Relativistic Velocity Composition Derived using 4-Vectors

4-Vector (generic) $\mathbf{V} = V^{\mu} = $ (temporal component, spatial component)
4-Velocity $\mathbf{U} = U^{\mu} = \gamma(c,\mathbf{u}) = c\mathbf{T}$ [m/s] $\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$ [1] : $\boldsymbol{\beta} = \mathbf{u}/c$ [1]
4-"Unit"Temporal $\mathbf{T} = T^{\mu} = \gamma(1,\boldsymbol{\beta}) = \mathbf{U}/c$ [1] The 4-"Unit"Temporal is dimensionless

with

Speed of Light c
3-velocity $\mathbf{u} = (u_x,u_y,u_z)$
Relativistic beta $\boldsymbol{\beta} = \mathbf{u}/c = (\beta_x,\beta_y,\beta_z)$
Relativistic gamma $\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]} = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]}$

Lorentz Scalar Products give invariant magnitudes
$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = c^2$
$\mathbf{T}\cdot\mathbf{T} = \gamma(1,\boldsymbol{\beta})\cdot\gamma(1,\boldsymbol{\beta}) = \gamma^2(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}) = 1$

However, one can also take the Lorentz Scalar Products of the 4-Vectors of different particles.
$\mathbf{U}_1\cdot\mathbf{U}_2 = \gamma_1(c,\mathbf{u}_1)\cdot\gamma_2(c,\mathbf{u}_2) = \gamma_1\gamma_2(c^2 - \mathbf{u}_1\cdot\mathbf{u}_2) = \gamma_2(c^2)$ {when $\mathbf{u}_1 = \mathbf{0}$} $= \gamma_1(c^2)$ {when $\mathbf{u}_2 = \mathbf{0}$} $= \gamma_{12}(c^2) = \gamma_{rel}(c^2)$
$\gamma_1\gamma_2(c^2 - \mathbf{u}_1\cdot\mathbf{u}_2) = \gamma_{rel}(c^2)$
When measuring two different 4-Velocities, the Lorentz Scalar Product is proportional to their relative Lorentz gamma factor $\gamma_{12} = \gamma_{rel}$.

These are all Lorentz Scalar Invariants. All observers must agree on their values.

$\mathbf{U}_1\cdot\mathbf{U}_2 = \gamma_{12}(c^2) = \gamma_{rel}(c^2)$	$\mathbf{U}\cdot\mathbf{U} = c^2$
$\mathbf{T}_1\cdot\mathbf{T}_2 = \gamma_{12} = \gamma_{rel}$	$\mathbf{T}\cdot\mathbf{T} = 1$

One can obtain the relativistic "addition" of velocities using these relations:
$\mathbf{U}_1\cdot\mathbf{U}_2 = \gamma_1\gamma_2(c^2 - \mathbf{u}_1\cdot\mathbf{u}_2) = \gamma_{rel}c^2$
$\mathbf{T}_1\cdot\mathbf{T}_2 = \gamma_1\gamma_2(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2) = \gamma_{rel}$
$\gamma_1\gamma_2(c^2 - \mathbf{u}_1\cdot\mathbf{u}_2) = \gamma_{rel}c^2$
$\gamma_1\gamma_2(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2) = \gamma_{rel}$
$\gamma_1^{-2}\gamma_2^{-2}(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^{-2} = \gamma_{rel}^{-2}$
$(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_1)(1-\boldsymbol{\beta}_2\cdot\boldsymbol{\beta}_2)(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^{-2} = (1-\boldsymbol{\beta}_{rel}\cdot\boldsymbol{\beta}_{rel})$
$(\boldsymbol{\beta}_{rel}\cdot\boldsymbol{\beta}_{rel}) = 1-(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_1)(1-\boldsymbol{\beta}_2\cdot\boldsymbol{\beta}_2)(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^{-2}$
$(\boldsymbol{\beta}_{rel})^2 = [(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2-(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_1)(1-\boldsymbol{\beta}_2\cdot\boldsymbol{\beta}_2)](1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^{-2}$
$(\boldsymbol{\beta}_{rel})^2 = [(\boldsymbol{\beta}_1)^2-2(\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)+(\boldsymbol{\beta}_2)^2](1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^{-2}$
$(\boldsymbol{\beta}_{rel})^2 = [(\boldsymbol{\beta}_1)-(\boldsymbol{\beta}_2)]^2(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^{-2}$
$\boldsymbol{\beta}_{rel} = [\boldsymbol{\beta}_1-\boldsymbol{\beta}_2]/(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)$: $\boldsymbol{\beta}_{additive} = [\boldsymbol{\beta}_1+\boldsymbol{\beta}_2]/(1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)$ by letting $\boldsymbol{\beta}_2 \to -\boldsymbol{\beta}_2$
$\mathbf{u}_{rel} = [\mathbf{u}_1-\mathbf{u}_2]/(1-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)$: $\mathbf{u}_{additive} = [\mathbf{u}_1+\mathbf{u}_2]/(1+\mathbf{u}_1\cdot\mathbf{u}_2/c^2)$ by letting $\mathbf{u}_2 \to -\mathbf{u}_2$
The relativistic composition of velocities law

