**Relativistic Lorentz Factor ($\gamma$) Derived using 4-Vectors**

Start with classical time (t), a scalar, and classical position (**r**)=(x,y,z), a 3-vector. Together, they label <events>, when & where.
These are tensors in a 3D Euclidean world: (t) is a scalar invariant 3D (0,0)-tensor; (**r**) is a 3D (1,0)-tensor.
The scalar product $(\mathbf{r}\cdot\mathbf{r}) = r^j\, \delta_{jk}\, r^k = |\mathbf{r}|^2 = (r)^2$ is a 3D Euclidean invariant. $\delta_{jk} = \mathrm{Diag}[+1,+1,+1]$ is the 3D Euclidean Metric.

However, imagine that the world is 4D-Tensorial, not 3D-tensorial. How would that work? Why is it needed?

This idea is prompted because there exist physical time-dilation and length-contraction effects, which invalidate the 3D Euclidean invariants idea. In other words, (t) and $(\mathbf{r}\cdot\mathbf{r})$ are still invariants in a 3D mathematical model, but this 3D model doesn't seem to apply to our physical world. Also, Maxwell's EM theory and the EM experiments by several physicists in last half of 1800s indicated a new invariant, $(\varepsilon_o\mu_o)$, which became associated with a finite free-space speed-of-light (c), a velocity-type invariant. $(\varepsilon_o\mu_o = 1/c^2)$.

Let's start by assuming a 4-Vector=4D (1,0)-Tensor which combines the classical components (t) and (**r**), utilizing the idea of (c).
4-Position **R** = (ct,**r**)

A dimensional factor of velocity=[length/time] is required to make overall dimensional units work correctly.
(c) is strongly hinted to be just such an invariant.

| In SI Units: | In Dimensional Types: |
|---|---|
| **R** = [m] | **R** = [length] |
| ct = [m/s]*[s] = [m] | ct = [length/time]*[time] = [length] |
| **r** = [m] | **r** = [length] |

Assume/postulate that (c) is both the constant and invariant that we need for the dimensional factor in the 4-Vector.
A constant is a value that doesn't change over time. ( d[c]/dt = 0 ) in 3D or ( d[c]/d$\tau$ = 0 ) in 4D.
An invariant is a value that all inertial observers will agree on its measured value. (c) is a 4D (0,0)-Tensor.

Using tensorial rules: $(\mathbf{R}\cdot\mathbf{R}) = (ct,\mathbf{r})\cdot(ct,\mathbf{r}) = (R^\mu\, g_{\mu\nu}\, R^\nu)$ is a 4D invariant.
$g_{\mu\nu}$ is a 4D Metric, and it's choice can be determined experimentally.
Flat spacetime = inertial reference-frames (IRF's) gives a couple of possibilities to check:

| | | |
|---|---|---|
| 4D Euclidean Metric | $g_{\mu\nu} \rightarrow \delta_{\mu\nu} = \mathrm{Diag}[+1,+1,+1,+1]$ | $(\mathbf{R}\cdot\mathbf{R}) = (R^\mu\, \delta_{\mu\nu}\, R^\nu) = (ct,\mathbf{r})\cdot(ct,\mathbf{r}) = (c^2t^2 + \mathbf{r}\cdot\mathbf{r})$ |
| 4D Minkowski Metric | $g_{\mu\nu} \rightarrow \eta_{\mu\nu} = \mathrm{Diag}[+1,-1,-1,-1]$ | $(\mathbf{R}\cdot\mathbf{R}) = (R^\mu\, \eta_{\mu\nu}\, R^\nu) = (ct,\mathbf{r})\cdot(ct,\mathbf{r}) = (c^2t^2 - \mathbf{r}\cdot\mathbf{r})$ |

In either of these cases, the Rest Value or Proper Value gives $(\mathbf{R}\cdot\mathbf{R}) = (ct,\mathbf{r})\cdot(ct,\mathbf{r}) = c^2t_o^2 = c^2\tau^2$
Define the 4D scalar invariant $(\tau = t_o)$, a 4D (0,0)-Tensor, as RestTime designated with a naught $(t_o)$ or ProperTime ($\tau$).
It is the "proper" time as measured by a clock moving with the object/frame, i.e. At-Rest with respect to the object/frame.

