RestMass & Other Rest Quantities, based on 4-Vectors and Tensor Mathematics

Some important factors are used frequently in Special Relativity (SR):

Invariant LightSpeed	c	: Speed of Light same in all inertial reference frames
Lorentz Beta Factor	$\boldsymbol{\beta} = \mathbf{u}/c$	: The dimensionless 3-velocity factor
Lorentz Gamma Factor	$\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = 1/\sqrt{[1 - u^2/c^2]} = 1/\sqrt{[1 - \beta^2]}$	: The "scaling factor" between inertial frames of SR
Lorentz Transformation	$\Lambda^\mu{}_\nu = \partial R^\mu/\partial R^\nu$	
Minkowski Metric	$\eta_{\mu\nu} = \text{Diagonal}[+1,-1,-1,-1] = \text{Diag}[+1,-\delta_{\mu\nu}]$	: The "flat" 4D SpaceTime of SR

SR 4-Vectors = 4D (1,0)-Tensors have some very simple, fundamental, and Invariant Lorentz Scalar relations between one another:

4-Position	$\mathbf{R} = R^\mu = (ct,\mathbf{r}) \in$ <Event> $\in$ <Time·Space>
4-Velocity	$\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) = (\mathbf{U}\cdot\partial)\mathbf{R} = (d/d\tau)\mathbf{R}$
4-Momentum	$\mathbf{P} = P^\mu = (E/c,\mathbf{p}) = (E_o/c^2)\mathbf{U} = (m_o)\mathbf{U}$
4-WaveVector	$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k}) = (\omega_o/c^2)\mathbf{U} = (1/\hbar)\mathbf{P}$
4-Gradient	$\partial = \partial^\mu = (\partial_t/c,-\nabla) = (-i)\mathbf{K}$
4-NumberFlux	$\mathbf{N} = N^\mu = (nc,\mathbf{n}) = (n_o)\mathbf{U}$
4-CurrentDensity	$\mathbf{J} = J^\mu = (\rho c,\mathbf{j}) = (\rho_o)\mathbf{U}$
4-EM_VectorPotential	$\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a}) = (\varphi_o/c^2)\mathbf{U}$

$\mathbf{U} = \gamma(c,\mathbf{u})$ → (E_o/c^2) ↠ $\mathbf{P} = (E/c,\mathbf{p})$; $\mathbf{U}$ → (ω_o/c^2) ⟿ $\mathbf{K} = (\omega/c,\mathbf{k})$; $(\hbar)\uparrow \; \downarrow(1/\hbar)$

Start with a generic SR 4-Vector:
$\mathbf{A} = A^\mu = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$: 4-Vector $\mathbf{A}$, 4D (1,0)-Tensor A^μ , temporal scalar a^0 , spatial 3-vector $\mathbf{a} = (a^1,a^2,a^3)$

The internal components a^0 & $\mathbf{a}$ may vary in different inertial frames, but the 4-Vector $\mathbf{A}$ is an invariant geometric tensor object.
The inertial frames are connected by Lorentz Transformations $\Lambda^\mu{}_\nu$ which have the following "invariant magnitude" properties:
$A^{\mu'} = \Lambda^\mu{}_\nu A^\nu$: $\mathbf{A'}\cdot\mathbf{A'} = A^{\mu'} \eta_{\mu\nu} A^{\nu'} = (\Lambda^\mu{}_\alpha A^\alpha)\eta_{\mu\nu}(\Lambda^\nu{}_\beta A^\beta) = (\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta)(A^\alpha A^\beta) = (\Lambda^\mu{}_\alpha \Lambda_{\mu\beta})(A^\alpha A^\beta) = (\eta_{\alpha\beta})(A^\alpha A^\beta) = \mathbf{A}\cdot\mathbf{A}$

For example, a Lorentz Boost along the (t,x)-direction:
$(a^0,a^1,a^2,a^3)' = [\, [\gamma,-\gamma\beta,0,0] , [-\gamma\beta,\gamma,0,0] , [0,0,1,0] , [0,0,0,1] \,] * (a^0,a^1,a^2,a^3)$

The Lorentz Scalar Product is an Invariant quantity, with any and all inertial reference frames getting the same final result.
$\mathbf{A}\cdot\mathbf{A} = A^\mu \eta_{\mu\nu} A^\nu = (a^0,\mathbf{a})\cdot(a^0,\mathbf{a}) = (a^0)^2 - \mathbf{a}\cdot\mathbf{a} = (a^0{}_o)^2$
$\mathbf{A}\cdot\mathbf{B} = A^\mu \eta_{\mu\nu} B^\nu = (a^0,\mathbf{a})\cdot(b^0,\mathbf{b}) = (a^0b^0) - \mathbf{a}\cdot\mathbf{b} = (a^0{}_o b^0{}_o)$

The final scalar quantities $(a^0{}_o)^2$ or $(a^0{}_o b^0{}_o)$ are also known as "at-rest" quantities, or Rest Quantities, and are denoted with a naught ($_o$)
This is because you can pick an "at-rest" frame to do the calculation, which quite often makes the math simpler.

