4-Vectors (spin=1), all of which are Lorentz Invariant.
SRQM Interpretation: Quantum Information

We should not be surprised by the “quantum” probabilities being correct instead of “classical” probabilities in the EPR/Bell-Inequalities experiments. Classical thinking (in both CM and QM) has a number of fallacies when it is mistakenly applied outside of its range-of-validity.

Quantum information (qubits) differs strongly from classical information, epitomized by the bit, in many striking and unfamiliar ways. Among these are the following:

A unit of quantum information is the qubit. Unlike classical digital states (which are discrete), a qubit is continuous-valued, describable by a direction on the Bloch sphere. Despite being continuously valued in this way, a qubit is the smallest possible unit of quantum information, as despite the qubit state being continuously-valued, it is impossible to measure the value precisely.

A qubit cannot be (wholly) converted into classical bits; that is, it cannot be "read". This is the no-teleportation theorem.

Despite the awkwardly-named no-teleportation theorem, qubits can be moved from one physical particle to another, by means of quantum teleportation. That is, qubits can be transported, independently of the underlying physical particle. SRQM: Ket states are informational, not physical.

An arbitrary qubit can neither be copied, nor destroyed. This is the content of the no-cloning theorem and the no-deleting theorem. SRQM: Conservation of worldlines.

Although a single qubit can be transported from place to place (e.g. via quantum teleportation), it cannot be delivered to multiple recipients; this is the no-broadcast theorem, and is essentially implied by the no-cloning theorem. SRQM: Conservation of worldlines.

Qubits can be changed, by applying linear transformations or quantum gates to them, to alter their state. While classical gates correspond to the familiar operations of Boolean logic, quantum gates are physical unitary operators that in the case of qubits correspond to rotations of the Bloch sphere.

Due to the volatility of quantum systems and the impossibility of copying states, the storing of quantum information is much more difficult than storing classical information. Nevertheless, with the use of quantum error correction quantum information can still be reliably stored in principle. The existence of quantum error correcting codes has also led to the possibility of fault tolerant quantum computation.

Classical bits can be encoded into and subsequently retrieved from configurations of qubits, through the use of quantum gates. By itself, a single qubit can convey no more than one bit of accessible classical information about its preparation. This is Holevo’s theorem. However, in superdense coding a sender, by acting on one of two entangled qubits, can convey two bits of accessible information about their joint state to a receiver.

Quantum information can be moved about, in a quantum channel, analogous to the concept of a classical communications channel. Quantum messages have a finite size, measured in qubits; quantum channels have a finite channel capacity, measured in qubits per second.
Minkowski still applies in local GR

QM is a local phenomenon

The QM Schrodinger Equation is not fundamental. It is just the low-energy limiting-case of the RQM Klein-Gordon Equation. All of the standard QM Axioms are shown to be empirically measured constants or emergent properties of SR. It is a bad approach to start with NRQM as an axiomatic starting point and try to generalize it to RQM, in the same way that one cannot start with CM and derive SR. Since QM *can* be derived from SR, this partially explains the difficulty of uniting QM with GR: QM is not a “separate formalism” outside of SR that can be used to “quantize” just anything...

Strictly speaking, the use of the Minkowski space to describe physical systems over finite distances applies only in the SR limit of systems without significant gravitation. In the case of significant gravitation, SpaceTime becomes curved and one must abandon SR in favor of the full theory of GR.

Nevertheless, even in such cases, based on the GR Equivalence Principle, Minkowski space is still a good description in a local region surrounding any point (barring gravitational singularities). More abstractly, we say that in the presence of gravity, SpaceTime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski Space. Thus, the structure of Minkowski Space is still essential in the description of GR.

So, even in GR, at the local level things are considered to be Minkowskian:
i.e. SR → QM “lives inside the surface” of this local SpaceTime, GR curves the surface.
SRQM Interpretation: Main Result

QM is derivable from SR!

Hopefully, this interpretation will shed light on why Quantum Gravity has been so elusive. Basically, QM rules of “quantization” don’t apply to GR. They are a manifestation-of/derivation-from SR. Relativity *is* the “Theory of Measurement” that QM has been looking for.

This would explain why no one has been able to produce a successful theory of “Quantum Gravity”, and why there have been no violations of Lorentz Invariance, CPT, or the Equivalence Principle.

If quantum effects “live” in Minkowski SpaceTime with SR, then GR curvature effects are at a level above the RQM description, and two levels above standard QM. SR+QM are “in” SpaceTime, GR is the “shape” of SpaceTime...

Thus, this SRQM Treatise explains the following:

- Why GR works so well in it’s realm of applicability {large scale systems}.
- Why QM works so well in it’s realm of applicability {micro scale systems and certain macroscopic systems}.
  - i.e. The tangent space to any point in GR curvature is locally Minkowskian, and thus QM is typically found in small local volumes...
- Why RQM explains more stuff than QM without SR {because QM is just an approximation: the low-velocity limiting-case of RQM}.
- Why all attempts to "quantize gravity" have failed {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}.
- Why all attempts to modify GR keep conflicting with experimental data {because GR is apparently fundamental – passed all tests to-date}.
- Why QM works perfectly well with SR as RQM but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}.
- How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, and give the SRQM Interpretation of Quantum Mechanics.

