Quantum Mechanics is *derivable* from Special Relativity. Using the 4-Vectors of SR, plus a few empirical facts, one can derive the principles of QM. Therefore, the standard QM “axioms” are actually emergent from the rules of SR. Hence, [SR→QM]

The 4D <Time Space>-splitting into 1D Temporal + 3D Spatial components plays an integral role in understanding these relations.  

Key Words: Special Relativity (SR), Quantum Mechanics (QM), Quantum-Classical relation, Quantum Emergence, SRQM, SR→QM

Introduction:
There are currently two main foundational bodies of physics theory used to model reality as we know it:

Relativity: (general (GR) ~ astrophysical scale + Special (SR) ~ connects all scales) & Quantum Mechanics (QM) ~ atomic scale.

1) Relativity uses tensor mathematics to calculate the properties of 4D Spacetime. GR is used for physical systems in which the mass of objects is very large. Essentially this means: “Mass tells spacetime how to curve, spacetime curvature tells mass how to move”. SR is a “special” limiting-case of GR for low curvature ~ low mass. SR shows that certain physical properties once thought totally separate are actually dual to one another: (time:space), (energy:momentum), etc. Tensor mathematics has the concept that tensor component-values can relativistically vary such that measurements between events are independent of arbitrarily-imposed coordinate-systems, e.g. consider 2D pictures of a 3D object from different angles & distances. The pics look different, yet are all the same object. Integral to physical relativistic tensors is that 4D <Time Space> = 1D Temporal (t) + 3D Spatial (x,y,z) entities have specific ways of splitting into their various natural, measurable-components, depending on the type of tensor that they are represented by:

- 4-D Vector V, uses the mathematics of 4-Vectors and other 4D Tensors to describe the properties and relations of physical systems, ex. consider 2D pictures of a 3D object from different angles & distances. The pics look different, yet are all the same object.

- 4-D Scalar S, has internal symmetry principles, with all quantum particles obeying some portion of a universal { U(1) → QM → EM & CM} Symmetry.

- 4-D Tensor, uses tensor mathematics to calculate the properties of 4D Spacetime. GR is used for physical systems in which the mass of objects is very large. Essentially this means: “Mass tells spacetime how to curve, spacetime curvature tells mass how to move”. SR uses the mathematics of 4-Vectors and other 4D Tensors to describe the properties and relations of physical objects and concepts of both relativistic and quantum physics. Quantum “axioms” are instead actual SR derived principles.

Consider the following idea, with each level a special limiting-case subset of the former physical theory:

GR → {limit-case g^μν → δ^μν} → SR → derives → RQM → {limit-case |v| << c} → QM → {limit-case h\|p\| << (p\cdot p)} → (EM & CM) 

The main idea of [SRQM] is that the rules of RQM can be derived from the rules of SR. Thus, [SR→QM]
Notation / Conventions / Fundamentals:

{Temporal, 0\textsuperscript{th} Component, Positive(+)}, SI = Metric Signature (+,−,−,−) with [SI Dimensional-Units]. The “Metric System” :-) aka. (“Time-Positive”, “Particle-Physics”, “West-Coast”, “Mostly-Minuses”) Metric Convention

SR <Time-Space>-splitting Component Coloring Mnemonic: Temporal (blue) + Spatial (red) give Mixed SpaceTime (purple)

4D “Flat” <Time-Space> SR:Minkowski Metric

\( \eta_{\nu \sigma} = \text{Diagonal}[+1,-1,-1,-1] \) Cartesian : Generally, \( \{g_{\nu \sigma}\} = 1/\{\eta^{\nu \sigma}\} \) for non-zero

Mixed 4D (1,1)-Tensor form Minkowski Metric

\( \eta^\nu = \text{Diagonal}[+1,+1,+1,+1] \) Always \( I_{\nu} = g^{\nu}_{\nu} = \text{Kronecker Delta} = \text{Identity} \)

4-Position \( R = R^\nu = (ct, r) \) : 4D Position-OneForm \( R_\nu = (ct, r) \) = \( \eta_{\nu \rho}R^\rho \) \[m\]

4-Gradient \( \partial = (\partial/c, \nabla) = (\partial/R_\nu) \) : 4D Gradient-OneForm \( \partial_\nu = (\partial/c, \nabla) = (\partial/R^\rho) = \eta^{\rho}_{\nu} \partial_\rho \) \[1/m\]

\( \partial[R^\nu] = \partial[R] = (\partial/c, \nabla)(ct, r) \) \( \rightarrow \) \( \partial/c, \nabla(\partial/c, \nabla, \partial/c, \nabla) = \text{Diagonal}[+1,-1,-1,-1] \)

\( \eta_{\nu \sigma}R^\nu = \text{Diagonal}[+1,+1,+1,+1] \) \( \rightarrow \) \( \text{Scalar} = \text{New.Ref.Frame' = LorentzTransf. contracted w/ Old.Ref.Frame} \)

4D Tensors use Greek indices: ex. \( \{\mu, \nu, \sigma, \rho, \ldots\} \) : ex. 4-Position \( R^\nu = (r, c) = (r^\nu, c^\nu) \) with 4 possible index-values \{0,1,2,3\}

3D tensors use Latin indices: ex. \( \{i, j, k, \ldots\} \) : ex. 3-position \( r^i = (r^i, c^i) \) with 3 possible index-values \{1,2,3\}

4-Vector (4D) \( \mathbf{A} = A^\mu = (a^\mu) = (a_0, \mathbf{a}^i) = (a_0, \mathbf{a}^0, \mathbf{a}^i) \rightarrow (a_0, \mathbf{a}^0, \mathbf{a}^i, \mathbf{a}^j) \) Cartesian:rectangular \( \rightarrow (a_0, \mathbf{a}^0, \mathbf{a}^i, \mathbf{a}^j) \) spherical \( \rightarrow \) other coordinate basis

3-vector (3D) \( \mathbf{a} = a^\mu = (a_0, \mathbf{a}^i) = (a_0, \mathbf{a}^0, \mathbf{a}^i) \rightarrow (a_0, \mathbf{a}^0, \mathbf{a}^i, \mathbf{a}^j) \) Cartesian:rectangular \( \rightarrow (a_0, \mathbf{a}^0, \mathbf{a}^i, \mathbf{a}^j) \) spherical \( \rightarrow \) other coordinate basis

4-Scalar \( S = S^\mu = 1 \) Invariant Lorentz Scalar, same for all frames \{s\} or \{s_s\}

4-Vector \( V = V^\nu = (1^0 + 3^i) \)-splitting into \{v^\nu, v_i, v^i\}

4-Tensor, Anti-Symmetric \( T_{\text{asym}} = T^{\nu}_{\text{asym}} = (3^0 + 3^i) \)-splitting into \{v^\nu, v_0, v^0, v^i, v^j, v^k\}

4-Tensor, Symmetric \( T_{\text{sym}} = T^{\nu}_{\text{sym}} = (1^0 + 3^i + 3^j) \)-splitting into \{v^\nu, v_0 + v_1, v_0^0 + v_1^1, v^i, v^j, v^k\}

\( S = \{s\} = \text{number (1)} \) : \( V = V^\nu = (v^\nu, v_i, v^i) \) = vector (4) : \( T = T_{\text{sym}} + T_{\text{asym}} = T^{\nu}_{\text{sym}} = (\{t^0, t^i\}, \{t^0, t^j\}) \) = matrix (16) : etc.

Technically, these are all 4-Tensors = 4D Tensors; specify precisely using the #(m,n)-Tensor notation \{#(dims, upper, lower bounds)\} 

All SR 4-Tensors obey \( T_{\nu+i}^{\mu-j} = \Lambda^\nu_{\nu+i} \Lambda^{\mu-j}_{\nu+i} \ldots \Lambda^{\mu-i}_{\nu-j} \) \( \text{invariance over } T^{\nu-j} \) \text{ : } m = \# of indices and a separate Lorentz Transform \( \Lambda \) for each index

<Time-Space> 4-Vector Name matches its spatial 3-vector component name: ex. 4-Position \( R = (c^\nu, t^\nu, 3^i) \) \( \text{[length]} \rightarrow [m] \)

LightSpeed Factor (c) in temporal component as required to make all [dimensional-units] of a 4-vector’s components match

SR 4-Vector = (4D SpaceTime 4-Vector) = (1D temporal 3-vector, 3D spatial 3-vector) \( \rightarrow \) (4D (1+3))-splitting into \( \{v^\nu, v_0, v^0, v^i\} \) \( \leftarrow \) \( V \)

Tensor-index-notation in non-bold: ex. \( A^\nu = (a^\nu) = (a^0, a_1, a_2, a_3) \)

4-Vectors (4D) in Bold UPPERCASE: ex. \( \mathbf{A} = \mathbf{A}^{\nu} = (\mathbf{a}^\nu) = (\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3) \)

3-vectors (3D) in Bold LOWERCASE: ex. \( \mathbf{a} = (\mathbf{a}^\nu) = \mathbf{a} = (\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3) \)

Temporal scalars (1D) in non-bold, usually lowercase, 0\textsuperscript{th} component: ex. \( a_0, a_0 \) \text{”Count from 1, but index from 0 :-)”}

Individual non-grouping components of 4-Tensors in non-bold: ex. \( a_0^\nu = (a_0, a_1, a_2, a_3) = (a_0, a) \) : Vectors are grouping structures, thus <Bold> Rest scalars (invariants) in non-bold, denoted with naught (\( ; \)): ex. \( m, U = \mathbf{P} = (m, U_0) \)

A rest-frame is a valid restful conceptual concept

Upper indices 4-Vector \( A^\nu = (a^\nu) = (a_0, a) \) : Lower indices 4-Vector \( B_\nu = (b_\nu) = (b_0, b_i) \)

Index lowering/raising via Minkowski Metric \( \eta_{\nu \rho} \): ex. \( R_\nu = \eta_{\nu \rho}R^\rho \) or \( \partial^\nu = \eta^{\nu \rho} \partial_\rho \)

Relativistic Gamma \( \gamma = 1/\sqrt{1 - \beta \cdot \beta} \) : Relativistic \( \beta = u/c = \{0..1\} \) : ProperTimeDerivative (d/dt) = \( (U/c) \)

4D (1,0)-Tensor = 4-Vector: \( \mathbf{A} = \mathbf{A}^{\nu} = \mathbf{a} = \{0..3\} \) \( \rightarrow \) \( \text{4D GradientOneForm} \) \( \partial_\nu = (\partial/c, \nabla) \) \( \text{or} \) \( \text{4D GradientOneForm} \) \( \partial^\nu = \eta^{\nu \rho} \partial_\rho \)

“Unit”Temporal 4-Vector \( \mathbf{T} = \gamma(1, \beta) \), with Lorentz Scalar Invariant \( \mathbf{T} \cdot \mathbf{T} = T^\nu T_\nu = \gamma(1 - \beta \cdot \beta) \) \( +1 \rightarrow \mathbf{T} = U/c \)

Null 4-Vector \( N = \{\pm a, a\} = \{\pm a, a\} \), with Lorentz Scalar Invariant \( N \cdot N = N_\nu N^\nu = a_0^2 + a_i^2 \) \( = 0 \)

“Unit”Spatial 4-Vector \( \mathbf{S} = \gamma a (\beta \cdot \beta, \mathbf{n}) \), with Lorentz Scalar Invariant \( \mathbf{S} \cdot \mathbf{S} = S^\nu S_\nu = \gamma a^2 (\beta \cdot \beta) - \mathbf{n} \cdot \mathbf{n} \) \( = -1 \rightarrow \mathbf{T} = U/c \)

Space-like separated \( \mathbf{S} \) : Invariant Spatial Topology=Space-ordering

Null-like separated \( \mathbf{S} \) : Invariant Null LightCone

Time-like separated \( \mathbf{S} \) : Invariant LightSpeed (c)
Alternate ways/styles of writing 4-Vector and 4-Tensor expressions in Physics:

(A · B) is a 4-Vector style, which uses vector-notation \( \{ \text{ex. bold vector-grouping, inner product } "\cdot", \text{ exterior product } "\wedge" \} \), and can show relations very compactly. There are 4D analogs to the standard 3D vector rules, some of which are shown below:

- Use **bold** UPPERCASE to represent 4-Vectors: \( A = (A^0, \ldots, A^3) = (a^0, \ldots, a^3) \) & **bold** lowercase for 3-vectors: \( a = (a^0, \ldots, a^3) \)
- 4D Lorentz Scalar Product (A · B) = \( (A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3) = (a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3) = (a^0 b^0 - a \cdot b) \)

The 4-AngularMomentum uses the exterior wedge product \( (\wedge) \) and can be written as \( M = R^s P = \{[[0,-cN], [cN, I = r^p P]]\} = (R^p P - R^0 P^0) \). The 4D Gauss’ Theorem in SR is: \( \int_\Omega d^4x \left( \partial^\alpha V^\alpha \right) = \frac{1}{3!} \oint dS(N) \) \( \text{Divergence integral of vector field = boundary flux integral with:} \)

\( \Omega \) as a 4D simply-connected region of Minkowski SpaceTime \( \int \partial V \cdot N \) as its 3D boundary with its own 3D Volume element \( dS \) and outward-pointing 4-Unit"HyperSurface"Normal \( N \)

\( \text{d}^4 \mathbf{x} \) as a 4D Infinitesimal Volume Element

\( \mathbf{V} \) as an arbitrary 4-Vector

(\( A^\nu \eta_{\nu\rho} B^\rho \)) is a Ricci-Calculus style, which uses tensor-indexotation and is useful for more complicated expressions, especially to clarify those expressions involving tensors with more than one index, such as the Faraday EM Tensor \( F^\mu = \left( \partial^\nu A^\nu - \partial^\nu A^\nu \right) = (\partial V A) = F \).