Using the 4-Velocity as an example:
$\mathbf{U}\cdot\mathbf{U} = U^\mu \eta_{\mu\nu} U^\nu = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2 [c^2 - \mathbf{u}\cdot\mathbf{u}]$

Pick the rest-frame in which the 3-velocity $\mathbf{u} \to \mathbf{0}$, which gives $\gamma = 1/\sqrt{[1 - u^2/c^2]} \to 1$.
$\mathbf{U}\cdot\mathbf{U} = U^\mu \eta_{\mu\nu} U^\nu = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2 [c^2 - \mathbf{u}\cdot\mathbf{u}] = 1^2 [c^2 - \mathbf{0}\cdot\mathbf{0}] = c^2$

Or, do it the long way, using the relativistic Lorentz Gamma Factor $\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]}$:
$\mathbf{U}\cdot\mathbf{U} = U^\mu \eta_{\mu\nu} U^\nu = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2 [c^2 - \mathbf{u}\cdot\mathbf{u}] = (1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]})^2 * [c^2 - \mathbf{u}\cdot\mathbf{u}] = (c^2/[c^2 - \mathbf{u}\cdot\mathbf{u}]) * [c^2 - \mathbf{u}\cdot\mathbf{u}] = c^2$

Using the 4-Position and 4-Velocity:
$\mathbf{R}\cdot\mathbf{U} = R^\mu \eta_{\mu\nu} U^\nu = (ct,\mathbf{r})\cdot\gamma(c,\mathbf{u}) = \gamma [c^2t - \mathbf{r}\cdot\mathbf{u}]$

Pick the rest-frame in which the 3-velocity $\mathbf{u} \to \mathbf{0}$, which gives $\gamma \to 1$.
$\mathbf{R}\cdot\mathbf{U} = R^\mu \eta_{\mu\nu} U^\nu = (ct,\mathbf{r})\cdot\gamma(c,\mathbf{u}) = \gamma [c^2t - \mathbf{r}\cdot\mathbf{u}] = 1 [c^2t_o - \mathbf{r}\cdot\mathbf{0}] = c^2t_o = c^2\tau$
The resulting $t_o = \tau$ is known as: the Rest Time, or the Proper Time, or the Invariant Time

Now, let's derive really nice mathematics, using the generic 4-Vector **A**:
$\mathbf{A} = A^\mu = (a^0,\mathbf{a}) = (a^0,a^1,a^2,a^3)$

Take the Lorentz Scalar Product of this generic 4-Vector **A** with the SR 4-Velocity **U**
$\mathbf{A}\cdot\mathbf{U} = A^\mu\ \eta_{\mu\nu}\ U^\nu = (a^0,\mathbf{a})\cdot\gamma(c,\mathbf{u}) = \gamma(a^0c - \mathbf{a}\cdot\mathbf{u})$

Pick the rest-frame in which the 3-velocity $\mathbf{u} \to \mathbf{0}$, which gives $\gamma \to 1$.
$\mathbf{A}\cdot\mathbf{U} = A^\mu\ \eta_{\mu\nu}\ U^\nu = (a^0,\mathbf{a})\cdot\gamma(c,\mathbf{u}) = \gamma(a^0c - \mathbf{a}\cdot\mathbf{u}) = 1(a^0{}_oc - \mathbf{a}\cdot\mathbf{0}) = a^0{}_oc$

c is invariant, and $a^0{}_o$ is an { invariant = proper = rest } quantity, the same for all inertial frames, denoted with the naught ($_o$)

Now, we can do something very clever:
$\mathbf{A}\cdot\mathbf{U} = a^0{}_oc = a^0{}_oc * (c/c) = (a^0{}_o/c) * (c^2) = (a^0{}_o/c)\mathbf{U}\cdot\mathbf{U}$: because $\mathbf{U}\cdot\mathbf{U} = (c^2)$ generally

