Triple Integrals in various coordinate systems, and reductions using Dirac Delta functions

Cartesian/Rectangular

$\text{Vol} = \int_0^X dx \int_0^Y dy \int_0^Z dz = x|_0^X\; y|_0^Y\; z|_0^Z = (X-0)\cdot(Y-0)\cdot(Z-0) = X\cdot Y\cdot Z$

$\text{Area} = \int_0^X dx \int_0^Y dy \int_0^Z \delta(z-Z_o)\cdot dz = x|_0^X \cdot y|_0^Y \cdot(1 \text{ if } Z_o\in[0..Z], \text{ else } 0) = X\cdot Y$

$\text{Length} = \int_0^X dx \int_0^Y \delta(y-Y_o)\cdot dy \int_0^Z \delta(z-Z_o)\cdot dz = x|_0^X \cdot(1 \text{ if } Y_o\in[0..Y], \text{ else } 0)\cdot(1 \text{ if } Z_o\in[0..Z], \text{ else } 0) = X$

Polar/Cylindrical

$\text{Vol} = \int_0^R dr \int_0^{2\pi} (r\cdot d\theta) \int_0^Z dz = \int_0^R r\cdot dr \int_0^{2\pi} d\theta \int_0^Z dz = (½)r^2|_0^R\; \theta|_0^{2\pi}\; z|_0^Z = (½)R^2\cdot 2\pi\cdot Z = \pi R^2 Z$

$\text{Area}_{disc} = \int_0^R dr \int_0^{2\pi} (r\cdot d\theta) \int_0^Z \delta(z-Z_o)\cdot dz = \int_0^R r\cdot dr \int_0^{2\pi} d\theta \int_0^Z \delta(z-Z_o)\cdot dz = (½)r^2|_0^R\; \theta|_0^{2\pi}\; (1 \text{ if } Z_o\in[0..Z], \text{ else } 0) = (½)R^2\cdot 2\pi = \pi R^2$

$\text{Area}_{½plane} = \int_0^R dr \int_0^{2\pi} [\delta(\theta-\theta_o)/r]\cdot(r\cdot d\theta) \int_0^Z dz = \int_0^R dr \int_0^{2\pi} [\delta(\theta-\theta_o)]d\theta \int_0^Z dz = r|_0^R\; (1 \text{ if } \theta_o\in[0..2\pi], \text{ else } 0)\; z|_0^Z = R\cdot Z$

$\text{Area}_{cylinder} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} (r\cdot d\theta) \int_0^Z dz = \int_0^R r\cdot\delta(r-R_o)\cdot dr \int_0^{2\pi} d\theta \int_0^Z dz = \int_0^R R_o\cdot\delta(r-R_o)\cdot dr \int_0^{2\pi} d\theta \int_0^Z dz = (R_o\cdot 1 \text{ if } R_o\in[0..R], \text{ else } 0)\; \theta|_0^{2\pi}\; z|_0^Z = 2\pi R_o Z$

$\text{Length}_{radial} = \int_0^R dr \int_0^{2\pi} [\delta(\theta-\theta_o)/r]\cdot(r\cdot d\theta) \int_0^Z \delta(z-Z_o)\cdot dz = \int_0^R dr \int_0^{2\pi} \delta(\theta-\theta_o)\cdot d\theta \int_0^Z \delta(z-Z_o)\cdot dz = r|_0^R\; (1 \text{ if } \theta_o\in[0..2\pi], \text{ else } 0)\; (1 \text{ if } Z_o\in[0..Z], \text{ else } 0) = R$

$\text{Length}_{circle} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} (r\cdot d\theta) \int_0^Z \delta(z-Z_o)\cdot dz = \int_0^R r\cdot\delta(r-R_o)\cdot dr \int_0^{2\pi} d\theta \int_0^Z \delta(z-Z_o)\cdot dz = (R_o\cdot 1 \text{ if } R_o\in[0..R], \text{ else } 0)\; \theta|_0^{2\pi}\; (1 \text{ if } Z_o\in[0..Z], \text{ else } 0) = 2\pi R_o$

$\text{Length}_{height} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} [\delta(\theta-\theta_o)/r]\cdot(r\cdot d\theta) \int_0^Z dz = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} \delta(\theta-\theta_o)\cdot d\theta \int_0^Z dz = (1 \text{ if } R_o\in[0..R], \text{ else } 0)\; (1 \text{ if } \theta_o\in[0..2\pi], \text{ else } 0)\; z|_0^Z = Z$