Tensor rules include concepts such as:

- Kronecker Delta \( (\delta_\mu^\nu) = \text{Diagonal}[+1,+1,+1,+1] = \text{Diagonal}[+1,+1,1,0] \) = 4D Identity \( I_4 = \eta^{\mu\nu} = g^{\mu\nu} \)
- Levi-Civita Symbol \( (\epsilon_\mu\nu\rho) = [+1 \text{ for even permutation, -1 for odd permutation, else 0}] = \text{Totally Anti-Symmetric Tensor, also (e^{\mu\nu\rho}) \} \)
- Index lowering/raising with a Metric \( g^{\mu\nu} \) \( \text{ex. } A^\mu B_\nu = a^\nu b_\nu \) \( \text{+ a}_1 b_1 + a^\nu b^\nu \) \( \text{B} = (A^{\eta\nu} \eta_{\nu\rho} B^\rho) \)

Symmetric/anti-Symmetric Tensor decomposition \( [T^\mu = S^\mu + A^\mu], \) with \( S^\mu = (T^\mu + T^\nu)/2 \) and \( A^\mu = (T^\mu - T^\nu)/2 \)

Tensor Contraction of Symmetric with Anti-Symmetric yields zero \( \{S^\mu A_\mu = 0\} \), from \( \{ S^\mu = +S^\mu \} \) and \( \{ A^\mu = -A^\mu \} \) parts of \( T^\mu \)

- Proof: 10 components + 6 components = 16 comps
- \( S^\mu A_\mu = \{\text{swapping dummy indices}\} \rightarrow S^\mu A_\mu = (S^\mu)(A_\mu) = (+S^\mu)(-A_\mu) = -S^\mu A_\mu = 0 \) since \( \{ C = -C = 0 \} \)

The Symmetric Tensor is further decomposed into an Isotropic part \( S_{iso}^\mu = (S^\mu/4)\eta^\mu \) and zero-trace Anisotropic part \( S_{aniso}^\mu = S^\mu - S_{iso}^\mu \)

So, \( \{ T^\mu = S_{iso}^\mu + S_{aniso}^\mu + A^\mu \} \) this is manifestly invariant: The Poincaré Group Symmetry operations respect these decompositions, meaning that boosts, rotations, etc. don’t intermix them, unlike the \( \text{(temporal+} \text{mixed+spatial)-splittings, which can get intermixed.} \)

Notes on Poincaré SpaceTime Group, Tensor Linear Mapping, Lie Group, etc.:

SR 4-Vectors have a Poincaré Group linear mapping \( (V^\nu = \Lambda^\nu_\mu, V^\mu + \Delta V^\mu \) which preserves interval-magnitude: \( (V^\nu V_\nu = V^0 V_0) \).

For a given mapping, the Lorentz-Transform (\( \Lambda^\nu_\mu \)) \( \text{[dimensionless]} \) & SpaceTime-Translation-Transform (\( \Delta V^\mu \)) \( \text{[m]} \) are constants. Since the SpaceTime-Translation part is not dimensionless, one has to take care handling it. Temporarily set multiplier \( \{ \Lambda^\nu_\mu \rightarrow \delta^\nu_\mu \} \)

4-Position \( X^\nu \rightarrow \Lambda^\nu_\mu X^\mu + \Delta X^\mu \rightarrow X^\nu + \Delta X^\mu \rightarrow \text{Not SpaceTime Translation Invariant, hence not Poincaré Invariant} \rightarrow \Delta X^\mu \rightarrow 0 \)

4-Displacement \( \Delta X_{i\nu} \rightarrow X^\nu_X - X^\nu_i = \left( \Lambda^\nu_\mu X^\mu_i + \Delta X^\mu \right) - \left( \Lambda^\nu_\mu X^\nu_i + \Delta X^\nu \right) = (X^\mu_i - X^\nu_i) \rightarrow \text{Poincaré Invariant} \rightarrow \Delta X^\nu \rightarrow \text{unrestricted} \)

4-Velocity \( U^\nu \rightarrow (d/d\tau)[X^\nu + \Delta X^\nu] = (d/d\tau)[X^\nu] + (d/d\tau)[\Delta X^\nu] = U^\nu + 0^\nu = U^\nu \rightarrow \text{Poincaré Invariant} \rightarrow \Delta X^\nu \rightarrow \text{unrestricted} \)

Likewise, any 4-Vector that is based on 4-Velocity \( \{ \text{lots of them, since 4-Velocity only has 3 independent components, it can be multiplied by a Lorentz Scalor to make a new 4-Vector \} \} \) will be Poincaré Invariant. Basically, only the 4-Position is not, although it is still Lorentz Invariant. You can think of 4-Position as a 4-Displacement in which one of the endpoints is “pinned” to the 4-Origin \( O = (0,0,0) \) [Cartesian]. Since it is “pinned”, it can’t be SpaceTime-Translated, but it can still be Lorentz-Transform “Rotated” about the 4-Origin.

\( \text{[ ]} \rightarrow \) The Poincaré Group is a Lie Group, and can be written as a Unitary Operation: \( U(\Lambda^\nu_\mu, \Delta X^\mu) = e^{i[1/2\omega_{\mu}\Omega]} e^{i[\Omega \Delta X^\mu P\mu]} \) with:

- 4-LinearMomentum \( P^\mu \) \( \text{[kg \cdot m/s]} \) as generator of SpaceTime-Translation-Transforms with \( \Delta X^\mu \) \( \text{[m]} \) encoding the \( 1 \rightarrow 3 \rightarrow 4 \) displacements.
- 4-AngularMomentum \( M^\mu \) \( \text{[kg \cdot m^2/s]} \) as generator of Lorentz-Transforms with anti-symmetric \( \omega_{\mu\nu} \) \( \text{[1]} \) encoding the 3 angles + 3 boosts. Both \( \{ \Delta X^\mu P^\mu \} \) and \( \{ \omega_{\mu\nu} M^\mu \} \) have dimensional-units of \( \{ \text{Action} = \text{kg \cdot m^2/s = J \cdot s} \} \)

\( \text{Infinitesimal} \Lambda^\nu_\mu = \delta^\nu_\mu + \omega_{\nu\rho} A^\rho + \ldots \)
The following 4-Vectors \{ 4D (1,0)-Tensors \} are all elements of classical SR and EM:

4-Position \( \mathbf{R} = (ct, \mathbf{r}) \) \[\text{[m]}\]  
Alt. \( \mathbf{X} = (ct, \mathbf{x}) \) only Lorentz, not Poincaré Invariant

4-Displacement \( \Delta \mathbf{R} = \Delta \mathbf{r} \) \[\text{[m]}\]  
Finite \( \Delta \mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1 \) fully Poincaré Invariant

4-Differential \( d\mathbf{r} = d\mathbf{r}^\alpha \) \[\text{[m]}\]  
Infiniteesimal d\mathbf{R}

4-Velocity \( \mathbf{U} = U^\alpha = (\gamma(c, \mathbf{u}) \) \[\text{[m/s]}\]

4-Momentum \( \mathbf{P} = \mathbf{p} = (E/c, \mathbf{p}) \) \[\text{[kg·m/s = N·s]}\]

4-WaveVector \( \mathbf{K} = K^\alpha = (\alpha/c, \mathbf{k}) \) \[\text{[rad/m]}\]
Invariant SR WavePhase \( \Phi = -\mathbf{k} \cdot \mathbf{X} \) \[\text{[rad]}\]

4-Gradient \( \mathbf{\partial} = \mathbf{\partial}^\alpha = (\partial/c, \mathbf{\nabla}) \) \[1/m\]
Invariant: \( \delta = \mathbf{\partial}_\mathbf{R} \): See also 4-Velocity Gradient \( \mathbf{\partial}_\mathbf{U} \) \[\text{s/m}\]

4-(Dust)NumberFlux \( \mathbf{N} = N^\alpha = (n/c, n) \) \[\text{[#/m² = N]}\]

4-Current(Density)=4-ChargeFlux \( \mathbf{J} = J^\alpha = (pc_j, c) \) \[\text{[C/(m²·s) = A/m²]}\]

4-(EM)VectorPotential \( \mathbf{A} = A^\alpha = (\phi/c, \mathbf{a}) \) \[\text{[kg·m/(C·s) = T·m]}\]

4-(Minkowski)Force \( \mathbf{F} = F^\alpha = (\gamma(E/c, \mathbf{f})) \) \[\text{[kg·m/s² = N]}\]

The mathematical Lorentz Scalar Product of two generic 4-Vectors \( \mathbf{A} = A^\alpha \) and \( \mathbf{B} = B^\alpha \) is:

\[
\langle \mathbf{A}, \mathbf{B} \rangle = A^\alpha B^\beta \delta_{\alpha \beta} = (\phi/c, \mathbf{a}) \cdot (\phi/c, \mathbf{a})
\]

which is a Lorentz Scalar Invariant \{ 4D (0,0)-Tensor \}

Applying this rule to various SR 4-Vectors gives the following 4Lorentz Invariants (rest values indicated with naught _o):

\[
\begin{align*}
\langle \mathbf{R}, \mathbf{R} \rangle &= (ct)^2 - r^2 = (ct_0)^2 = (ct_0)^2 \\
\langle \mathbf{U}, \mathbf{U} \rangle &= \gamma(c^2 - u^2) = c^2 \\
\langle \mathbf{P}, \mathbf{P} \rangle &= (E/c)^2 - p^2 = (E/c_0)^2 - p_0^2 = (m_0 c)^2 \\
\langle \mathbf{K}, \mathbf{K} \rangle &= (\alpha/c)^2 - k^2 = (\alpha_0/c_0)^2 \\
\langle \mathbf{J}, \mathbf{J} \rangle &= (\rho/c)^2 - j^2 = (\rho_0/c_0)^2 \\
\langle \mathbf{A}, \mathbf{A} \rangle &= (\phi/c)^2 - a^2 = (\phi_0/c_0)^2 \\
\langle \mathbf{F}, \mathbf{F} \rangle &= (E/c)^2 - f^2 = (E_0/c_0)^2 \\
\langle \mathbf{U}, \mathbf{J} \rangle &= \gamma(\partial/c - \mathbf{u} \cdot \mathbf{\nabla}) = \gamma(\partial/c_0 - \mathbf{u}_0 \cdot \mathbf{\nabla}) \\
\langle \mathbf{K}, \mathbf{R} \rangle &= (\alpha/c_0 - \mathbf{\nabla}) = (\partial/c_0 - \mathbf{\nabla}) \\
\langle \mathbf{P}, \mathbf{R} \rangle &= (E/c_0 - \mathbf{f}) = (E_0/c_0 - \mathbf{f}) = -\Phi
\end{align*}
\]