If **A** ~ **U**, i.e. if **A** is temporal like **U**, then we can write a manifestly invariant tensor equation for 4-Vector **A**
$\mathbf{A} = (a^0{}_o/c)\mathbf{U}$

Taking the Lorentz Scalar Self-Product
$\mathbf{A}\cdot\mathbf{A} = (a^0,\mathbf{a})\cdot(a^0,\mathbf{a}) = (a^0)^2 - (\mathbf{a}\cdot\mathbf{a}) = (a^0{}_o/c)^2\mathbf{U}\cdot\mathbf{U} = (a^0{}_o/c)^2\ (c^2) = (a^0{}_o)^2$

This is often seen rewritten as:
$(a^0)^2 = (a^0{}_o)^2 + (\mathbf{a}\cdot\mathbf{a})$

Now, separated into temporal scalar and spatial 3-vector components:
$\mathbf{A} = (a^0,\mathbf{a}) = (a^0{}_o/c)\mathbf{U} = (a^0{}_o/c)\gamma(c,\mathbf{u})$
Temporal: $(a^0) = (a^0{}_o/c)\gamma(c) = \gamma a^0{}_o$
Spatial: $(\mathbf{a}) = (a^0{}_o/c)\gamma(\mathbf{u}) = \gamma a^0{}_o\ \mathbf{u}/c = a^0\mathbf{u}/c$

Summarizing:

4-Vector $\mathbf{A} = (a^0,\mathbf{a})$
Temporal: $(a^0) = \gamma a^0{}_o$
Spatial: $(\mathbf{a}) = a^0\mathbf{u}/c = \gamma a^0{}_o\mathbf{u}/c$
Manifestly Invariant Form: $\mathbf{A} = (a^0{}_o/c)\mathbf{U}$
$(a^0)^2 = (a^0{}_o)^2 + (\mathbf{a}\cdot\mathbf{a})$

This is ****general**** for all SR 4-Vectors.
The ($\mathbf{A} = A^\mu$) is a 4-Vector = a 4D (1,0)-Tensor, which is an invariant geometric physical object.
The (a^0) and ($\mathbf{a}$) are relativistic tensor components, which may vary by Lorentz transforms based on differing inertial reference frames.
The ($a^0{}_o$) is an invariant 4D (0,0)-Tensor, an invariant scalar, the same across all inertial reference frames.

The naught ($_o$) notation applies to all SR 4-Vectors. It is notationally simpler and easier to use { E and E_o } than {$E_{relativistic}$ and $E_{invariant}$}

Now, using this general concept on the physical SR 4-Vectors that we know about:

4-Vector $\mathbf{A} = (a^0,\mathbf{a})$
Temporal: $(a^0) = \gamma a^0{}_o$
Spatial: $(\mathbf{a}) = a^0\mathbf{u}/c = \gamma a^0{}_o\mathbf{u}/c$
Manifestly Invariant Form: $\mathbf{A} = (a^0{}_o/c)\mathbf{U}$
$(a^0)^2 = (a^0{}_o)^2 + (\mathbf{a}\cdot\mathbf{a})$

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4-Position $\mathbf{R} = R^\mu = (ct,\mathbf{r})$
Temporal: $(t) = \gamma t_o = \gamma\tau$: The SR Time Dilation Effect

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4-Momentum $\mathbf{P} = P^\mu = (E/c,\mathbf{p}) = (mc,\mathbf{p})$
Temporal: $(E) = \gamma E_o$
Spatial: $(\mathbf{p}) = E\mathbf{u}/c^2 = \gamma E_o\mathbf{u}/c^2$
Manifestly Invariant Form: $\mathbf{P} = (E_o/c^2)\mathbf{U}$
$(E)^2 = (E_o)^2 + (\mathbf{p}\cdot\mathbf{p})c^2$

This matches the known quantum relations:
$\mathbf{P} = \hbar\mathbf{K} = (E/c,\mathbf{p}) = \hbar(\omega/c,\mathbf{k}) = (\hbar\omega/c,\hbar\mathbf{k})$
Temporal: $\{E = \hbar\omega\}$ & $\{E_o = \hbar\omega_o\}$ give Einstein's photoelectric relations
Spatial: $\{\mathbf{p} = \hbar\mathbf{k}\}$ gives de Broglie's matter-wave relation
Manifestly Invariant Form: $\mathbf{P} = (\hbar)\mathbf{K}$