SRQM: Special Relativistic Quantum Measurement, Special Relativistic Quantum Mechanics

SRQM: A treatise of SR→QM by John B. Wilson (SciRealm@aol.com)
SRQM Chart:
Special Relativity $\rightarrow$ Quantum Mechanics
SR$\rightarrow$QM Interpretation Simplified

SRQM: The [SR$\rightarrow$QM] Interpretation of Quantum Mechanics

Special Relativity (SR) Axioms: Invariant Interval + LightSpeed (c) as Universal Physical Constant lead to SR, although technically SR is itself the Minkowski-SpaceTime low-curvature:"flat" limiting-case of GR.

\{c,\tau,m_0,\hbar,i\} = \{c:\text{SpeedOfLight}, \tau:\text{ProperTime}, m_0:\text{RestMass}, \hbar:\text{Dirac/PlanckReducedConstant}(\hbar=h/2\pi), i:\text{ImaginaryNumber}\sqrt{-1}\}:

are all Empirically Measured SR Lorentz Invariant Physical Constants and/or Mathematical Constants

4-Vector SRQM Interpretation of QM

SR + Empirically Measured Physical Constants lead to RQM via the Klein-Gordon Quantum Eqn, and thence to QM via the low-velocity limit \{ |v| << c \}, giving the Schrödinger Eqn. This fundamental KG Relation also leads to the other Quantum Wave Equations:

spin=0 boson field = 4-Scalar:

\{ |v| = c : m_0 = 0 \}  
Free Scalar Wave (Higgs)

spin=1/2 fermion field = 4-Spinor:

\{ 0 <= |v| < c : m_0 > 0 \}  
Klein-Gordon

spin=1 boson field = 4-Vector:

\{ 0 <= |v| << c : m_0 > 0 \}  
Schrödinger (regular QM)

Weyl

Maxwell (EM photonic)

Proca

SRQM: A treatise of SR$\rightarrow$QM by John B. Wilson (SciRealm@aol.com)
SR → QM
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR & QM Invariant Waves
SR → QM Klein-Gordon Relativistic Quantum Particle in EM Potential d’Alembertian Wave Equation
\[ \frac{\partial^2}{\partial t^2} \psi - \nabla^2 \psi = m^2 \psi \]
Limit: \( |V| < c \)
(with potential \( V = q \phi + (m_c)^2 \))
= Schrödinger QM Equation (EM potential)
= Lorentz potential \( A = (\phi/c, a) \)

4-Vector SRQM Interpretation of QM
4-Gradient = Alteration of SR <Events>
SR SpaceTime Dimension=4
SR Lorentz Transforms
SR Action → 4-Momentum
SR Phase → 4-WaveVector
SR Proper Time Derivative
SR & QM Invariant Waves

**[ SR → QM ]**

4-Gradient \( \partial^\mu \partial^\nu = (\partial/c, -\nabla) = -iK \)
\[ \delta [R^\nu] = \delta^\nu_\nu \]
Minkowski Metric
\[ \delta R = 4 \]
Space Time Dim

4-WaveVector K^\nu
\[ K = (\omega/c, k) = (\omega/c)^2 U = P/h \]

4-Momentum P^\mu
\[ P = (mc, p) = (E/c, p) = m_0 U \]

4-EMVectorPotential A^\mu
\[ A = (\phi/c, a) \]

4-PotentialMomentum Q^\mu
\[ Q = (V/c, q) = q(\phi/c, a) \]

4-TotMomentum Conservation
\[ P_t = (E/c, p_t) = (E + q\phi/c, p + q\mathbf{a}) = P + Q \]

SR 4-Tensor
(2,0)-Tensor T^{\mu\nu}
(1,1)-Tensor T^\nu_\nu
4-CoVector:OneForm
(0,1)-Tensor V^\nu = (v^\nu, \mathbf{v})
SR 4-Vector
(1,0)-Tensor V = (v^\nu, \mathbf{v})
SR 4-Scalar
(0,0)-Tensor S or \( S_0 \)
Lorentz Scalar

Existing SR Rules
Quantum Principles

SR QM Physics
A Tensor Study of Physical 4-Vectors
See also:
http://scirealm.org/SRQM.html (alt discussion)
http://scirealm.org/SRQM-RoadMap.html (main SRQM website)
http://scirealm.org/4Vectors.html (4-Vector study)
http://scirealm.org/SRQM-Tensors.html (Tensor & 4-Vector Calculator)
http://scirealm.org/SciCalculator.html (Complex-capable RPN Calculator)

or Google “SRQM”

http://scirealm.org/SRQM.pdf (this document: most current ver. at SciRealm.org)
The 4-Vector SRQM Interpretation

QM is derivable from SR!

The SRQM or [SR→QM] Interpretation of Quantum Mechanics
A Tensor Study of Physical 4-Vectors

quantum relativity

SRQM = SciRealm QM?  A happy coincidence… :)  Ambigrams

SR 4-Tensor
(2,0)-Tensor $T_{\mu}^{\nu}$
(1,1)-Tensor $T^\mu_\nu$, or $T^\nu_\mu$
(0,2)-Tensor $T_{\mu\nu}$

SR 4-Vector
(1,0)-Tensor $V^\mu = V = (v^0, v)$

SR 4-CoVector: OneForm
(0,1)-Tensor $V_\mu = (v_0, -v)$

SR 4-Scalar
(0,0)-Tensor $S$ or $S_0$
Lorentz Scalar

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