$\boldsymbol{\beta}_{rel\ 12} = [\boldsymbol{\beta}_{10}-\boldsymbol{\beta}_{20}]/(1-\boldsymbol{\beta}_{10}\cdot\boldsymbol{\beta}_{20})$ with 0 considered the origin frame
by letting $\boldsymbol{\beta}_2 \to -\boldsymbol{\beta}_2$
$\boldsymbol{\beta}_{additive\ 12} = [\boldsymbol{\beta}_{10}+\boldsymbol{\beta}_{02}]/(1+\boldsymbol{\beta}_{10}\cdot\boldsymbol{\beta}_{02})$ with 2 considered the origin frame

The definition of the Lorentz Gamma Factor is contained in this.
$\mathbf{U}\cdot\mathbf{U} = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = c^2$
Solving for gamma factor:
$\gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = (c^2)$
$\gamma^2 = (c^2)/(c^2 - \mathbf{u}\cdot\mathbf{u})$
$\gamma^2 = 1/(1 - \mathbf{u}\cdot\mathbf{u}/c^2) = 1/(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})$
$\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$

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$\mathbf{u}_{\mathbf{additive}} = [\mathbf{u_1}+\mathbf{u_2}]/(1+\mathbf{u_1}\cdot\mathbf{u_2}/c^2)$
Taking the classical limiting-case of $(\mathbf{u_1}\cdot\mathbf{u_2}) << (c^2)$
$\mathbf{u}_{\mathbf{additive}} = [\mathbf{u_1}+\mathbf{u_2}]/(1 + \sim 0)$
$\mathbf{u}_{\mathbf{additive}} = [\mathbf{u_1}+\mathbf{u_2}]_{classical}$

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4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) = c\mathbf{T}$	[m/s]	$\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$ [1] : $\boldsymbol{\beta} = \mathbf{u}/c$ [1]
4-"Unit"Temporal	$\mathbf{T} = T^\mu = \gamma(1,\boldsymbol{\beta}) = \mathbf{U}/c$	[1]	The 4-"Unit"Temporal is dimensionless
4-ProbabilityFlux	$\mathbf{J} = J^\mu = (\rho c,\mathbf{j}=\rho\mathbf{u}) = \rho_o\mathbf{U}$	[#m/s]	

$\mathbf{U_1}\cdot\mathbf{U_2} = \gamma_{12}(c^2) = \gamma_{rel}(c^2)$
$\rho_{o1}\mathbf{U_1}\cdot\rho_{o2}\mathbf{U_2} = \rho_{o1}\ \rho_{o2}\ \gamma_{rel}(c^2)$
$\mathbf{J_1}\cdot\mathbf{J_2} = \rho_{o1}\ \rho_{o2}\ \gamma_{rel}(c^2)$

$\gamma_1\gamma_2(1-\boldsymbol{\beta_1}\cdot\boldsymbol{\beta_2}) = \gamma_{rel}$

$\rho_{o1}\ \rho_{o2}\ \gamma_1\gamma_2(1-\boldsymbol{\beta_1}\cdot\boldsymbol{\beta_2}) = \rho_{o1}\ \rho_{o2}\ \gamma_{rel}$

$\rho_1\ \rho_2\ (1-\boldsymbol{\beta_1}\cdot\boldsymbol{\beta_2}) = \rho_{o1}\ \rho_{o2}\ \gamma_{rel}$

4-Vector (generic)	$\mathbf{V} = V^\mu = (\text{temporal component}, \textbf{spatial component})$			
4-Momentum	$\mathbf{P} = P^\mu = (E/c, \mathbf{p}) = (mc, \mathbf{p}) = m_o\mathbf{U}$	[kg·m/s = N·s]		
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c, \mathbf{u}) = c\mathbf{T}$	[m/s]	$\gamma = 1/\sqrt{[1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}]}$ [1] : $\boldsymbol{\beta} = \mathbf{u}/c$ [1]	
4-"Unit"Temporal	$\mathbf{T} = T^\mu = \gamma(1, \boldsymbol{\beta}) = \mathbf{U}/c$	[1]	The 4-"Unit"Temporal is dimensionless	