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4-WaveVector $\mathbf{K} = K^\mu = (\omega/c,\mathbf{k})$
Temporal: $(\omega) = \gamma\omega_o$
Spatial: $(\mathbf{k}) = \omega\mathbf{u}/c^2 = \gamma\omega_o\mathbf{u}/c^2$
Manifestly Invariant Form: $\mathbf{K} = (\omega_o/c^2)\mathbf{U}$
$(\omega)^2 = (\omega_o)^2 + (\mathbf{k}\cdot\mathbf{k})c^2$

$\{E,\omega,\mathbf{p},\mathbf{k}\}$ are relativistic tensor components, which can vary among frames
$\{E_o,\omega_o,\hbar\}$ are invariant = proper = rest quantities, which are the same

$E = \hbar\omega = \hbar\gamma\omega_o = \gamma E_o$
$\mathbf{p} = \hbar\mathbf{k} = \hbar\omega\mathbf{u}/c^2$

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4-Momentum $\mathbf{P} = P^\mu = (mc,\mathbf{p}) = (E/c,\mathbf{p})$
Temporal: $(m) = \gamma m_o$
Spatial: $(\mathbf{p}) = m\mathbf{u} = \gamma m_o\mathbf{u}$
Manifestly Invariant Form: $\mathbf{P} = (m_o)\mathbf{U}$
$(E)^2 = (m_oc^2)^2 + (\mathbf{p}\cdot\mathbf{p})c^2$

This matches the known relativistic relations:
$\mathbf{P} = m_o\mathbf{U} = (E/c,\mathbf{p}) = m_o\gamma(c,\mathbf{u}) = \gamma m_o(c,\mathbf{u}) = m(c,\mathbf{u}) = (mc,m\mathbf{u}) = (mc,\mathbf{p})$
Temporal: $\{E = mc^2\}$ & $\{E_o = m_oc^2\}$ give Einstein's Energy-mass relations
Spatial: $\{\mathbf{p} = m\mathbf{u} = \gamma m_o\mathbf{u}\}$ gives Einstein's relativistic momentum
Manifestly Invariant Form: $\mathbf{P} = (m_o)\mathbf{U}$

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4-Velocity $\mathbf{U} = U^\mu = \gamma(c,\mathbf{u}) = (\gamma c,\gamma\mathbf{u})$
Temporal: $(\gamma c) = \gamma c$
Spatial: $(\gamma\mathbf{u}) = \gamma c\mathbf{u}/c = \gamma\mathbf{u}$
Manifestly Invariant Form: $\mathbf{U} = (c/c)\mathbf{U} = \mathbf{U}$
$(\gamma c)^2 = (c)^2 + \gamma^2(\mathbf{u}\cdot\mathbf{u})$: $(\gamma)^2 = 1 + (\gamma\beta)^2$

$\{E,m,\mathbf{p}\}$ are relativistic tensor components, which can vary among frames
$\{E_o,m_o,c\}$ are invariant = proper = rest quantities, which are the same

$E = mc^2 = \gamma m_oc^2 = \gamma E_o$
$\mathbf{p} = m\mathbf{u} = \gamma m_o\mathbf{u}$

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4-NumberFlux $\mathbf{N} = N^\mu = (nc,\mathbf{n}) = (nc,n\mathbf{u})$
Temporal: $(n) = \gamma n_o$
Spatial: $(\mathbf{n}) = n\mathbf{u} = \gamma n_o\mathbf{u}$
Manifestly Invariant Form: $\mathbf{N} = (n_o)\mathbf{U}$
$(nc)^2 = (n_oc)^2 + (\mathbf{n}\cdot\mathbf{n})$

$\partial\cdot\mathbf{N} = (\partial_t/c,-\nabla)\cdot(nc,\mathbf{n}) = (\partial_t n + \nabla\cdot\mathbf{n}) = 0$
Conservation of Particle #

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4-CurrentDensity $\mathbf{J} = J^\mu = (\rho c,\mathbf{j})$
Temporal: $(\rho) = \gamma\rho_o$
Spatial: $(\mathbf{j}) = \rho\mathbf{u} = \gamma\rho_o\mathbf{u}$
Manifestly Invariant Form: $\mathbf{J} = (\rho_o)\mathbf{U}$
$(\rho c)^2 = (\rho_oc)^2 + (\mathbf{j}\cdot\mathbf{j})$