Now, derive the 4D Velocity **U**. Since ($\tau$) is invariant, we can make a new 4-Vector from the 4-Position **R** using standard calculus.
In the same way that 3D vector 3-velocity **u** = d**r**/dt:

4-Velocity **U**
= d**R**/d$\tau$
= (d/d$\tau$)(ct,**r**)
= (d/d$\tau$ ct,d/d$\tau$ **r**)
= (cdt/d$\tau$,d**r**/d$\tau$)
= (dt/d$\tau$)*(c,(d$\tau$/dt)(d**r**/d$\tau$)) : taking the (dt/d$\tau$) factor outside
= (dt/d$\tau$)*(c,d**r**/dt)
= (dt/d$\tau$)*(c,**u**) : again using classical **u** = d**r**/dt
= $\gamma$(c,**u**) : assigning $\gamma$ = (dt/d$\tau$) = The Lorentz Gamma factor, but we don't have its main formula yet...

This 4-Vector is the SR 4-Velocity $\mathbf{U} = \gamma(c,\mathbf{u})$.

Now, take the invariant tensorial scalar product of the 4-Velocity **U**.
$(\mathbf{U}\cdot\mathbf{U}) = (U^\mu \delta_{\mu\nu} U^\nu) = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 + \mathbf{u}\cdot\mathbf{u})$ for 4D Euclidean
$(\mathbf{U}\cdot\mathbf{U}) = (U^\mu \eta_{\mu\nu} U^\nu) = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u})$ for 4D Minkowski

Is there a way to get a formula for the Lorentz gamma ($\gamma$)? Yes!

Go back to the Lorentz Scalar Product of the 4-Position.
$(\mathbf{R}\cdot\mathbf{R}) = c^2\tau^2$
$d(\mathbf{R}\cdot\mathbf{R}) = d\mathbf{R}\cdot\mathbf{R} + \mathbf{R}\cdot d\mathbf{R} = 2(\mathbf{R}\cdot d\mathbf{R}) = d[c^2\tau^2] = c^2\, 2\tau\, d\tau$
$(\mathbf{R}\cdot d\mathbf{R}) = c^2\tau\, d\tau$
$(\mathbf{R}\cdot d\mathbf{R}/d\tau) = c^2\tau\, d\tau/d\tau$
$(\mathbf{R}\cdot\mathbf{U}) = c^2\tau$
$(\mathbf{R}\cdot\mathbf{U}\tau) = c^2\tau^2 = (\mathbf{R}\cdot\mathbf{R})$
$\mathbf{U}\tau = \mathbf{R}$
$(\mathbf{R}\cdot\mathbf{R}) = (\mathbf{U}\tau\cdot\mathbf{U}\tau) = c^2\tau^2$
$(\mathbf{U}\cdot\mathbf{U})\tau^2 = c^2\tau^2$
And finally...
$(\mathbf{U}\cdot\mathbf{U}) = c^2$
The Lorentz Scalar Product of the 4-Velocity is an invariant.
Thus, (c) is invariant ($c = \sqrt{[\mathbf{U}\cdot\mathbf{U}]}$), and was already a constant from the original assumption.

Now solve for the Lorentz gamma ($\gamma$):
$(\mathbf{U}\cdot\mathbf{U}) = (U^\mu \delta_{\mu\nu} U^\nu) = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 + \mathbf{u}\cdot\mathbf{u}) = c^2$ for 4D Euclidean
$(\mathbf{U}\cdot\mathbf{U}) = (U^\mu \eta_{\mu\nu} U^\nu) = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u}) = c^2$ for 4D Minkowski

$\gamma^2 = c^2/(c^2 + \mathbf{u}\cdot\mathbf{u}) = 1/(1 + u^2/c^2)$ for 4D Euclidean
$\gamma^2 = c^2/(c^2 - \mathbf{u}\cdot\mathbf{u}) = 1/(1 - u^2/c^2)$ for 4D Minkowski

$\gamma = 1/\sqrt{[1 + \mathbf{u}\cdot\mathbf{u}/c^2]}$ for 4D Euclidean
$\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]}$ for 4D Minkowski

Experiment tells us that the world is essentially Minkowski $g_{\mu\nu} \to \eta_{\mu\nu} = \mathrm{Diag}[+1,-1,-1,-1]$, except in high gravity situations.
Thus, the Lorentz Relativistic Gamma factor is: $\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = 1/\sqrt{[1 - u^2/c^2]} = (dt/d\tau)$