Invarient 4D Scalar Products (details follow):

- Proper Time \( t = t \):
  \( s = t - \sqrt{s^2} \)

- Proper Proper Length \( l = l \):
  \( m = \sqrt{c^2} \)

- Rest Mass \( m = m \):
  \( \text{kg·m/s²} = J \)

- Energy \( E = mc^2 \):
  \( \text{kg·m²/s²} = \gamma \)

- Rest Energy \( E_0 = mc_0^2 \):
  \( \text{kg·m²/s²} = \gamma \)

- Rest Mass \( m_0 = m_0 \):
  \( \text{kg·m/s²} = \gamma \)

- Prop. Time Partial \( \partial/c = \partial/c_0 \):
  \( \text{[s]} \)

- d’Alembertian 4D Wave:
  \( \nabla^2 \)

- Proper Number Density \( n = n \):
  \( \text{#/m²} \)

- Equation \( \mathbf{\partial} \cdot \mathbf{\nabla} \):
  \( \text{[1/m²]} \)

- Invariant SR WavePhaze \( \Phi \):
  \( \text{[rad]} \)

- Invariant SR Action \( S \):
  \( \text{[kg·m²/s² = J·s = Action]} \)

The 4D 4-Vectors also have some fundamental relations between one another (again using rest naught _o notation):

4-Position \( \mathbf{R} = R^\alpha = (ct_0, \mathbf{r}_0) \) \( \in \langle \text{Event Space} \rangle \)

4-Velocity \( \mathbf{U} = U^\alpha = (c, \mathbf{u}_0) \) \( \mathbf{\partial}_\mathbf{R} \)

4-Momentum \( \mathbf{P} = P^\alpha = (E_0/c_0, \mathbf{p}_0) \) \( = (m_0, \mathbf{0}) \) \( \mathbf{\partial}_\mathbf{U} \)

4-WaveVector \( \mathbf{K} = K^\alpha = (\alpha_0/c_0, \mathbf{k}_0) \) \( \mathbf{\partial}_0 \)

4-Gradient \( \mathbf{\partial} = \mathbf{\partial}^\alpha = (\partial/c_0, \mathbf{\nabla}_0) \) \( \mathbf{\partial}_\mathbf{K} \)

4-(Dust)NumberFlux \( \mathbf{N} = N^\alpha = (n_0/c_0, n_0) \) \( \mathbf{\partial}_\mathbf{N} \)

4-Current(Density)=4-ChargeFlux \( \mathbf{J} = J^\alpha = (pc_j, c) \) \( \mathbf{\partial}_\mathbf{J} \)

4-(EM)VectorPotential \( \mathbf{A} = A^\alpha = (\phi_0/c_0, \mathbf{a}_0) \) \( \mathbf{\partial}_\mathbf{A} \)

4-(Minkowski)Force \( \mathbf{F} = F^\alpha = (\gamma(E_0/c_0, \mathbf{f}_0) \) \( \mathbf{\partial}_\mathbf{F} \)

Faraday EM 4-Tensor \( \mathbf{\mathbf{F}} \)

Maxwell Equation with Source \( \nabla \times \mathbf{F} = \mathbf{E} + \mathbf{\partial} \times \mathbf{J} \)
\( \nabla \cdot \mathbf{F} = 0 \)

4D Lorentz Force Equation \( \mathbf{\partial} \times \mathbf{F} = 0 \) \( \mathbf{\partial}_\mathbf{F} \)

Conservation of EM Current(Density) \( \mathbf{\partial} \cdot \mathbf{J} = 0 \) \( \mathbf{\partial}_\mathbf{J} \)

Beautiful proof based on Symmetric:AntiSymmetric tensors

Fundamental Constants (Lorentz Invariants) which efficiently appear from SR 4-Vector formalism:

<table>
<thead>
<tr>
<th>Physical Property</th>
<th>Value</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light Speed ( c )</td>
<td>( c/m/s )</td>
<td>( n ) material index-of-refraction</td>
</tr>
<tr>
<td>Rest Mass ( m_0 )</td>
<td>( \text{kg} )</td>
<td>Varies depending on particle type</td>
</tr>
<tr>
<td>EM charge ( e )</td>
<td>( \text{C} )</td>
<td>Varies depending on particle type</td>
</tr>
<tr>
<td>Electric Constant/Permittivity ( \varepsilon_0 )</td>
<td>( \text{F/m = C²/s²·kg/m³} )</td>
<td>( \varepsilon_0·\mu_0 = 1/c^2 )</td>
</tr>
<tr>
<td>Magnetic Constant/Permeability ( \mu_0 )</td>
<td>( \text{H/m = kg·m/C²} )</td>
<td>( \varepsilon_0·\mu_0 = 1/c^2 )</td>
</tr>
<tr>
<td>Planck's constant ( \hbar )</td>
<td>( \text{J·s = kg·m²/s = Action)} )</td>
<td>We shall see this in the next section...</td>
</tr>
<tr>
<td>Boltzmann's constant ( k_b )</td>
<td>( \text{kg·m³·K·s⁻²} )</td>
<td>See the full presentation SRQM-Roadmap</td>
</tr>
</tbody>
</table>

Constants with \( \varepsilon \) are considered in their non-interacting state, their effective values change with matter-interaction.

<Photon speed> varies in medium \( = c/n \) due to atomic interaction times, \( \text{ex. going through a prism, which causes refraction effects.} \\ Note: (c) is large, but never \( \to \infty \); (h) is small, but never \( \to 0 \); Always use realistic limits, ex. \( \{ \text{ or } \} \) for SR→CM, RQM→QM
One can divide two generic 4-Vectors \( \mathbf{A} = A^\mu (a^\mu, a) \) and \( \mathbf{B} = B^\mu (b^\mu, b) \) by using an intermediary 4-Vector \( \mathbf{V} = V^\mu (v^\mu, v) \) for the following mathematical relations:

\[
|A|/|B| = (A^\mu V_\mu)/(B^\mu V_\mu) = (a^\mu, a^\mu)/(b^\mu, b^\mu) \quad \text{which is a Lorentz Scalar Invariant} \quad \{4D\ (0,0)-Tensor\}
\]

Applying the division rule with certain SR 4-Vector combinations leads to the following very suggestive relations:

\[
(P\cdot U)/(K\cdot U) = \gamma E / (p\cdot u)/[\gamma(\omega - k\cdot u)] = E/\omega, \quad \quad \rightarrow \left|P\right|/\left|K\right| = E/\omega,
\]

\[
(P\cdot K)/(K\cdot K) = (E/c^2 - p\cdot k)/(\omega/c^2 - k\cdot k) = m_0/c^2, \quad \quad \rightarrow \left|P\right|/\left|K\right| = E/\omega,
\]

\[
(P\cdot P)/(K\cdot P) = (E/c^2 + p\cdot p)/(\omega/c^2 + k\cdot k) = m_0/c^2, \quad \quad \rightarrow \left|P\right|/\left|K\right| = E/\omega,
\]

\[
(P\cdot R)/(K\cdot R) = (E + pr)/(\omega - kr) = (-S_{\text{phase,planewave}})/(\omega/c), \quad \quad \rightarrow \left|P\right|/\left|K\right| = E/\omega,
\]

Also, from the definitions:

\[
K = (\omega/c^2)\mathbf{U} \quad \text{or} \quad \mathbf{U} = (c^2/\omega)\mathbf{K}
\]

Thus:

\[
P = (E/c^2)\mathbf{U} = (E/c^2)(c^2/\omega)\mathbf{K} = (E/\omega)\mathbf{K} \quad \quad \text{or} \quad \quad \left|P\right|/\left|K\right| = E/\omega,
\]

Analysis of Dirac’s Constant (\( h = \hbar/2\pi \)) : Planck’s Constant (\( h = 2\pi\hbar \)) and (\( E/\omega_0 \)) in the context of SRQM:

It is an empirical (observational) fact that the Lorentz Scalar Invariant (\( E/\omega_0 = \gamma E/\gamma_0 \omega_0 = (E/\omega) \rightarrow (h) \quad \left[\text{J-s/\text{rad}}\right] \)) for all known experimental measurements. The SR 4D Tensor rules show that one doesn’t need a quantum axiom for this. (h) is actually an empirically-measurable quantity, just like (c), (e), (G), (k), (μ), or the other fundamental constants, which are also Lorentz Scalar Invariants. (b) can be measured classically (without a need for quantum axioms) from the photoelectric effect, from the inverse photoelectric effect, from LED’s (injection electroluminescence), from Atomic Line Spectra (Bohr Atom), from the Duane-Hunt Law in Bremsstrahlung, from Electron Diffraction in crystals, from the Watt/Kibble-Balance, from the Sagnac Effect, from Incandescent Blackbody Intensity-Temperature Relations, from Compton Scattering, from the Stern-Gerlach experiment, etc.

Regarding physics simulations of measurements of Dirac’s : Planck’s Constant (\( h = \hbar/2\pi \)), and (\( E/\omega_0 \)) in the context of SRQM:

http://scirealm.org/Physics-PlanckConstantViaSternGerlach.html
http://scirealm.org/Physics-PlanckConstantViaSagnacEffect.html
http://scirealm.org/Physics-PlanckConstantViaComptonScattering.html
http://scirealm.org/Physics-PlanckConstantViaIncandescence.html
http://scirealm.org/Physics-PlanckConstantViaLEDs.html
http://scirealm.org/Physics-PlanckConstantViaAtomicLineSpectra.html
http://scirealm.org/Physics-PlanckConstantViaElectronDiffraction.html
http://scirealm.org/Physics-PlanckConstantViaGravityInducedInterferometry.html
http://scirealm.org/Physics-PlanckConstantViaBremsstrahlung.html
http://scirealm.org/Physics-PlanckConstantViaVacuumFluctuation.html

This implies the following SR ( wave-like \( \psi \)) relation:

\[
P = \hbar \mathbf{K} = (E/c, p) = h (\omega/c, k) = h \omega/c, h k
\]

The temporal part \( \{E = \hbar \omega\} \) gives Einstein’s photoelectric quantum postulate.

The spatial part \( \{p = \hbar k\} \) gives de Broglie’s matter-wave quantum postulate.

This is very similar to Einstein’s other SR ( particle-like \( \psi \)) relation:

\[
P = m\mathbf{U} = (E/c, p) = m\gamma (c, u) = m(c, u) = (mc, mu) = (E/c, Eu/c^2) = (E/c^2)\gamma (c, u) = (E/c^2)U
\]

The temporal part \( \{E = m\gamma c^2 = mc^2 = \gamma E\} \) gives Einstein’s famous energy:mass relation (in both rest \( \frac{\gamma}{c} \) & relativistic forms). The spatial part \( \{p = mc, u = mu = Eu/c^2 = \gamma_E u/c^2\} \) gives Einstein’s relativistic momentum.

The 4-WaveVector \( \mathbf{K} \) exists in all physics/mathematical contexts as a solution of the 4D Invariant \( \Delta \psi \). It is an empirical & mathematical fact that all waves (classical/relativistic/EM/quantum/purely-mathematical) can be modeled using complex planewaves which obey a principle of superposition: Wave \( \psi = \sum \psi_n \) = sum of complex planewaves.