$\partial\cdot\mathbf{J} = (\partial_t/c,-\nabla)\cdot(\rho c,\mathbf{j}) = (\partial_t\rho + \nabla\cdot\mathbf{j}) = 0$
Conservation of EM Charge

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4-EM_VectorPotential $\mathbf{A} = A^\mu = (\varphi/c,\mathbf{a}) = (\varphi_o/c^2)\mathbf{U}$
Temporal: $(\varphi) = \gamma\varphi_o$
Spatial: $(\mathbf{a}) = \varphi\mathbf{u}/c^2 = \gamma\varphi_o\mathbf{u}/c^2$
Manifestly Invariant Form: $\mathbf{A} = (\varphi_o/c^2)\mathbf{U}$
$(\varphi)^2 = (\varphi_o)^2 + (\mathbf{a}\cdot\mathbf{a})c^2$

$\partial\cdot\mathbf{A} = (\partial_t/c,-\nabla)\cdot(\varphi/c,\mathbf{a}) = (\partial_t\varphi/c^2 + \nabla\cdot\mathbf{a}) = 0$
Lorenz Gauge Condition
Conservation of EM potential

The standard Schrödinger QM Relations derived from SR:

We can examine these SR 4-Vector relations above… and by simply combining them...
$\mathbf{P} = \hbar\mathbf{K}$ (a relation which is entirely empirical, based on just SR arguments, shown above)
$\mathbf{K} = i\partial$ (which is a relation for complex plane-waves, used in classical EM and elsewhere)

$\mathbf{P} = i\hbar\partial = (E/c,\mathbf{p}) = i\hbar(\partial_t/c,-\nabla)$
Temporal: $\{E = i\hbar\partial_t = i\hbar\ \partial/\partial t\}$ gives unitary QM time evolution operator.
Spatial: $\{\mathbf{p} = -i\hbar\nabla\}$ gives the QM momentum operator.

So, what does all this mean for RestMass (m_o)?

It is a logical, mathematically sound concept based on tensor math which applies to the actual physical world.

It is a fact that there are relativistic frame effects.
In the same way that we "see" Time-Dilated ($t = \gamma t_o$) muons…
In the same way that atmospheric muons "see" a Length-Contracted ($L = L_o/\gamma$) atmosphere…
Muons also have an intrinsic, invariant RestMass (m_o).
The Lorentz Gamma Factor (γ) which causes this relativistic time dilation and length contraction is just a reference frame effect.
There are also a Proper Time (t_o) and a Proper Length (L_o), which are frame invariants.

There is a { Rest = Proper = Invariant } quantity for all SR 4-Vector objects.
However, each of these 4-Vectors will have inner components which can "appear" different in differing inertial frames.

Each fundamental particle has a { Rest = Proper = Invariant } Mass (m_o).
The Total Relativistic Energy E of a fundamental particle is given by $(E)^2 = (m_oc^2)^2 + (\mathbf{p}\cdot\mathbf{p})c^2$
The Total Relativistic Energy E is thus a combination of the invariant RestMass energy (m_oc^2) and the energy of motion ($|\mathbf{p}|c$).
In the frame where the 3-momentum **p** goes to zero, this reduces to $(E_o)^2 = (m_oc^2)^2$: $\mathbf{p} \to \mathbf{0}$ implies $E \to E_o$

The RestMass Energy of each fundamental particle never changes for that given particle.
It can only be split or added to by a change in the nature of the fundamental particle, ie. Feynman diagrams, in which particles transform into other particles.

The energy of motion is really just a by-product Lorentz Gamma Factor (γ) that relates the differing inertial frames.

The Total Energy of Incoming particles = the Total Energy of Outgoing particles in such reactions.

This also still allows a "relativistic mass" (m).
$(E)^2 = (m_oc^2)^2+(\mathbf{p}\cdot\mathbf{p})c^2$
$(mc^2)^2 = (m_oc^2)^2+(\gamma m_o)^2(\mathbf{u}\cdot\mathbf{u})c^2$
$(m)^2 = (m_o)^2+(\gamma m_o)^2(\mathbf{u}\cdot\mathbf{u})/c^2$
$(m)^2 = (m_o)^2+(\gamma m_o)^2(\boldsymbol{\beta}\cdot\boldsymbol{\beta})$
$(m)^2 = (m_o)^2[1+(\gamma)^2(\boldsymbol{\beta}\cdot\boldsymbol{\beta})]$
$(m)^2 = (m_o)^2[1+(\gamma\beta)^2]$
$(m)^2 = [1+(\gamma\beta)^2](m_o)^2$
$(m)^2 = \gamma^2(m_o)^2$ because $(\gamma)^2 = 1 + (\gamma\beta)^2$: from $\gamma = 1/\sqrt{[1-\beta^2]}$
$m = \gamma m_o$

However, again, the Lorentz Gamma (γ) is just a frame effect. Particles still retain their invariant = proper = RestMass (m_o).
For instance, whizzing past a mass with huge $m = \gamma m_o$ doesn't suddenly cause it to become a black hole.