As a double check:
$(\mathbf{U}\cdot\mathbf{U}) = (U^\mu \eta_{\mu\nu} U^\nu) = \gamma(c,\mathbf{u})\cdot\gamma(c,\mathbf{u}) = \gamma^2(c^2 - \mathbf{u}\cdot\mathbf{u})$
Setting $\mathbf{u} \to \mathbf{0}$ for the At-Rest limit-case makes $\gamma = 1/\sqrt{[1 - u^2/c^2]} \to 1$. Hence $(\mathbf{U}\cdot\mathbf{U})_{rest} = c^2$
But since Lorentz Scalar Product is a tensorial expression, it holds generally. $(\mathbf{U}\cdot\mathbf{U}) = c^2$

Result: The form of Lorentz Relativistic Gamma factor ($\gamma$) is fixed by the assumptions: constant (c) & Minkowski Metric.

Note: The ($|\mathbf{u}| \ll c$) limit-case provides a path back to classical physics, the 3D Euclidean model, and provides a way to understand its limited realm of validity. In other words, it still works as an excellent approximation, but only within its realm of validity ($|\mathbf{u}| \ll c$).

4-Differential $d\mathbf{R} = (cdt,d\mathbf{r})$
$(d\mathbf{R}\cdot d\mathbf{R}) = (cdt,d\mathbf{r})\cdot(cdt,d\mathbf{r}) = c^2dt^2 - d\mathbf{r}\cdot d\mathbf{r} = c^2dt^2\,[1 - d\mathbf{r}\cdot d\mathbf{r}/(c^2dt^2)] = c^2dt^2\,[1 - \mathbf{u}\cdot\mathbf{u}/c^2] = c^2dt^2\,[1/\gamma^2] = c^2d\tau^2$
$dt^2/d\tau^2 = \gamma^2$
$dt/d\tau = \gamma$

Assumptions:
(c) is constant. Experimentally we find that photons seem to have this property, a constant and invariant velocity.
The metric we physically experience is Minkowski ( $g_{\mu\nu} \to \eta_{\mu\nu}$ ), at least in weak gravity situations.
Tensor mathematics is consistent and appears to correctly describe our physical reality.

Results:
(c) is a 4D invariant, a 4D (0,0)-Tensor

Lorentz Relativistic Gamma: $\gamma = 1/\sqrt{[1 - \mathbf{u}\cdot\mathbf{u}/c^2]} = (dt/d\tau)$

| Name | Definition | LSP | SR Relation | Components : 4D (0,0)-Tensors |
|---|---|---|---|---|
| 4-Position | $\mathbf{R} = (ct,\mathbf{r})$ | $(\mathbf{R}\cdot\mathbf{R}) = c^2\tau^2$ | $\mathbf{R}$ describes <event> | $(t)$,$(\mathbf{r})$ relativistic : $(\tau=t_o)$ invariant |
| 4-Velocity | $\mathbf{U} = \gamma(c,\mathbf{u})$ | $(\mathbf{U}\cdot\mathbf{U}) = c^2$ | $\mathbf{U} = d\mathbf{R}/d\tau$ | $(\gamma)$,$(\mathbf{u})$ relativistic : $(c=c_o)$ invariant |

Addendum:

| | | | | |
|---|---|---|---|---|
| 4-Momentum | $\mathbf{P} = (E/c=mc,\mathbf{p})$ | $(\mathbf{P}\cdot\mathbf{P}) = m_o^2c^2$ | $\mathbf{P} = m_o\mathbf{U} = (E_o/c^2)\mathbf{U}$ | $(m)$,$(E)$ relativistic : $(m_o)$,$(E_o)$ invariant |
| 4-WaveVector | $\mathbf{K} = (\omega/c,\mathbf{k})$ | $(\mathbf{K}\cdot\mathbf{K}) = m_o^2c^2/\hbar^2$ | $\mathbf{P} = \hbar\mathbf{K}$ | $(\omega)$,$(\mathbf{k})$ relativistic : $(\hbar=\hbar_o)$ invariant |