An individual wavefunction of form \( \psi = (a)e^{i(K\cdot X)} \) \( = (a)e^{i(\phi)} \) \( = (a)e^{iS_{\text{action}}/\hbar} \) has amplitude (a) that can be:

\[
4D\ (0,0)-\text{Tensor} \quad \text{A} \quad \text{ex. Quantum Scalar}
\]

\[
4D\ (1,0)-\text{Tensor} \quad \text{A}^\phi \quad \text{ex. EM/Photonic/Proca}
\]

\[
4D\ (2,0)-\text{Tensor} \quad \text{A}^\nu \quad \text{ex. Gravitational Wave}
\]

This gives the mathematical 4-Vector relation:

\[
\mathbf{\partial} = i\mathbf{K} = (\partial/c, -\hat{V}) = -i(\omega/c, k)
\]

The temporal part \( \{\partial = -i\omega\} \) or \( \{\omega = i\partial\} \) gives temporal:frequency complex planewave change:operator

The spatial part \( \{\hat{V} = i\mathbf{k}\} \) or \( \{\mathbf{k} = -i\hat{V}\} \) gives spatial:wavenumber complex planewave change:operator

\[
\psi = (a)e^{i(K\cdot X)}; \quad \text{There exists also} \quad \psi^* = (a^*e^{i(K\cdot X)}), \quad \text{giving} \quad \psi^* = |\psi|^2, \quad \text{independent of the phase part} \quad \Theta = \Theta(K\cdot X)
\]

\[
\partial\psi = \partial(a)e^{i(K\cdot X)} = \partial iK(a)e^{i(K\cdot X)} = \pm iK(\psi) = \pm i\Theta(\psi), \quad \text{with the minus sign} \quad \{\partial = -i\mathbf{K}\} \text{ typically chosen for historical reasons.}
\]

\[
\partial[\psi^* \psi] = (\partial[\psi^*]) \psi + \psi^* (\partial[\psi]) = (\pm iK\psi^* \psi + iK\psi \psi) = 0 = \partial[(a^*)(a)], \quad \text{giving conservation of probability} \quad \{|\psi^* \psi| = |\psi|^2\}.
\]
SR Wave Energy-Momentum, Dispersion, & Velocity Relations using 4D Tensors (esp. 4-Vectors):

4-Vectors = 4D (1,0)-Tensors  Properties
4-Position \( \mathbf{R} = R^\mu = (ct, \mathbf{r}) \)  [m]  \( c = \) Invariant LightSpeed
4-Velocity \( \mathbf{U} = U^\mu = \gamma(c, u) \)  [m/s]  \( \gamma = 1/\sqrt{1 - (u/c)^2} \), \( u = \) particle:group velocity
4-Momentum \( \mathbf{P} = P^\mu = (E/c, \mathbf{p}) \)  [kg·m/s]  \( E = \) relativistic energy, \( \mathbf{p} = \) relativistic 3-momentum
4-WaveVector \( \mathbf{K} = K^\mu = (\omega/c, \mathbf{k} = \omega \mathbf{n}/v_{\text{phase}}) \)  [rad/m]  \( v_{\text{phase}} = \omega/k = \) phase velocity

4-Vectors = 4D (1,0)-Tensors  Relations
4-Velocity \( \mathbf{U} = d\mathbf{R}/dt \)  \( \tau = \) Proper Time, \( d/d\tau = \) Proper Time Derivative
4-Momentum \( \mathbf{P} = m\mathbf{U} = h\mathbf{K} \)  \( E_0 = \) Rest Energy, \( m_0 = \) Rest mass, \( h = \) Dirac's const
4-WaveVector \( \mathbf{K} = (\omega_0/c, \mathbf{c}) \mathbf{U} \)  \( \omega_0 = \) Rest Angular Frequency

\[ \mathbf{K} = (\omega/c, \mathbf{k} = \omega \mathbf{n}/v_{\text{phase}}) = (\omega_0/c)\mathbf{U} = (\omega_0/c)^2\gamma(c, \mathbf{u}) = (\gamma\omega_0/c^2)\gamma(c, \mathbf{u}) = (\gamma\omega_0/c, \gamma\omega_0 \mathbf{u}/c^2) \]  wave-motion [\( \mathbf{n} \)]

The temporal part: \( \omega = \gamma\omega_0 \)  \( \{\text{Relativistic } \omega : \text{Rest } \omega_0\} \) ang.-frequency
The spatial part: \( \mathbf{k} = \gamma\omega_0 \mathbf{n}/v_{\text{phase}} = \gamma\omega_0 \mathbf{u}/c^2 \)
\( \mathbf{n}/v_{\text{phase}} = \mathbf{u}/c^2 \)
\( \mathbf{u} = c\mathbf{n}/v_{\text{phase}} \)
\( |v_{\text{phase}} \times \mathbf{u}| = c^2 \)

\[ \mathbf{P} = (E/c, \mathbf{p}) = (E_0/c)\mathbf{U} = (E_0/c)^2\gamma(c, \mathbf{u}) = (\gamma E_0/c, \gamma E_0 \mathbf{u}/c^2) \]  particle-motion [\( \mathbf{c} \)]

The temporal part: \( E = \gamma E_0 \)  \( \{\text{Relativistic } E : \text{Rest } E_0\} \) energy
The spatial part: \( \mathbf{p} = \gamma E_0 \mathbf{u}/c^2 = E\mathbf{u}/c^2 = E^2\mathbf{n}/(c^2 v_{\text{phase}}) = E\mathbf{n}/v_{\text{phase}} \)
\( \gamma m_0 \mathbf{u} = m\mathbf{u} \)
\( \{\text{Relativistic } \mathbf{p} : \text{3-momentum}\} \)

\[ \mathbf{P} = (E/c, \mathbf{p}) = h\mathbf{K} = h(\omega/c, \mathbf{k} = \omega \mathbf{n}/v_{\text{phase}}) \]  particle-wave duality [\( \mathbf{\cdot} \)]

The temporal part: \( E = h\omega \)
The spatial part: \( \mathbf{p} = h\mathbf{k} = h\omega \mathbf{n}/v_{\text{phase}} = E\mathbf{n}/v_{\text{phase}} \)
The Einstein photoelectric eqn.

The de Broglie matter-wave eqn.

So, 3-momentum \( |\mathbf{p}| = E \mathbf{U} / \text{phase-velocity } v_{\text{phase}} \) in general. Interesting to note that matter at-rest \( \{\mathbf{p} = 0\} \) has \( v_{\text{phase}} = \infty \)

\( (\mathbf{U}\cdot\mathbf{U}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = (c^2)^2\gamma^2[1 - \mathbf{u}\cdot\mathbf{u}/c^2] = (c^2)^2 \)

\( (\mathbf{P}\cdot\mathbf{P}) = [(E/c)^2 - \mathbf{p}\cdot\mathbf{p}] = (E_0/c)^2 \)
\( (E/c)^2 = \mathbf{p}\cdot\mathbf{p} + (E_0/c)^2 \)
\( (E)^2 = \mathbf{p}\cdot\mathbf{p} + (E_0)^2 \)
\( E^2 = (pc)^2 + (E_0)^2 \)
\( E = \sqrt{[(pc)^2 + (E_0)^2]} \)  : If photonic, \( (E_0, 0) \), then \( E_{\text{photon}} = |\mathbf{p}|c \), which gives \( u_{\text{photon}} = v_{\text{phase,photon}} = c \), from \( |v_{\text{phase}} \times \mathbf{u}| = c^2 \)

\( (\mathbf{K}\cdot\mathbf{K}) = [(\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}] = (\omega_0/c)^2 \)
\( (\omega/c)^2 = k^2 + (\omega_0/c)^2 \)
\( d[(\omega/c)^2] = dk^2 + (\omega_0/c)^2 \)
\( 2\omega d\omega/c^2 = 2k dk + 0 \)
\( \omega d\omega/c^2 = k dk \)
\( d\omega/dk = c^2 k/\omega = c^2/v_{\text{phase}} = \mathbf{u} \)

Recapping:
\( \omega/k = v_{\text{phase}} \)  From the formal definition of 4-WaveVector \( \mathbf{K} = K^\mu = (\omega/c, \mathbf{k} = \omega \mathbf{n}/v_{\text{phase}}) \)
\( d\omega/dk = \mathbf{u} = v_{\text{group}} = v_{\text{particle}} \)  Derived from Lorentz Scalar Product \( (K\cdotK) = [(\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}] = (\omega_0/c)^2 \)
\( \mathbf{u} = c\mathbf{n}/v_{\text{phase}} \)  The relation between particle and wave velocities, from \( \mathbf{K} = (\omega_0/c)^2 \mathbf{U} \)

One of the main points here is that many physics relations are easily derivable from simple 4-Vector tensorial rules and that many of these physics formulas are really just the temporal & spatial parts of these 4D <Time, Space> relations.
SR Sagnac Effect, Measuring Absolute Spatial Rotation, using 4D Tensors (esp. 4-Vectors):

4-Vectors = 4D (1,0)-Tensors

Properties
4-Position \( R = R^\mu = (ct, \mathbf{r}) \) \([m]\) \( c = \) Invariant LightSpeed
4-Differential \( \mathbf{d}R = dR^\mu = (c dt, d\mathbf{r}) \) \([m]\)
4-Velocity \( \mathbf{U} = U^\mu = (\gamma c, \mathbf{u}) \) \([\text{m/s}]\)
4-Momentum \( \mathbf{P} = P^\mu = (\gamma mc, \mathbf{p}) \) \([\text{kg·m/s}]\)
4-WaveVector \( \mathbf{K} = K^\mu = (\omega/c, \mathbf{k}) = \omega n/\nu_{\text{phase}} \) \([\text{rad/m}]\)
4-TotalMomentum \( \mathbf{P}_T = P_T^\mu = (\mu/c, \mathbf{p}_T) \) \([\text{kg·m/s}]\)
4-TotalWaveVector \( \mathbf{K}_T = K_T^\mu = (\gamma \omega/c, \mathbf{k}_T) \) \([\text{rad/m}]\)

Waves are additive, as are Momenta (superposition)

\[ P = (E/c, \mathbf{p}) = h\mathbf{K} = h(\omega/c, \mathbf{k} = \omega n/\nu_{\text{phase}}) \quad \text{particle-wave duality} \quad [\text{γ/λ}] \]

\[ \mathbf{P} \cdot d\mathbf{R} = h\mathbf{K} \cdot d\mathbf{R} \quad \text{[kg·m²/s] = J = Action} \quad \text{Action:Phase Relation} \]

\[ dS = h\Phi = h(\omega/c, \mathbf{k} = \omega n/\nu_{\text{phase}}) \quad \text{S = action} \]

Wavefunction

\[ \Psi = \Psi_0 e^{-i\Phi} = \Psi_0 e^{-iS/\hbar} \]

\[ \text{3-TotalMomentum} = -4-\text{Gradient}[\text{Action}] \{ \Phi_T = -\partial[\text{S}_\text{action,total}] \} \]

\[ \text{Action (relativistic invariance, Lorentz Scalar Product)} \]

\[ S = \int \mathbf{P} \cdot d\mathbf{R} = -\int \mathbf{P} \cdot d\mathbf{R} = \int (\mathbf{P} \cdot \mathbf{U}) dr = \mathbf{E} \cdot d\mathbf{r} = -m, c^2 dr \]

\[ dS = \mathbf{P} \cdot d\mathbf{R} = -\mathbf{K} \cdot d\mathbf{r} = (-\mathbf{K} \cdot d\mathbf{r}) = \mathbf{K} \cdot d\mathbf{r} \]

\[ dS_{\text{temporal part}} = -\mathbf{Edr} \quad dS_{\text{spatial part}} = \mathbf{K} \cdot d\mathbf{r} \]