Spherical

$\text{Vol} = \int_0^R dr \int_0^{2\pi} (r\cdot d\theta) \int_0^\pi (r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R r^2\cdot dr \int_0^{2\pi} d\theta \int_0^\pi (\sin[\varphi]\cdot d\varphi) = (1/3)r^3|_0^R\; \theta|_0^{2\pi}\; (-\cos[\varphi])|_0^\pi = (1/3)R^3\cdot 2\pi\cdot 2 = (4/3)\pi R^3$

$\text{Area}_{sphere} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} (r\cdot d\theta) \int_0^\pi (r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R r^2\cdot\delta(r-R_o)\cdot dr \int_0^{2\pi} d\theta \int_0^\pi (\sin[\varphi]\cdot d\varphi) = R_o^2\; \theta|_0^{2\pi}\; (-\cos[\varphi])|_0^\pi = R_o^2\cdot 2\pi\cdot 2 = 4\pi R_o^2$

$\text{Area}_{½circle} = \int_0^R dr \int_0^{2\pi} [\delta(\theta-\theta_o)/(r\cdot\sin[\varphi])]\cdot(r\cdot d\theta) \int_0^\pi (r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R r\cdot dr \int_0^{2\pi} [\delta(\theta-\theta_o)]\cdot d\theta \int_0^\pi d\varphi = (½)r^2|_0^R\; 1\; \varphi|_0^\pi = (½)R^2\cdot 1\cdot\pi = (\pi/2)R^2$

$\text{Area}_{cone} = \int_0^R dr \int_0^{2\pi} (r\cdot d\theta) \int_0^\pi [\delta(\varphi-\varphi_o)/r]\cdot(r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R r\cdot dr \int_0^{2\pi} d\theta \int_0^\pi (\sin[\varphi_o]\delta(\varphi-\varphi_o)\cdot d\varphi) = (½)r^2|_0^R\; \theta|_0^{2\pi}\; (\sin[\varphi_o]) = (½)R^2\cdot 2\pi\cdot(\sin[\varphi_o]) = \sin[\varphi_o]\pi R^2 = (x/R)\pi R^2 = (x)\pi R$

$\text{Length}_{circle} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} (r\cdot d\theta) \int_0^\pi [\delta(\varphi-\varphi_o)/r]\cdot(r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R R_o\cdot\delta(r-R_o)dr \int_0^{2\pi} d\theta \int_0^\pi \sin[\varphi_o]\delta(\varphi-\varphi_o)d\varphi = R_o\; 2\pi\; (\sin[\varphi_o]) = \sin[\varphi_o]2\pi R_o$

$\text{Length}_{½arc} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} [\delta(\theta-\theta_o)/(r\cdot\sin[\varphi])]\cdot(r\cdot d\theta) \int_0^\pi (r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R r\cdot\delta(r-R_o)\cdot dr \int_0^{2\pi} \delta(\theta-\theta_o)d\theta \int_0^\pi d\varphi = \int_0^R R_o\cdot\delta(r-R_o)\cdot dr \int_0^{2\pi} \delta(\theta-\theta_o)\cdot(d\theta) \int_0^\pi (r\cdot d\varphi) = R_o\; 1\; \pi = \pi R_o$

$\text{Length}_{radial} = \int_0^R dr \int_0^{2\pi} [\delta(\theta-\theta_o)/(r\cdot\sin[\varphi])]\cdot(r\cdot d\theta) \int_0^\pi [\delta(\varphi-\varphi_o)/r]\cdot(r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R dr \int_0^{2\pi} \delta(\theta-\theta_o)d\theta \int_0^\pi \delta(\varphi-\varphi_o)d\varphi = r|_0^R\; 1\; 1 = R$

$\text{Point} = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} [\delta(\theta-\theta_o)/(r\cdot\sin[\varphi])]\cdot(r\cdot d\theta) \int_0^\pi [\delta(\varphi-\varphi_o)/r]\cdot(r\cdot\sin[\varphi]\cdot d\varphi) = \int_0^R \delta(r-R_o)\cdot dr \int_0^{2\pi} \delta(\theta-\theta_o)d\theta \int_0^\pi \delta(\varphi-\varphi_o)d\varphi = 1\; 1\; 1 = 1$