$\mathbf{P} = m_o\mathbf{U} = (E/c=mc,\mathbf{p}) = m_o\gamma(c,\mathbf{u}) = m(c,\mathbf{u})$
Temporal part: $E = mc^2 = \gamma m_o c^2 = \gamma E_o$ : Einstein's Energy-Mass Relation
Spatial part: $\mathbf{p} = m\mathbf{u} = \gamma m_o\mathbf{u} = E\mathbf{u}/c^2$ : Relativistic 3-momentum $\mathbf{p}$

$\mathbf{P} = \hbar\mathbf{K} = (E/c,\mathbf{p}) = \hbar(\omega/c,\mathbf{k})$
Temporal part: $E = \hbar\omega$ : Einstein's PhotoElectric Effect
Spatial part: $\mathbf{p} = \hbar\mathbf{k}$ : de Broglie's Matter-Wave Duality Relation

============================================

GR "Weak-Field" limiting-case… or alternately viewed as a small perturbation field ($h_{\mu\nu}$) on the SR Minkowski Metric ($\eta_{\mu\nu}$):
SpaceTime Metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ : $\eta_{\mu\nu} = \text{Diag}[+1,-1,-1,-1]$ : $h_{\mu\nu} = (2\varphi/c^2)\delta_{\mu\nu}$ : $g_{00} = (1+2\varphi/c^2)$ & $g_{ii} = (-1+2\varphi/c^2)$ : Gravity potential ($\varphi$)

$c d\tau = \sqrt{[g_{\mu\nu}dX^\mu dX^\nu]} = \sqrt{[g_{\mu\nu}(dX^\mu/dt)(dX^\nu/dt)]}dt = \sqrt{[(1+2\varphi/c^2)c^2+(-1+2\varphi/c^2)\mathbf{u}\cdot\mathbf{u}]}dt = \sqrt{[c^2+2\varphi-\mathbf{u}\cdot\mathbf{u}+2\varphi\mathbf{u}\cdot\mathbf{u}/c^2]}dt$

$d\tau = \sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2+2\varphi\mathbf{u}\cdot\mathbf{u}/c^4]}dt \sim \sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2+\{0...\}]}dt = dt/\gamma_{WeakGrav}$ Assume $O[1/c^4]$ factor $\sim \{0...\}$

So, "Weak-Gravity" Lorentz Factor $\gamma_{WeakGrav} = 1/\sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2]}$ and the Metric effectively is: $g_{00} = (1+2\varphi/c^2)$ & $g_{ii} \to (-1)$

4D Minkowski Metric $g_{\mu\nu} \to \eta_{\mu\nu} = \text{Diag}[+1,-1,-1,-1]$
4D WeakGravity Metric $g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} \to \text{Diag}[(1+2\varphi/c^2),-1,-1,-1]$

$\Delta\tau = \Delta t/\gamma_{WeakGrav} = \sqrt{[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2]}\Delta t$ : Aging rate slows for: lower in a gravity potential ($\varphi = -\mathbf{g}\cdot\mathbf{z}$) and at higher velocity ($\mathbf{u}$)

$\mathbf{U}\cdot\mathbf{U} = U^\mu g_{\mu\nu}U^\nu = g_{\mu\nu}U^\mu U^\nu = (1+2\varphi/c^2)(u^0)^2 + (-1)(u^i\cdot u^j) = \gamma^2[(1+2\varphi/c^2)c^2 + (-1)\mathbf{u}\cdot\mathbf{u}] = \gamma^2[(c^2+2\varphi c^2/c^2)-(\mathbf{u}\cdot\mathbf{u})] = \gamma^2[c^2+2\varphi-\mathbf{u}\cdot\mathbf{u}]$
$= \gamma^2c^2[1+2\varphi/c^2-\mathbf{u}\cdot\mathbf{u}/c^2] = (c^2)$ if $\gamma = \gamma_{WeakGrav}$

Then $\gamma \to \gamma_{WeakGrav} = 1/\sqrt{[1 + 2\varphi/c^2 - \mathbf{u}\cdot\mathbf{u}/c^2]} = (dt/d\tau)$

Very similar to the regular Lorentz factor but with a gravitational potential ($\varphi = -\mathbf{g}\cdot\mathbf{z}$) component, but only valid for ( $2\varphi\mathbf{u}\cdot\mathbf{u}/c^4 \sim 0$ )