$(E)^2 = (m_o c^2)^2+(\mathbf{p\cdot p})c^2$
$(E)^2 = (m_o c^2)^2+(pc)^2$
$E = \sqrt{[m_o^2 c^4+p^2c^2]} = (m_o^2c^4+p^2c^2)$^(1/2)

$dE = (1/2)(2pc^2)(m_o^2c^4+p^2c^2)$^(-1/2)dp
$dE = (pc^2)(m_o^2c^4+p^2c^2)$^(-1/2)dp
$dE = (pc^2)/\sqrt{[m_o^2c^4+p^2c^2]}dp$
$dE = (pc^2)/\sqrt{[m_o^2c^4+p^2c^2]}dp$
$dE = (pc^2)/c\sqrt{[m_o^2c^2+p^2]}dp$
$dE = (pc)/\sqrt{[m_o^2c^2+p^2]}dp$
$dE/dp = (pc)/\sqrt{[m_o^2c^2+p^2]}$

$E = \gamma m_o c^2$
$p = \gamma m_o u$
$E = pc^2/u$
$Eu = pc^2$
$u\ dE + E\ du = dp\ c^2$

$p = \gamma m_o u$
$p = m_o u/\sqrt{[1-u^2/c^2]}$
$p = m_o u(1-u^2/c^2)$^(-1/2)
$dp = m_o\ [(1-u^2/c^2)$^(-1/2)+u(-1/2)(-2u/c²)(1-u²/c²)^(-3/2)]du
$dp = m_o\ [\gamma+\gamma^3\, u^2/c^2)]du$
$dp = \gamma\ m_o\ [1+\gamma^2\, \beta^2)]du$
$dp = \gamma\ m_o\ [\gamma^2\ ]du$
$dp = \gamma^3\, m_o\ du$
$du/dp = 1/(\gamma^3 m_o)$

$\gamma = (1 - \beta^2)$^(-1/2)
$d\gamma = (-1/2)(-2\beta)(1 - \beta^2)$^(-3/2)dβ
$d\gamma = \beta\gamma^3 d\beta$
$d\gamma = u\gamma^3\ du\ /\ c^2$
$d\gamma/du = u\gamma^3/c^2$

$dp = m_o[u\ d\gamma + \gamma\ du]$
$dp/du = m_o[u\ d\gamma/du + \gamma\ du/du]$
$dp/du = m_o[u^2\gamma^3/c^2 + \gamma]$
$dp/du = \gamma\ m_o[u^2\gamma^2/c^2 + 1]$
$dp/du = \gamma\ m_o[\gamma^2\beta^2 + 1]$
$dp/du = \gamma\ m_o[\gamma^2]$
$dp/du = \gamma^3 m_o$

$E = \gamma m_o c^2$
$dE = d\gamma\ m_o c^2$
$dp/du = \gamma^3 m_o$
$dp = \gamma^3 m_o\ du$
$d\gamma/du = u\gamma^3/c^2$
$d\gamma = u\gamma^3/c^2\ du$
$dE = m_o\, u\gamma^3\ du$
$dE = u\ dp$
$u = dE/dp$

4-WaveVector $\mathbf{K} = K^\mu = (\omega/c=1/c$Ŧ$,\mathbf{k}=\omega\hat{\mathbf{n}}/v_{phase}=\hat{\mathbf{n}}/$ƛ$=\omega\mathbf{u}/c^2) = (\omega_o/c^2)\mathbf{U} = (1/\hbar)\mathbf{P}$ $\hat{\mathbf{n}}$ = unit-direction 3-vector {2 DoF}