$\mathbf{T_1}\cdot\mathbf{T_2} = \gamma_{12} = \gamma_{rel}$	:4-Vector form, Particle 1 and Particle 2 as seen from the frame of **T**
$T_1^\mu \cdot T_2^\nu = \gamma_{12} = \gamma_{rel}$	:Tensor form
$T_1^\mu\, \eta_{\mu\nu}\, T_2^\nu = \gamma_{12} = \gamma_{rel}$	:4-Vector Lorentz Scalar Product = Minkowski Metric
$(\, \Lambda^\mu{}_\rho T'_1{}^\rho\,)\, \eta_{\mu\nu}\, (\, \Lambda^\nu{}_\sigma\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Identical Lorentz Transform can be applied to both frames, T' → T
$(\, \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\,)\, \eta_{\mu\nu}\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:These are all just commutative numbers in tensor format
$(\, \Lambda^{-1}{}_\rho{}^\mu\, \Lambda^\nu{}_\sigma\,)\, \eta_{\mu\nu}\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Swapping index positions of Lorentz $\Lambda^\mu{}_\rho$ gives the Inverse Lorentz Transform $\Lambda^{-1}{}_\rho{}^\mu$
$(\, \Lambda^{-1}{}_{\rho\nu}\, \Lambda^\nu{}_\sigma\,)\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Minkowski can lower an index
$(\, \eta_{\rho\alpha}\;\; \Lambda^{-1\alpha}{}_\nu\,)(\, \Lambda^\nu{}_\sigma\,)\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Minkowski can raise an index
$\eta_{\rho\alpha}\;\; (\, \Lambda^{-1\alpha}{}_\nu\, \Lambda^\nu{}_\sigma\,)\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Commutative
$\eta_{\rho\alpha}\;\; (\, \delta^\alpha{}_\sigma\,)\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:$(\Lambda^{-1})^\beta{}_\mu\, \Lambda^\mu{}_\nu = \delta^\beta{}_\nu$
$\eta_{\rho\sigma}\, (\, T'_1{}^\rho\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Kronecker Delta renames an index
$(\, T'_1{}^\rho\, \eta_{\rho\sigma}\, T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:Commutative
$(\, T'_1{}^\rho \cdot T'_2{}^\sigma\,) = \gamma_{12} = \gamma_{rel}$	:4-Vector Lorentz Scalar Product = Minkowski Metric
$T'_1{}^\rho \cdot T'_2{}^\sigma = \gamma_{12} = \gamma_{rel}$	:Tensor form
$\mathbf{T'_1}\cdot\mathbf{T'_2} = \gamma_{12} = \gamma_{rel}$	:4-Vector form, Particle 1 and Particle 2 as seen from frame **T'**

$(\Lambda^{-1})^\beta{}_\mu\, \Lambda^\mu{}_\nu = \delta^\beta{}_\nu$
$(\Lambda)^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu$

The point of all this is that γ_{rel} is a Lorentz Scalar Invariant, which is actually clear from the very first line, $\mathbf{T_1}\cdot\mathbf{T_2} = \gamma_{12} = \gamma_{rel}$
The Lorentz Scalar Product of any two SR 4-Vectors is a Lorentz Invariant Scalar.

The Relativistic Gamma γ_{rel} between any two particles/waves/events remains an invariant as seen from any frame.
To be clear, this means the difference between two specified frames remains invariant, as measured from any other frame.

$\mathbf{P_1}\cdot\mathbf{U_2} = m_{o1}\mathbf{U_1}\cdot\mathbf{U_2} = m_{o1}c^2\mathbf{T_1}\cdot\mathbf{T_2} = \gamma_{rel}\, E_{o1} = \gamma_{12}\, E_{o1} = E_{12}$ = Relativistic Energy of Particle 1 as measured by observer in frame 2.
Note, this is not the Rest Energy E_o of the particle, unless the frame 2 observer is at rest wrt. frame 1.

E_o is an invariant, the Energy as measured in the RestFrame.
$E = \gamma E_o$ is not an invariant. The gamma is unspecified, and therefore the E is unspecified. Different in different frames.
$E_{\text{frame wrt. o}} = \gamma_{rel}\, E_o$ is an invariant. The gamma level is specified to a particular frame from the rest frame.

$\gamma_{rel} = \gamma_1\gamma_2(c^2 - \mathbf{u_1}\cdot\mathbf{u_2}) = \mathbf{U_1}\cdot\mathbf{U_2}$
γ_1 is not invariant, γ_2 is not invariant, but the combo $\gamma_{rel} = \gamma_1\gamma_2(c^2 - \mathbf{u_1}\cdot\mathbf{u_2})$ is invariant.

$\mathbf{P} = \hbar\mathbf{K} = (E/c, \mathbf{p}) = \hbar(\omega/c, \mathbf{k})$
Temporal part: $E = \hbar\omega$: Einstein Photoelectric
Spatial part: $\mathbf{p} = \hbar\mathbf{k}$: de Broglie Wave-Particle

E is not invariant, ω is not invariant, but the combo $E/\omega = \hbar$ is invariant.
p is not invariant, **k** is not invariant, but the combo $|\mathbf{p}| / |\mathbf{k}| = \hbar$ is invariant.

$E_{\text{frame wrt. o}} = \gamma_{rel}\, E_o$ is an invariant.
Typically, we just use the frame that is the rest frame. Then $E_{\text{rest-frame}} = (1)\, E_o$ is an invariant.