4-Momentum subpart \( P_{\text{rotational}} \) due to Rotation (not Angular Momentum, which is \( \mathbf{M} = \mathbf{R} \times \mathbf{p} \))

\[ P_{\text{rotational}} = m, U_{\text{rotational}} = E_{\text{rotational}} \quad m, \gamma(c, U_{\text{rotational}}) = m(c, U_{\text{rotational}}) \]

\[ \Delta \Phi_{\text{rotational}} \quad \text{The rotational part of the Relativistic Phase} \]

\[ = \mathfrak{f} \mathbf{d}\Phi_{\text{rotational}} \quad \text{in differential Relativistic Phase form} \]

\[ = (1/h) \mathfrak{f} \mathbf{d}\Phi_{\text{rotational}} \quad \text{in differential Relativistic Action form} \{ \Phi_{\text{action}} = h\Phi_\text{phase} \} \quad \text{from} \{ \mathbf{P} = h\mathbf{K} \} \]

\[ = (1/h) \mathbf{p}_{\text{rotational}} dr \quad \text{in 3-momentum form} \]

\[ = (1/h) m, v_{\perp} dr \quad \text{in 3-tangential-velocity form} \]

\[ = (1/h) \mathbf{m}(\Omega \times r) \cdot dr \quad \text{by Vector Triple Product Rule} \]

\[ = (m/h) \mathbf{K} \cdot (r \times dr) \Omega \quad \text{assume} \Omega \text{ is constant} \]

\[ = (2m/h) \mathbf{A} \quad \text{Def. of Area of closed planar curve} \quad [\text{kg}]*[1/(\text{J·s})]*[\text{rad}]*[\text{m}] = \{\text{rad}\} \]

\[ = \text{Sagnac Effect (single-beam full-circle or split dual-beam half-circle, matter-wave form} \{m_0 > 0\} \)

\[ = (2m/c^2) \Omega \quad \{ \text{mc} = h\omega \} \quad \text{particle-wave duality} \{ \gamma/c \} \]

\[ = (4\pi/c^2) \Omega \quad \text{[angular relation]} \]

\[ = (4\pi/2c) \Omega \quad \text{[wavelength:frequency relation for photons]} \quad \{	ext{[m]}^4* [s/m]*[\text{rad}]/[s]*[\text{m}] = \{\text{rad}\} \]

\[ = \text{Sagnac Effect (single-beam full-circle or split dual-beam half-circle, photonic form} \{m_0 = 0\} \)

\[ = \text{Often seen as} (4m/h) \Omega \quad \{\text{8π/2c} \Omega \} \quad \text{when using split dual-beam full-circle, which gives (2×) the phase shift} \]
SR Aharonov-Bohm: Aharonov-Casher Effect, Measuring Absolute EM Potentials, using 4D Tensors (esp. 4-Vectors):

4-Vectors = 4D (1,0)-Tensors Properties

4-Position \( \mathbf{R} = [ct, \mathbf{r}] \)

4-Differential \( d\mathbf{R} = d[ct, \mathbf{r}] \)

4-Velocity \( \mathbf{U} = \gamma \mathbf{c} + \mathbf{u} \)

4-Momentum \( \mathbf{P} = \frac{m}{c} \mathbf{P} \)

4-WaveVector \( \mathbf{K} = \frac{\omega}{c} \mathbf{k} \)

4-TotalMomentum \( \mathbf{P}_T = \mathbf{P} + \mathbf{P}_EM \)

4-VectorPotential \( \mathbf{A} = \mathbf{A}^e + \mathbf{A}^m \)

4-PotentialMomentum \( \mathbf{Q} = \mathbf{Q}^e + \mathbf{Q}^m \)

\( \mathbf{P} = (E/c, \mathbf{p}) = h\mathbf{K} = h (\omega/c, \mathbf{k} = \omega \mathbf{n}/v_{\text{phase}}) \)

Action: Phase Relation \( \partial \Phi = -\int d\Phi \)

\( S_{\text{action}} = h\Phi_{\text{phase}} \)

\( \Phi \) Lorentz Scalar Invariant Phase \( \Phi \) (additive subparts):

4-TotalWaveVector = -4-Gradient[Phase] \( \{ \mathbf{K}_T = -\hat{\mathbf{e}}[\Phi_{\text{phase,total}}] \} \)

Action S (relativistic Invariant, Lorentz Scalar Product):

\( \mathbf{P} \cdot d\mathbf{R} \)

\( dS_{\text{temporal part}} = -Edt \)

\( dS_{\text{spatial part}} = p \cdot dr \)

For the AB Effect, examine:

\( \mathbf{P}_T = \mathbf{P} + \mathbf{P}_{EM} = \mathbf{q} \mathbf{A}_{EM} \)

\( \mathbf{K}_T = K_{\text{dyn}} + K_{AB} \)

The Aharonov-Casher is similar, except using point dipoles instead of point charges

Mixed (3-vector) part: \( \Delta \Phi_{AC} = (1/4\pi\epsilon_0) \oint (\mathbf{e} \times \mathbf{m}) \cdot d\mathbf{r} \) {AC Effect, EM magnetic-dipole \( \mathbf{m} \)}

Magnetic Flux \( \Phi_B = \oint (\mathbf{a}_{EM} \cdot d\mathbf{r}) \)

\( \Delta \Phi_{AB} = (q\Phi_B) \) {AB Magnetic Effect}
Gravitationally-induced neutron interference, Measuring Relative Gravity Potential, using 4D Tensors (esp. 4-Vectors):

4-Vectors = 4D (1,0)-Tensors  

Properties & Relations

(+) metric signature : \((cdt)^2 = g_{\mu\nu}dx^\mu dx^\nu\)

4-Differential \(dX = dx^i = (cdt, dx)\)  

[m]

4-Velocity \(U^i = U^\mu = \gamma (c,u)\)  

[m/s] \(\gamma = 1/\sqrt{[1 - (u\cdot u/c)^2]}\) in regular SR

4-WaveVector \(K^\mu = (\omega/c,k)\)  

[rad/m] \(\omega = mc/\sqrt{(c^2-m^2/c^2)}\)

4-Momentum \(P^\mu = (mc,p)\)  

[kg m/s]

GR “Weak-Field” limiting case... or alternatively viewed as a small perturbation field \((h_{\mu\nu})\) on the SR Minkowski Metric \((\eta_{\mu\nu})\):

Space-Time Metric \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\) : \(\eta_{\mu\nu} = \text{Diag}[-1,-1,-1,-1]\)  

\(h_{\mu\nu} = (2\phi/c)\delta_{\mu\nu}\) : Gravity potential \((\phi)\)

\[cd\tau = \sqrt{[g_{\mu\nu}dx^\mu dx^\nu]} \quad \gamma = \frac{1}{\sqrt{[1 - (u\cdot u/c)^2]}}\]

\[t = \sqrt{[1-2\phi/c \omega^2 - (1/c^2)(1 - (m/c)^2)]} dt \]

So, “Weak-Gravity” Lorentz Factor \(\gamma_{\text{WeakGrav}} = 1/\sqrt{[1 + 2\phi/c^2 - u\cdot u/c^2]}\) and the Metric effectively is: \(g_{\mu\nu} = (1+2\phi/c^2)\) & \(g_{\mu\nu} = (-1 - 2\phi/c^2)\) : Gravity potential \((\phi)\)

4-WaveVector: K Lorentz Invariant

\(K^\mu = (\omega/c,k)\)

\(U^\mu = (u^\mu/c, u)\)

Assume \(U = U_0\): (1+nx)^2 \(\sim (1+nx)\) for \(|nx| << 1\)

Assume that angular frequency \((\omega)\), and hence energy \((E = h\omega)\) of particle-wave, remains unchanged \((\omega = \omega')\) or \((E = h\omega = h\omega') = E'\). This is apparently a condition of neutrons having essentially elastic collisions with the reflectors.

Phase Difference: \(\Delta \phi = -(m/c^2) g_{\mu\nu} dx^\mu dx^\nu\)

This matches the COW (R. Collela, A.W. Overhauser, S.A. Werner) Gravitationally-Induced Quantum Interference result. This is also a micro-scale test of the Equivalence Principle.

Note: Binomial Approximation \((1+x)^n \sim (1+nx)\) for \(|nx| << 1\)

Note: By Green’s Theorem, I believe the shape of the simple-closed-path-area is independent of the result.
To recap, the SR 4-Vectors have some very simple, fundamental, and Invariant Lorentz Scalar relations between one another:

4-Momentum \( P = P^\mu = (E/c, p) = (m, U) \)

4-WaveVector \( K = K^\mu = (\omega/c, k) = (1/\hbar)P \)

4-Gradient \( \partial = \partial^\mu = (\partial_t/c, \nabla) = (-i\hbar)K \)

The main point of work above is to show that the last two relations, \( \{ K = (1/\hbar)P \} \) and \( \{ \partial = -iK \} \), alternately \( \{ P = \hbar K \} \) and \( \{ K = i\partial \} \), are of the same character as the other relations in this group. They are derivable from SR and/or purely-mathematical principles, and do not require quantum axioms for their existence. These will then lead to wave-particle duality and other derived quantum “axioms”.

The Lorentz Scalar Products of these SR-related 4-Vectors gives the following chain of Invariants:

\((R \cdot R) = (cr)^2\)

\((U \cdot U) = (c)^2\)

\((P \cdot P) = (m, c)^2\)

\((K \cdot K) = (m, c/h)^2\)

\((\partial \cdot \partial) = (m, c/h)^2\)

\((\partial \cdot \partial) = (1/c)^2\)

\((\partial \partial) = (\omega/c)^2\)

: \((h/m, c) = (c/\omega) = \lambda\) is the reduced Compton wavelength \([\text{length} = m]\)

: The fundamental quantum Klein-Gordon (KG) RQM wave relation: [SR→QM]

Each step is a logical progression, taking into account the simple relation between each of these SR 4-Vectors.

In the same way that the Relativistic 4D Euler-Lagrange Relation \((U \partial^\mu R) = (d/dt)(\partial R) = \partial R\) [itself a variation of \((d/dt)[R] = U\)] implies that there can exist a Lagrangian function \(L\) that solves it, the KG relation \((\partial^\mu \partial^\nu) - (m, c/h)^2 = -(m, c/h)^2\) implies that there can exist a “wavefunction” \(\Psi\) which solves it. One does not need a presupposed quantum axiom. The Klein-Gordon relation gives a relativistic, 2nd order, linear Partial Differential Equation (PDE). The fact that it is a linear PDE leads to the principle of quantum superposition. The standard Schrödinger quantum wave equation \(\{ (i\hbar \partial_t) = [(V) - (h/(2m)) \} \) is the non-relativistic \((\nu<<c)\) limit-case of the KG relativistic quantum wave equation, which continues to show superposition.

The Klein-Gordon Eqn. \((\partial^\mu \partial^\mu - (im/c)\hbar^2 = 0)\) is itself the Relativistic Quantum (RQM) Equation for spin\=0 particles (4-Scalars).

Factoring the KG Eqn. \((\partial^\mu + im/c)\hbar^2 (\partial^\mu - im/c) = 0\) leads to the RQM Dirac Eqn. for spin\=1/2 particles (4-Dirac Particles).

Applying the KG Eqn. to a 4-Vector field leads to the RQM Proca Eqn. for spin\=1 particles (4-Vector Particles).

Taking the low-velocity-limit \((\nu<<c)\) of the KG leads to the standard QM non-relativistic Schrödinger Eqn., for spin\=0 (4-Scalar).

The Klein-Gordon relation gives a relativistic, 2nd order, linear Partial Differential Equation (PDE). The fact that it is a linear PDE leads to the principle of quantum superposition. The standard Schrödinger quantum wave equation \(\{ (i\hbar \partial_t) = [(V) - (h/(2m)) \} \) is the non-relativistic \((\nu<<c)\) limit-case of the KG relativistic quantum wave equation, which continues to show superposition.

The standard Schrödinger QM Relations derived from SR:

Again, we examine these SR 4-Vector relations derived above… and by simply combining them...

\( P = \hbar K \) (a relation which is entirely empirical, based on just SR arguments, shown above)

\( K = i\partial \) (which is a relation for complex plane-waves, used in classical EM)

\( P = i\hbar \partial = (E/c, p) = i\hbar (\partial_t/c, \nabla) \)

The temporal part \(\{E = \hbar \partial_t = i\hbar \partial_\tau\} \) gives unitary QM time evolution.

The spatial part \(\{p = -i\hbar \nabla\} \) gives the QM momentum operator.

These are the main quantum relations used in standard QM calculations, as well as in RQM.

Just as a note, to emphasize again the SR origin of the 4-Vectors:

The 4-Momentum \( P \) is used in purely-relativistic particle collision calculations.

The 4-WaveVector \( K \) is used in purely-relativistic Doppler effect calculations, and in classical EM wave descriptions.

Both \( P \) and \( K \) are used in the relativistic Compton effect photon-electron scattering calculations.