$\mathbf{K} = K^\mu = (\omega/c,\mathbf{k}=\omega\hat{\mathbf{n}}/v_{phase}) = (\omega_o/c^2)\mathbf{U} = (\omega_o/c^2)\gamma(c,\mathbf{u}) = (\gamma\omega_o/c^2)(c,\mathbf{u}) = (\omega/c^2)(c,\mathbf{u}) = (\omega/c,\omega\mathbf{u}/c^2)$
$(\omega/c,\mathbf{k}=\omega\hat{\mathbf{n}}/v_{phase}) = (\omega/c,\omega\mathbf{u}/c^2)$
$\omega\hat{\mathbf{n}}/v_{phase} = \omega\mathbf{u}/c^2$
$c^2\,\hat{\mathbf{n}} = \mathbf{u}*v_{phase}$

Now, separated into temporal scalar and spatial 3-vector components:
$\mathbf{A} = (a^0,\mathbf{a}) = (a^0_o/c)\mathbf{U} = (a^0_o/c)\gamma(c,\mathbf{u}) = (a^0,a^0\mathbf{u}/c) = a^0(1,\boldsymbol{\beta}) = \gamma a^0_o(1,\boldsymbol{\beta}) = a^0_o\gamma(1,\boldsymbol{\beta}) = a^0_o\bar{\mathbf{T}}$: $\bar{\mathbf{T}}$ = 4-Unit"Temporal" = $\gamma(1,\boldsymbol{\beta})$
Temporal: $(a^0) = (a^0_o/c)\gamma(c) = \gamma a^0_o$
Spatial: $(\mathbf{a}) = (a^0_o/c)\gamma(\mathbf{u}) = \gamma a^0_o\,\mathbf{u}/c = a^0\mathbf{u}/c = a^0\boldsymbol{\beta}$

Taking the Lorentz Scalar Self-Product:
$\mathbf{A}\cdot\mathbf{A} = (a^0,\mathbf{a})\cdot(a^0,\mathbf{a}) = (a^0)^2 - (\mathbf{a}\cdot\mathbf{a}) = (a^0_o/c)^2\,\mathbf{U}\cdot\mathbf{U} = (a^0_o/c)^2\,(c^2) = (a^0_o)^2$
$(a^0)^2 = (a^0_o)^2 + (\mathbf{a}\cdot\mathbf{a})$
$(a^0)^2 = (a^0_o)^2 + |\mathbf{a}|^2$
$d[(a^0)^2] = d[(a^0_o)^2 + |\mathbf{a}|^2]$
$d[(a^0)^2] = 0 + d[|\mathbf{a}|^2]$
$2\,a^0\,d[(a^0)] = 2|\mathbf{a}|d[|\mathbf{a}|]$
$a^0\,d[a^0] = |\mathbf{a}|d[|\mathbf{a}|]$
$d[a^0]/d[|\mathbf{a}|] = |\mathbf{a}|/a^0 = \mathbf{u}/c = c/v_{phase}$: $u * v_{phase} = c^2$

Define $\text{Phase}_A = \varphi_A = \varphi = -\mathbf{A}\cdot\mathbf{X} = (\mathbf{a}\cdot\mathbf{x} - a^0ct)$
$\mathbf{a} = \partial\varphi/\partial\mathbf{x}$
$-a^0c = \partial\varphi/\partial t$
$v_{phase} = |\partial\mathbf{x}/\partial t| = |(\partial\mathbf{x}/\partial t)(\partial\varphi/\partial\varphi)| = |(\partial\mathbf{x}/\partial\varphi)(\partial\varphi/\partial t)| = |(1/\mathbf{a})(-a^0c)| = |(-a^0c/\mathbf{a})| = |(a^0c/\mathbf{a})|$: $\mathbf{a} = (a^0c\hat{\mathbf{n}}/v_{phase})$

4-Vector $\mathbf{A} = (a^0,\mathbf{a})$
Temporal: $(a^0) = \gamma a^0_o$
Spatial: $(\mathbf{a}) = a^0\mathbf{u}/c = \gamma a^0_o\mathbf{u}/c$
Manifestly Invariant Form: $\mathbf{A} = (a^0_o/c)\mathbf{U}$
$(a^0)^2 = (a^0_o)^2 + (\mathbf{a}\cdot\mathbf{a})$

Recapping:
$a^0/|\mathbf{a}| = v_{phase}/c$ From the formal definition of generic temporal 4-Vector $\mathbf{A} = A^\mu = (a^0,\mathbf{a} = a^0\mathbf{u}/c = a^0c\hat{\mathbf{n}}/v_{phase})$
$da^0/d\mathbf{a} = \mathbf{u}/c = v_{group}/c = v_{particle}/c$ Derived from Lorentz Scalar Product $(\mathbf{A}\cdot\mathbf{A}) = [(a^0)^2 - \mathbf{a}\cdot\mathbf{a}] = (a^0{}_o)^2$
$\mathbf{u} = c^2\hat{\mathbf{n}}/v_{phase}$ The relation between particle and wave velocities, from $\mathbf{A} = (a^0{}_o/c)\mathbf{U}$

$\mathbf{A} = A^\mu = (a^0,\mathbf{a}) = (a^0_o/c)\mathbf{U} = (a^0_o/c)\gamma(c,\mathbf{u}) = (a^0,a^0\mathbf{u}/c) = (a^0,a^0c\hat{\mathbf{n}}/v_{phase})$: then $\mathbf{u}/c = c\hat{\mathbf{n}}/v_{phase}$: $\mathbf{u}*v_{phase} = c^2\,\hat{\mathbf{n}}$
$(a^0,a^0\mathbf{u}/c) = (a^0,a^0c\hat{\mathbf{n}}/v_{phase})$
$(a^0,a^0\boldsymbol{\beta}) = (a^0,a^0\hat{\mathbf{n}}/\beta_{phase})$
$\boldsymbol{\beta} = \hat{\mathbf{n}}/\beta_{phase}$
$\boldsymbol{\beta} * \beta_{phase} = \hat{\mathbf{n}}$

$\mathbf{u}*v_{phase} = c^2\,\hat{\mathbf{n}}$

v_{phase} =?= $a^0c/|\mathbf{a}|$ =?= $c\partial|\mathbf{a}|/\partial a^0$
$v_{group} = |\mathbf{u}|$ =?= $c\partial a^0/\partial|\mathbf{a}|$
$v_{phase} * v_{group} = (c\partial|\mathbf{a}|/\partial a^0) * (c\partial a^0/\partial|\mathbf{a}|) = (c^2) * (\partial|\mathbf{a}|/\partial a^0) * (\partial a^0/\partial|\mathbf{a}|) = (c^2) * (\partial a^0/\partial a^0) * (\partial|\mathbf{a}|/\partial|\mathbf{a}|) = (c^2)$

$d[\gamma] = d[(1-\beta^2)\text{^}(-1/2)] = (-1/2)(1-\beta^2)\text{^}(-3/2)d[(1-\beta^2)] = (-1/2)(\gamma^3)d[(1-\beta^2)] = (-1/2)(\gamma^3)(-2)\beta d\beta = \beta\gamma^3\, d\beta$

$d[\gamma\beta] = [\beta\, d\gamma + \gamma\, d\beta] = [\beta\, \beta\gamma^3\, d\beta + \gamma\, d\beta] = [\beta^2\gamma^3\, d\beta + \gamma\, d\beta] = \gamma[\beta^2\gamma^2 + 1]\, d\beta = \gamma[\gamma^2]\, d\beta = \gamma^3\, d\beta$

$d[\gamma] = (\beta)\, d[\gamma\beta]$

$d[\gamma\beta] = (1/\beta)\, d[\gamma]$

$\mathbf{a} = (a^0{}_o)\gamma\beta = \gamma a^0{}_o\beta$

$d\mathbf{a} = d[\ \gamma\beta\ ]a^0{}_o$

$d\mathbf{a} = \gamma^3\, d\beta\ a^0{}_o$

$a^0 = \gamma a^0{}_o$

$da^0 = d[\gamma]a^0{}_o$

$da^0 = \beta\gamma^3\, d\beta\ a^0{}_o$

$da^0 = \beta\, d\mathbf{a}$

$da^0 / d\mathbf{a} = \beta$

$c\, da^0 / d\mathbf{a} = \beta c = u = v_{group}$

$\mathbf{R}\cdot\mathbf{R} = R^\mu\, \eta_{\mu\nu}\, R^\nu = (ct,\mathbf{r})\cdot(c,\mathbf{r}) = [c^2t^2 - \mathbf{r}\cdot\mathbf{r}] = c^2t_o{}^2$

$\Delta\mathbf{R}\cdot\Delta\mathbf{R} = \Delta R^\mu\, \eta_{\mu\nu}\, \Delta R^\nu = (c\Delta t,\Delta\mathbf{r})\cdot(c\Delta t,\Delta\mathbf{r}) = [c^2\Delta t^2 - \Delta\mathbf{r}\cdot\Delta\mathbf{r}] = c^2\Delta t_o{}^2$

$c^2\Delta t^2 = c^2\Delta t_o{}^2 + \Delta\mathbf{r}\cdot\Delta\mathbf{r}$