The 4-Gradient \( \partial \) is used in several purely-relativistic settings: charge conservation \((\partial^\mu J^\mu = 0)\), particle # conservation \((\partial^\nu N^\nu = 0)\)

Lorentz EM Gauge \((\partial^\mu A^\mu = 0)\), invariant d’Alembertian \((\partial^\nu \partial_\nu)\), proper time derivative \((U^\nu \partial_\nu)\), Euler-Lagrange Equation \((d/dt)[\partial R] = \partial R\)

Minkowski Metric \(\delta^\mu [R^\mu] = \eta^\nu\), SR SpaceTime Dimension \((\partial^\mu R) = 4\), Lorentz-Transform \(\partial[R^\nu] = \partial R^\nu/\partial R^\mu = N^\nu\) etc.

These facts show that the tensorial 4-Vectors and their relations are from SR, and not QM axioms.
Non-zero SR→QM Commutation Relation between 3-position $x = x'$ and 3-momentum $p = p'$:

4-Position $R = R^t = (ct, \mathbf{r}) = (ct, x, y, z) = X = X^a$

4-Gradient $\delta^a = (\delta_t/\partial t, \delta_x/\partial x, \delta_y/\partial y, \delta_z/\partial z) = (\delta_t/\partial c, \delta_x/\partial c, \delta_y/\partial c, \delta_z/\partial c) = \partial/\partial X_\mu$

Let \{ f \} be an arbitrary SR function.

$X[f] = Xf$ \& $\partial[f] = \partial[f]$ are the primitive relations. The following is basic calculus:

$X[\partial[f]] = X[\partial[f]]$: Apply 1st primitive rule rightward, with 2nd primitive rule as the argument

$\partial[X[f]] = \partial[X[f] + X[\partial[f]]]$: Calculus Product Rule, $d[ab] = d[a]*b + a*d[b]$, applies to differential & partial

$\partial[X[f]] - X[\partial[f]] = \partial[X[f]]$: Rearrange equation

$\partial[X[f]] - X[\partial[f]] = \partial[X[f]]$: Apply 1st primitive rule leftward to form a commutation relation

The 4-Vector parts of the left-hand side \{ $\partial[X[f]] - X[\partial[f]]$ \} form a commutator \{ [$A, B]f = A[B[f]] - B[A[f]]$ \}:

$\partial[X[f]] - X[\partial[f]] = [\partial, X]f$

The 4-Vector parts of the righthand side \{ $\partial[X]$ \} can be computed, which leads to a tensor product or dyadic with 2 indices:

$\partial[X] = \partial^a[X'] = (\partial_t/\partial c, \mathbf{\nabla})(ct, \mathbf{r}) = (\partial_t/\partial c, -\mathbf{\nabla}, \partial_x/\partial c, \partial_y/\partial c, \partial_z/\partial c)(ct, x, y, z) = \text{Diag}[1, -1, -1, -1]_{\text{Cartesian}} = \eta^{\mu\nu} = \text{Minkowski Metric}$

And since \{ f \} was an arbitrary SR function, we can remove it (or set it to unity), which leaves the functional form:

$[\partial, X] = [\partial^a, X^\prime] = \eta^{\mu\nu} = \text{Minkowski Metric}$ \hspace{1cm} [dimensionless units]

At this point, we have established purely mathematically, that there exists in SR a non-zero commutation relation between the SR 4-Gradient $\partial$ and SR 4-Position $X$.

Note also that \{ $X[f] = Xf$ \} does not actually say that $X$ is necessarily an operator. It just says that \{ an X next to an f \} = \{ X times f \}. $X$ could be an operator or just a numerical vector.

The 4-Gradient $\partial$ is definitely an operator, because it is already an operator:function in pure SR, and uses basic calculus rules.

Now, using these 4-Vectors and the relations between them derived from SR above:

4-Momentum $P = P^a = (E/c, \mathbf{p}) = (p^t, \mathbf{p}) = \hbar \mathbf{K}$

4-Wavevector $K = K^a = (\omega/c, \mathbf{k}) = (k^0, \mathbf{k}) = i\delta$

4-Gradient $\partial = (\partial_t/c, \mathbf{\nabla}) = (\partial^0, \partial^\mu) = \partial/\partial X_\mu$

$[\partial, X] = [\partial^a, X^\prime] = \eta^{\mu\nu}$

$[i\partial, X] = [i\partial^a, X^\prime] = i\eta^{\mu\nu}$

$[K, X] = [k^0, X^\prime] = -i\eta^{0\nu}$

$[\hbar K, X] = [\hbar k^0, X^\prime] = i\hbar \eta^{0\nu}$

$[P, X] = [p^0, X^\prime] = i\hbar \eta^{0\nu}$

$[X, P] = [X^\prime, P^0] = -i\hbar \eta^{0\nu}$

This is a major result of [SR→QM].

The temporal part $[x^0, p^0] = [ct, E/c] = [t, E] = -i\hbar \eta^{0\nu}$ is the "oft-misunderstood" time-energy commutation relation. The spatial part $[x^\mu, p^\mu] = i\hbar \delta^{\mu\nu}$ is the standard QM Canonical Commutation Relation, derived, not an axiom.

The mixed parts $[x^0, p^\mu] = [x^\mu, p^0] = \eta^{0\mu} = \eta^{\mu0} = 0$, meaning these parts commute normally, as expected classically.

Similar 4-Vector arguments lead to the standard angular-momentum quantum commutation relations via SR 4-AngularMomentum $M^\mu\nu = X^\mu \times P^\nu$.

The entire Poincaré Algebra (Lie Algebra of the Poincaré Group) can be generated in this fashion.

$P^\nu$ is generator of SpaceTime-Translations ($\Delta X^\nu$). $M^\mu\nu$ is the generator of Lorentz-Transformations ($\Lambda^\mu_\nu$). $\eta^{\mu\nu}$ is the Minkowski Metric.

Canonical (Momentum, Position):

$[P^\mu, X^\nu] = i\hbar \eta^{\mu\nu}$

$[P^\mu, X^\nu] = i\hbar \eta^{\mu\nu}$

$[P^\mu, P^\nu] = 0$ from [partial, partial] commutations as $[\partial^a, \partial^b] = 0$

$[P^\mu, P^\nu] = 0$ from partials commuting $[\partial^a, \partial^b] = 0$

$[M^\mu\nu, P^\rho] = i\hbar (\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu)$

$[M^\mu\nu, P^\rho] = i\hbar (\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu)$

$[M^\mu\nu, M^\rho_\sigma] = i\hbar (\eta^{\mu\rho} M^\nu_\sigma + \eta^{\nu\rho} M^\mu_\sigma + \eta^{\nu\sigma} M^\mu_\rho + \eta^{\mu\sigma} M^\nu_\rho)$

Also $[X^\mu, X^\nu] = 0$ because just numbers

The (i) and (h) again come from SR. The algebra is all real and overall dimensionless when using only \{ $X$ and $\partial$ \} in the definitions.

Likewise, the general mathematical uncertainty relations, \{ $\sigma_\mu \sigma_\nu \geq (1/2) \langle [A, B]\rangle$ \}, based on commutation relations, lead to the standard physical quantum Heisenberg uncertainty relations. Also note that the commutator order of operations is in accord with SR causality conditions. While spacelike-separated events \{ here \} and \{ there \} may occur in any temporal order, all observers will see the same temporal order of timelike-separated events. Thus, for time-like separations, if measurement-event $A$ occurs temporally before measurement-event $B$, then this would be written in operator notation as: $|\Psi'\rangle = A|\Psi\rangle$ then $|\Psi''\rangle = B|\Psi'\rangle = BA|\Psi\rangle$.

The operator order shows the timelike-separated order of measurement-events.

Due to non-zero commutation relations, $AB|\Psi\rangle$ would give a different result.
Using Green’s Vector Identity to establish a Conserved Current (could be # [dust or probability] or charged):
Consider the following purely mathematical argument:
\[ \hat{\nabla} \cdot (f \hat{E}[g] - \hat{A}[g] f) = f \hat{\nabla} \cdot \hat{E}[g] - \hat{A}[f] \hat{\nabla} \cdot f \]
with \( f \) and \( g \) as SR functions

Proof of the 4-Divergence relation:

\[ \hat{\nabla} \cdot (f \hat{E}[g] - \hat{A}[g] f) = \hat{\nabla} \cdot (f \hat{E}[g]) - \hat{\nabla} \cdot (\hat{A}[g] f) \]
\[ = (f \hat{\nabla} \cdot \hat{E}[g] + \hat{E}[f] \hat{\nabla} \cdot g) - (\hat{A}[f] \hat{\nabla} \cdot g + \hat{\nabla} \cdot \hat{A}[g] f) \]
\[ = f \hat{\nabla} \cdot \hat{E}[g] - \hat{A}[g] \hat{\nabla} \cdot f \]

We can also multiply this by a constant Lorentz Invariant Scalar Constant \((s)\), for dimensional-unit purposes.
\[ s (f \hat{\nabla} \cdot \hat{E}[g] - \hat{A}[g] \hat{\nabla} \cdot f) = s \hat{\nabla} \cdot [s (f \hat{E}[g] - \hat{A}[g] f)] = \hat{\nabla} \cdot J \]

Thus there mathematically exists a 4-Current(Density) \( J \) derivable from the SR d’Alembertian \((\hat{\nabla} \cdot \hat{\nabla})\).

Now, as applied to SR physics... Start with the Klein-Gordon relation derived above from the Lorentz Scalar Product:
\[ \hat{\nabla} \cdot \hat{\nabla} (-i/2m \partial_t \psi) = (\partial_t^2 - \partial^2 \psi) = -(i/2m \partial_t \psi) \]
and which also solves the Klein-Gordon relation, and gives unitary evolution and conservation of probability.

For generality, choose the SR 4-Vector relation \((\hat{\nabla} = -i\hat{K})\) as before with a complex planewave function \( g = (a) e^{-i(K \cdot X)} = \psi \), and choose \( f = g^* = (a^*) e^{i(K \cdot X)} = \psi^* \) as its complex conjugate.

At this point, we can choose \( s = (i/2m) = (i/2m_0) \), which is Lorentz Scalar Invariant, in order to make the probability have [dimensionless units = #] and be normalized to unity in the rest case. In 4-Vector form this gives probability-densities \( \rho_{\text{prob}} \) \([\text{#/m}^3]\).

\[ \rho_{\text{prob}} = (\rho_{\text{prob}}C_{\text{prob}}) = (i/2m_0)(\psi^* \hat{\phi}[\psi]) - \hat{\phi}[\psi^* \psi] = (i/2m_0)(\psi^* \hat{\phi}[\psi] - \hat{\phi}[\psi^* \psi]) \]

Examine the temporal component, the Relativistic Probability Density
\[ \rho_{\text{prob}} = (i/2m_0)(\psi^* \hat{\phi}[\psi] - \hat{\phi}[\psi^* \psi]) = (i/2m_0)(\psi^* \hat{\phi}[\psi] - \hat{\phi}[\psi^* \psi]) \]

Assume wave solution in following general form:
\[ \psi = A f[k] e^{i\omega t} \]
\[ \psi^* = A^* f[k] e^{-i\omega t} \]
then
\[ \rho_{\text{prob}} = (i/2m_0)(\psi^* \hat{\phi}[\psi] - \hat{\phi}[\psi^* \psi]) \]
Finally, multiply by charge \((q)\) to get standard SR EM 4-CurrentDensity = 4-ChargeFlux = \( J = (pc_j) = q\rho_{\text{prob}} = q(\rho_{\text{prob}}C_{\text{prob}}) \)
One can generalize (in this case) to include the effects of an EM VectorPotential \( A = (q/c_0) \)
4-ProbabilityCurrentDensity \( J_{\text{prob}} = (\rho_{\text{prob}}C_{\text{prob}}) = (i/2m_0)(\psi^* \hat{\phi}[\psi] - \hat{\phi}[\psi^* \psi]) + (q/m_0)(\psi^* \psi)A \)
Examine the temporal component:

\[ \rho_{\text{prob}} = \frac{\hbar}{2m} |\psi^*\rangle \langle \psi| \nabla^2 - \frac{\nabla}{\hbar} \psi^* \psi + (q/m_c)(\psi^*\psi)(v/c^2) \]

Typically, particle EM potential energy (\(q_\phi\)) is much less than particle rest energy (\(E_r\)), else it could generate new particles. So, take (\(q_\phi, << E_r\)), which gives the EM factor (\(q_\phi/E_r\)) \(\approx 0\)

Now, taking the low-velocity limit (\(\gamma \rightarrow 1\)), \(\rho_{\text{prob}} = \gamma (1 + 0)|\psi^*\rangle \langle \psi| \), \(\rho_{\text{prob}} \rightarrow \rho_{\text{prob}}(c)\) for \(|\psi| < < c\)

This is why the \{non-positive-definite\} probabilities and \{probabilities \(> 1\)} in the RQM Klein-Gordon equation puzzled physicists, and is the reason why one must regard the probabilities as charge conservation instead.

The original definition from SR is Continuity of Worldlines, (\(\partial J_{\text{prob}}/\partial t\)), which gives the EM factor (\(q_\phi\cdot \gamma\)) for \(|\psi| < < c\) LRQM limit-case.

Only the non-EM rest version has Total (\(\rho_{\text{prob}}\)) = Sum\([\langle \psi^*\psi \rangle] = 1\).

It is not a fundamental property, it is an emergent property which is valid only in the NRQM limit.

We now multiply by EM charge (\(q\)) to get:

4-“Charge”CurrentDensity \(J = (pc,i) = qJ_{\text{prob}} = q(q_{\text{prob}}c,J_{\text{prob}})\), which is the standard SR EM 4-CurrentDensity

**Comparison of SR 4-(Dust)NumberFlux \(N\) to QM 4-ProbabilityCurrent \(J_{\text{prob}}\), the same 4-Vector:**

**SR 4-Vector (properties & relations):**

4-Velocity \(U = \gamma(c,u)\)

4-Gradient \(\hat{\nabla} = (\partial/c, \nabla) = (\partial/\partial R)\)

4-(Dust)NumberFlux \(N = (nc,n=nu) = n(c,u) = n_\gamma(c,u) = n_U = J/q\)

4-Current(Density) = 4-ChargeCurrent \(J = (pc,i) = p(c,u) = p_\gamma(c,u) = qn_\gamma(c,u) = p_U = qn_U = qN\)

4-ProbabilityCurrentDensity \(J_{\text{prob}} = (\rho_{\text{prob}}c,J_{\text{prob}}=\rho_{\text{prob}}u) = (ih/2m,)(\psi^*\hat{\nabla}|\psi|\psi^*\psi) = \rho_{\text{prob}}c,\psi = (\rho_{\text{prob}}c,J_{\text{prob}})\) = \(\#/(m^3\cdot s) = \#/(m^3\cdot s) = \#-flux\)

Particle # Conservation (\(\partial N/c\)) = 0 \(\rightarrow\) Closed System Total Particle # [\(N\)] = constant

Charge Conservation \(\partial J/c\) = 0 \(\rightarrow\) Closed System Total Charge [\(Q\)] = constant

Probability Conservation (\(\partial J_{\text{prob}}/c\)) = 0 \(\rightarrow\) Closed System Total Probability Sum\([\langle \psi^*\psi \rangle]\) = 1 = constant

**SR \(\rightarrow\) RQM \(\rightarrow\) CM Classical Correspondence Principle:**

In SR, one finds the Newtonian classical limiting-case approximation by using \(|v| < < c\). In QM, there have been a variety of approaches to the Newtonian classical limiting-case approximation, including the idea of \{number of particles \(>> 1\}\), the physics action \(\{S >> \hbar\}\), divergence small compared to system magnitude \(\langle \hbar |\nabla \cdot p| < < (p\cdot p)\rangle\), etc. In the standard view of the theories of relativity and quantum mechanics, it is interesting to speculate on how the two “different” theories “conspire” to end up at the same classical mechanics physics as an approximation. However, in the SRQM view, this difficulty disappears. SR leads to RQM via the approach that has been shown. RQM then goes to CM as a limiting-case approximation by using \(|v| < < c\). CM then goes to CM as a limiting-case in its own manner. There is a single chain of relationships, rather than two different theories “amazingly” approaching the same classical limit-case.

\[ GR \rightarrow \{\text{limit-case } g^\mu_{\nu} \rightarrow \eta^\alpha_{\beta}\} \rightarrow \rightarrow RQM \rightarrow \{\text{limit-case } v |v| < c\} \rightarrow QM \rightarrow \{\text{limit-case } h \nabla \cdot p| < < (p\cdot p)\rangle\} \rightarrow (\text{EM & CM})\]
Standard Postulates of QM (i.e. the mathematical structure of quantum mechanics), which SRQM must provide:

I) Description of the state of a system: The state of an isolated physical system is represented, at a fixed time \( t \), by a state vector |\( \psi \rangle \) belonging to a Hilbert space \( \mathcal{H} \) called the state space: a vector space with an inner product operation (measurement & orthogonality).

II) Description of physical quantities: Every measurable physical quantity \( A \) is described by a Hermitian operator \( \hat{A} \) acting in the state space \( \mathcal{H} \). This operator is an observable, meaning that its eigenvectors form a basis for \( \mathcal{H} \).

III) Measurement of physical quantities: The result of measuring a physical quantity \( A \) must be one of the eigenvalues of the corresponding observable \( \hat{A} \).

IV) Measurement of physical quantities: When the physical quantity \( A \) is measured on a system in a normalized state |\( \psi \rangle \), the probability of obtaining an eigenvalue (denoted \( a_n \) for discrete spectra and \( \alpha \) for continuous spectra) of the corresponding observable \( A \) is given by amplitude squared of the appropriate wave function (projection onto corresponding eigenvector).

V) Effect of measurement on the state: If the measurement of the physical quantity \( A \) on the system in the state |\( \psi \rangle \) gives the result \( a_n \), then the state of the system immediately after the measurement is the normalized projection of |\( \psi \rangle \) onto the eigensubspace associated with \( a_n \).

VI) Time evolution of a system: The time evolution of the state vector |\( \psi(t) \rangle \) is governed by the Schrödinger relation, where \( \mathcal{H}[t] \) is the observable associated with the total energy of the system (the Hamiltonian): \( \imath \hbar \partial \psi(t) = \mathcal{H}[t] \psi(t) \).

Analysis of the Koopman-von Neumann (KvN) formalism, a Hilbert Space framework which can give QM or CM:

The idea that Hilbert Space requires a quantum axiom is disproved by the Koopman–von Neumann formulation of classical mechanics, in which Hilbert Space mathematical formulation is successfully applied and results the classical Liouville equation. This shows that the Hilbert Space framework is purely mathematical and can be applied to either/both classical and quantum systems. The main difference between which system emerges is the commutation relation between position and momentum. In the classical case, one assumes a zero-valued commutation relation. In the quantum case, there is a non-zero commutation relation.

SR, as shown above, gives a non-zero commutation relation, thus leading naturally to the QM case. One instead uses a limiting-case approximation to go from QM to CM, in the same way that there is a limiting-case approximation to go from GR to SR, RQM to QM.

From Wikipedia (with some modifications):

Derivation starting from operator axioms: It is possible to start from mathematical operator postulates, similar to the Hilbert space axioms of quantum mechanics, and derive the equation of motion by specifying how expectation values evolve.

The relevant axioms are that as in QM: (i) the states of a system are represented by normalized vectors of a complex Hilbert space \( \mathcal{H} \), and the observables are given by self-adjoint operators acting on that space, (ii) the expectation value of an observable is obtained in the manner as the expectation value in quantum mechanics, (iii) the probabilities of measuring certain values of some observables are calculated by the Born rule, and (iv) the state space of a composite system is the tensor product of the subsystem’s spaces.

Mathematical form of the operator axioms: The above axioms (i) to (iv), written in the \langle bra|\rangle\langle ket|\rangle notation, are

(i) \( \langle \psi(t) | \psi(t) \rangle = 1 \)

(ii) The expectation value of an observable \( \hat{A} \) at time \( t \) is \( \langle \hat{A} (t) \rangle = \langle \Psi(t) | \hat{A} |\Psi(t) \rangle \)

(iii) The probability that a measurement of an observable \( \hat{A} \) at time \( t \) yields \( A \) is \( \langle A |\Psi(t) \rangle^2 \), where \( \hat{A} | A \rangle = A | A \rangle \). (This axiom is an analogue of the Born rule in quantum mechanics.)

(iv) (see Tensor product of Hilbert spaces).

These axioms allow us to recover the formalism of both classical and quantum mechanics. Specifically, under the assumption that the classical position and momentum operators commute, the Liouville equation for the KvN wavefunction is recovered from averaged Newton's laws of motion. However, if the coordinate and momentum obey the canonical commutation relation, the Schrödinger equation of quantum mechanics is obtained.

Measurements: In the Hilbert space and operator formulation of classical mechanics, the Koopman von Neumann–wavefunction takes the form of a superposition of eigenstates, and measurement collapses the KvN wavefunction to the eigenstate which is associated the measurement result, in analogy to the wave function collapse of quantum mechanics.

Thus, Hilbert Space is not a “quantum” axiom. Instead, Hilbert Space and its properties are a set of “mathematical” axioms and formulations, independent of the physics. The formulation can be adapted by CM, and along with its zero commutation axiom, give classical Newtonian results. The formulation can be adapted by QM, and along with its non-zero commutation axiom, give quantum results. The formulation can be adapted by SRQM, and along with its non-zero commutation axiom, give relativistic quantum results.

That the Hilbert Space axioms can be used in SRQM is the result of SRQM providing a linear PDE \( \left( \partial \alpha \partial \right) = -(m_e c^2 / \hbar^2) \), which may be solved by generic mathematical Hilbert Space methods. Therefore, they are not quantum axioms, but emergent QM principles.
There is complete (+/-) symmetry, which agrees with all known experiments with NormalMatter = AntiMatter to-date.
Grouped and ordered by the trace values, one gets:

<table>
<thead>
<tr>
<th>Discrete Normal Matter (NM) Lorentz Transform Type</th>
<th>Trace</th>
<th>Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>NM-Minkowksi 4D Identity +I₄</td>
<td>Tr = +4</td>
<td>Det = +1 Proper</td>
</tr>
<tr>
<td>AM-Flip-xyz=AM-Combo(PT)=AM-NegateIdentity=AM-NegateCharge</td>
<td>Tr = +2</td>
<td>Det = -1 Improper</td>
</tr>
<tr>
<td>NM-Flip-xy=NM-Flip-xz=NM-Flip-yz=NM-Flip-z</td>
<td>Tr = 0</td>
<td>Det = +1 Proper</td>
</tr>
<tr>
<td>NM-Flip-xy=AM-ParityInverse</td>
<td>Tr = -2</td>
<td>Det = -1 Improper</td>
</tr>
<tr>
<td>NM-Flip-xyz=AM-Flip-t ,AM-Flip-x, AM-Flip-y, AM-Flip-z</td>
<td>Tr = -4</td>
<td>Det = +1 Proper</td>
</tr>
</tbody>
</table>

This clearly shows that Combo(PT) Transform is equivalent to a (C)harge Transform, which flips NormalMatter ← → AntiMatter. Also, this (C)harge Transform is Proper, with a determinant of +1, the same as the Boost, Rotation, Flip-TwoCoords Transforms, which means that it can occur in reality. Overall, this is the source of (CPT) Symmetry.

Conclusion:
Using standard Tensor calculus (especially the 4-Vector formulations) of Einstein-Minkowski Spacetime it is shown that foundational features of spacetime common to both Special Relativity (SR) and Quantum Mechanics (QM) exist. Many fundamental physical relations are encoded in the simple tensorial rules of 4-Vectors and their Lorentz Scalar Products.

\( (\hbar) \) is shown to be an empirically-measurable fundamental constant and a Lorentz Invariant Scalar, just like LightSpeed (c).

The 4-Vector relations \( P = m_o U \) \{particle view \[ \cdots \] \}, \( K = (o_o/c^2)U \) \{wave view \[ \cdots \] \}, \( P = hK \) \{particle-wave view \[ \cdots \] \}, are all shown to be isomorphic in the sense that all are derivable from SR and the rules of Tensor calculus.

The mathematical relation \( K = i\partial \) or \( \partial = -iK \) and the existence of a complex wavefunction \( \Psi \) is shown applicable to all types of waves: classical, quantum, relativistic, EM, purely-mathematical. \( K \) is a solution of the 4D Invariant d’Alembertian Wave Eqn. \((\partial^\dddot\cdot\partial^\dddot)\).

All waves are described by 4D Tensor amplitudes \( a = A_A, A_{\mu}, \text{etc.} \) and the Lorentz Scalar Product function \( \epsilon^{\mu\nu} (K^\dddot \cdot X) \) propagator.

The combination of these relations lead to a KG relativistic quantum wave relation \((\partial^\dddot\cdot\partial) = (im_o c/\hbar)^2 = -(m_o/c)^2 \) and to the 4-Vector form of the standard Schrödinger relations \( P = i\hbar \partial \), which give \{\( E = i\hbar \partial\)} and \{\( p = -i\hbar \nabla\)}.

There exists a non-zero commutation relation in SR: \([X^\mu, P^\nu] = -i\hbar \eta^\mu\nu\). which gives standard canonical QM commutation \([x^j, p^k] = i\hbar \delta^j_k\)

There exists a conserved current \( J_{\text{prob}} \) in SR, with \((\partial^\dddot\cdot J_{\text{prob}}) = 0\), based on Green’s vector identity applied to the KG relation.

The SR 4-(Dust)NumberFlux \( N \) appears to be equivalent to the RQM 4-ProbabilityCurrentDensity \( J_{\text{prob}} \).

The standard Born probability interpretation, \((\psi^* \psi) = \rho_{\text{prob}, o}\), emerges in the low-potential-energy and low-velocity limit.

\textbf{CPT} Symmetry emerges from an analysis of the mathematical properties of the Lorentz Transformations, particularly from the Invariant Tensor Trace \( \text{Tr} [A^\mu] \).

The correspondence principle of both SR and QM to Newtonian classical physics CM is discussed.

\textbf{One can derive the “axioms” of QM from the principles of SR, hence [SR→QM].}
Start with a few SR Physical 4-Vectors:

4-Position \( \mathbf{R} = (ct, \mathbf{r}) \)
4-Velocity \( \mathbf{U} = \gamma(c, \mathbf{u}) \)
4-Momentum \( \mathbf{P} = (E/c, \mathbf{p}) = (mc, \mathbf{p}) \)
4-Gradient \( \partial = \frac{\partial}{\partial t}/c - \nabla \)

Note the following relations between SR 4-Vectors:

\[
\mathbf{U} = dR/d\tau \\
\mathbf{P} = m\mathbf{U} = (E_0/c^2)\mathbf{U} \\
\mathbf{K} = \frac{1}{\hbar}\mathbf{P} = (\omega_0/c^2)\mathbf{U} \\
\partial = -im\mathbf{K} \\
\]

Form a chain of SR Lorentz Invariant Scalar Equations, based on those relations:

\[
R \cdot R = (c\tau)^2 \\
U \cdot U = (c)^2 \\
P \cdot P = (m/c)^2 = (E/c)^2 \\
K \cdot K = (m/c^2)^2 = (\omega/c)^2 \\
\partial \cdot \partial = -(m/c^2)^2 = -(\omega/c)^2 \\
\]

This is (RQM) = Relativistic Quantum Mechanics, derived from only:

5 of the Standard SR 4-Vectors
4 really simple empirical relations between them
1 SR rule for forming Lorentz Scalar Invariants, i.e. the Minkowski Metric \( \eta_{\mu\nu} \) which gives the Lorentz Scalar Product (\( \cdot \))

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4-Creativity \( \odot = (Music, Artwork) \)
4-Universality \( \infty = (Eternity, Infinity) \)
4-Origin \( O = (Now, Here) \)
Thus, this treatise explains (or at least gives a probable reason for) the following empirical observations:

Why GR works so well in its realm of applicability {massive large-scale systems}. Why QM works so well in its realm of applicability {micro-scale systems and special macroscopic systems, ex. superfluids}. i.e. The tangent space to GR curvature at any point is locally Minkowskian, and thus QM works for small volumes...

Why RQM explains physical effects that QM-without-SR cannot, and with greater accuracy those that basic QM can explain. Why attempts to “quantize gravity” fail {essentially, everyone has been trying to put the cart (QM) before the horse (GR)}. Why all attempts to modify GR keep conflicting with experimental data {because to-date, GR is apparently still fundamental}. Why QM works with SR as RQM, but not with GR {because QM is derivable from SR, hence a manifestation of SR rules}. In other words, the “special case” rules of QM are not something that can be imposed on GR, which is the “general” parent theory. How Minkowski Space, 4-Vectors, and Lorentz Invariants play vital roles in RQM, & give the SRQM Interpretation of QM.

Major clues from experiment, observation, and mathematics that is actually proven to be related to physical reality:
The components of 4-Vectors and 4-Tensors are experimentally measurable elements/properties of physical reality. Both General Relativity (GR) and Special Relativity (SR) have passed very stringent tests of multiple varieties. Relativistic Quantum Mechanics (RQM) and standard Quantum Mechanics (QM) have passed all tests within their realms of validity: RQM describes and explains phenomena that standard QM cannot. Mass **and** Spin are the Casimir Invariants of Poincaré Invariance, which comes from 4D SR Minkowski Space, not QM. Hence, neither mass nor spin require a quantum axiom for their existence, yet are the only “quantum numbers” of physical states.

To-date, there is no observational/experimental indication that quantum effects “alter” the fundamentals of either SR or GR. There have been no repeatable violations of Poincaré-Lorentz Invariance, nor of Local Position Invariance, nor of CPT Symmetry, nor of the Standard Model formulation (neutrino masses notwithstanding). This rules out many alternative gravity theories. In fact, in all known experiments where both SR/GR and QM are present, QM respects the principles of SR/GR, whereas SR/GR modify the results of QM, ex. Dirac RQM Wave Eqn. vs Schrödinger QM Wave Eqn. All tested quantum-level particles, atoms, isotopes, molecules, super-positions, spin-states, excited-states, etc. obey: GR's {Universality of Free-Fall & Equivalence Principle} and SR's { E = mc^2 & lightspeed (c) communication/signaling limit}. On the other hand, GR gravity *does* induce changes in quantum interference patterns and hence modifies QM. For instance, quantum-level atomic-clocks are used to measure gravitational “Doppler” red/blue-shift effects. i.e. GR gravitational frequency-shift (gravity time-dilation) alters atomic=quantum-level timing. Think about that for a moment...

While quantum entanglement is somewhat mysterious, there is no FTL-communication-with nor alteration-of distant particles. Getting a Stern-Gerlach “up” |here doesn’t cause the distant entangled-particle to suddenly start physically moving “down” |there. The universe appears to be both: Homogeneous = same all places [*] (SpaceTime-Translation ΔX = Symmetry) over displacement ΔX and Isotropic = same all directions (*) (Lorentz Λ = Symmetry) over {angle θ, hyperbolic angle φ}. The main Schrödinger relation is just a special case of complex plane waves on objects (4-Vectors) already connected in standard SR.

All Lorentz Scalar Product connections between SR 4-Vectors are Lorentz Invariants, and typically fundamental constants with, to-date, no consistent evidence of change over time during the age of the universe nor of varying value throughout spatial extent. CM typically uses a Phase Space description; QM typically uses a Hilbert Space description. However: CM can be done with a Hilbert Space description: see Koopman-von Neumann Classical Mechanics. QM can be done without a Hilbert Space description: see Phase Space Formulation of Quantum Mechanics. Hence, Hilbert space is mathematical: it and its associated properties do not require a quantum axiom for its existence. Particles of both CM and QM can be described by a wave-like theory: see the Hamilton-Jacobi Equations. Measurement of Planck's constant (h) can be done with experiments that do not need quantum theory for the measurement, just SR.

In other words, the actual measurement process uses empirical, non-QM-theory-dependent components {don't need Schrödinger eqn}. 50+ years of unsuccessful attempts to "quantize gravity".
To-date, no new particles found outside of the Standard Model by LHC or other experiments, just the expected SM Higgs Boson.

Other SRQM sources – most current versions can be found at SciRealm.org:
Main website: SRQM - QM from SR - Simple RoadMap (.html)
This Summary: http://scirealm.org/SRQM-Summary.pdf
Alternate discussion at: http://scirealm.org/SRQM.html
SRQM Flyer: http://scirealm.org/SRQM_Flyer.pdf
See: 4-Vectors & Lorentz Scalars Reference for more info on Four-Vectors (4-Vectors) in general
See: John's Online RPN Scientific Calculator, using Complex Math
See: SRQM - Online SR 4-Vector & Tensor Calculator

Trying out some math to get a term like “Weak-Gravity” Lorentz Factor $\gamma_{\text{WeakGrav}} = 1/\sqrt{1+2g\cdot z / c^2 - u \cdot u / c^2} = 1/\sqrt{1-2g \cdot z / c^2 - u \cdot u / c^2}$, but staying within SR framework:

Action S (relativistic Invariant, Lorentz Scalar Product)

$S = -\int \mathbf{P} \cdot d\mathbf{R} = -\int [\mathbf{P} \cdot d\mathbf{R}/d\tau] d\tau = -\int [\mathbf{P} \cdot \mathbf{U}] d\tau = -m_o c^2/\gamma d\tau$

$dS = -\mathbf{P} \cdot d\mathbf{R} = (-Edt + \mathbf{p} \cdot dr) = (-Edt + \mathbf{p} \cdot dr)$

$dS_{\text{temporal part}} = -Edt$

$dS_{\text{spatial part}} = \mathbf{p} \cdot dr$

$S = -m_o c^2/\gamma (1/\gamma) dt = [L dt with L = -m_o c^2/\gamma]

How about $\mathbf{P}$ variable with just $m_o$ constant?

$S = -\int [d(\mathbf{P} \cdot \mathbf{R})] = -\int [d(\mathbf{P}) \cdot \mathbf{R} + \mathbf{P} \cdot d(\mathbf{R})] = -\int [d(\mathbf{P}) \cdot d\tau \mathbf{R} + \mathbf{P} \cdot d(\mathbf{R})] d\tau = -\int [d(\mathbf{P}) \cdot d\tau \mathbf{R} + \mathbf{P} \cdot U] d\tau$

if $m_o$ = constant

$= -m_o [d(U)/d\tau \cdot \mathbf{R} + U \cdot U] d\tau$

$= -m_o [\mathbf{A} \cdot \mathbf{R} + U \cdot U] d\tau$

$= -m_o [-\mathbf{a}_o \cdot \mathbf{r}_o + c^2] d\tau$

$\Delta \Phi = -(m_o/h) [-\mathbf{a}_o \cdot \mathbf{r}_o + c^2] d\tau$

$\Delta \Phi = \int [\mathbf{K}_{AB} d\mathbf{R}_{AB} - \mathbf{K}_{CD} d\mathbf{R}_{CD} = (\omega_{AB} T - \mathbf{k}_{AB} L) - (\omega_{CD} T - \mathbf{k}_{CD} L) = (\mathbf{k}_{CD} - \mathbf{k}_{AB}) L, as temporal components cancel.$

$\Delta \Phi = (\mathbf{k}_{CD} - \mathbf{k}_{AB}) L = -(m/h)^2 \Delta \phi L/2\pi = -(m/h)^2 g \Delta z L/2\pi = -(m/h)^2 g H sin(\alpha) L/2\pi = -(m/h)^2 g A_o sin(\alpha) L/2\pi$

The $\mathbf{A} \cdot \mathbf{R}$ term (acceleration · position) is very similar to the gravitational potential term with $\phi = -\mathbf{g} \cdot \mathbf{z}$