Introduction to Quantum Mechanics Emergence from Special Relativity, SRQM

Preprint · September 2021

DOI: 10.13140/RG.2.2.12335.87205

William P. Rice
University of Wisconsin–Madison

Introduction to Quantum Mechanics Emergence from Special Relativity, SRQM

John B. Wilson[a]; William P. Rice[b]

[a] Principal Physicist at SciRealm, email: SciRealm@gmail.com
[b]Independent Researcher, Amarillo, Texas, USA., ricew0244@gmail.com
Version 1, Date: Sept. 02, 2021.

Abstract

Using the 4-vectors of Einstein-Minkowski spacetime as fundamental, these 4D (1,0)-tensors and their interrelations show that foundational features of spacetime common to both special relativity and quantum mechanics exist. The 4D spacetime splitting into 1D temporal + 3D spatial components plays an integral role in understanding these relations.

Key words: special relativity, quantum mechanics, quantum-classical relation, quantum emergence, SRQM, SR→QM

1. Introduction

Special relativity (SR) was derived by Einstein in 1905 to give Maxwell's electromagnetics (EM) a mathematically rigorous kinematic grounding. In 1907 Hermann Minkowski [1] initially developed his spacetime for Maxwell's EM.[2, 3] It is of some interest that in this formulation time is the imaginary coordinate, *ict,* of a complex manifold and the components of the 4-dimensional differential ∂/∂_h is an operator. The interval s was initially defined by $\mathrm{d}s^2 = (ic)^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. In 1909 Minkowski introduced Einstein's SR on a real pseudo-Riemannian 1+3-dimensional spacetime manifold.[4] Minkowski spacetime is equipped with a Lorentz scalar product, a type of inner product, defining the spacetime interval s by $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ which is invariant under Lorentz-Poincaré group transformations. Notice that in Minkowski's 1908 paper taking $i^2 = -1$ gives a real manifold with a space-positive metric signature (-,+,+,+) while in his 1909 paper the time-positive metric signature (+,-,-,-) is used. Both mathematical structures of 4-spacetime derive from the two Einstein postulates of SR [5] although it can also be developed from independent axioms.[6] 4-vectors, denoted generically as $\mathbf{A} = A^\mu = (a^\mu) = (a^0, a^i) = (a^0, \mathbf{a})$, provide a convenient representation of the standard Lorentz group $\Lambda^\mu{}_\nu$ in Minkowski spacetime, with tensor transformation rule $A'^\mu = \Lambda^\mu{}_\nu A^\nu$.

Here the time-positive metric signature (+,-,-,-) is used. Standard notation is used: $\gamma = 1/\sqrt{(1 - \mathbf{u}\cdot\mathbf{u}/c^2)}$, $\mathbf{u}$ = relative 3-velocity, τ = proper time, t = coordinate time. Invariant quantities are often appended with subscript '$_o$' to indicate they are invariant quantities. The Minkowski metric tensor is denoted by $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{Diagonal}[+1, -1, -1, -1]_{(\text{Cartesian})}$. The Lorentz scalar product of two arbitrary 4-vectors, $\mathbf{A} = A^\mu = (a^0, \mathbf{a})$ and $\mathbf{B} = B^\mu = (b^0, \mathbf{b})$ is provided by $(\mathbf{A}\cdot\mathbf{B}) = A^\mu\eta_{\mu\nu}B^\nu = A_\nu B^\nu = A^\mu B_\mu = (a^0b^0 - \mathbf{a}\cdot\mathbf{b}) = (a^0{}_o b^0{}_o)$, which is a Lorentz scalar invariant 4D (0, 0)-tensor. 3-vectors are denoted by lower case bold text. Index raising and lowering is accomplished using the Minkowski Metric, e.g., $R_\mu = \eta_{\mu\nu}R^\nu$.

SR is deterministic in the sense that given initial conditions it provides a definite and unique prediction for a measurement. In contrast QM is deterministic for time-evolution, but is non-deterministic for measurements. It does not provide a unique definite prediction, but instead merely probabilities for a spectrum of eigenvalues for a possible measurement result. One calculates these probabilities from the wavefunction using Born's rule [7, 8] which in many respects appears to be an add-on postulate to Schrödinger's formulation. In orthodox SR the temporal order of measurements is irrelevant and all observables commute; not so in QM. The literature contains a large number of proposals for formulating QM from classical foundations. The Koopman–von Neumann (KvN) mechanics [9, 10] of the 1930s provides a prototypical approach. Quantum Reconstruction attempts to reformulate QM from more intuitive principles or postulates than those proposed by Dirac and von Neumann.[11] These include reformulation of QM in terms of classical concepts such expressing wave functions in terms of appropriate Wigner distributions functions in phase space [12] and replacing quantum commutators with appropriate deformed Poisson brackets like Moyal brackets. More recently Gozzi, Reuter and Thacker [13, 14] propose a path integral formulation of KvN. Other proposals for QM to SR connections include Lam's [15] geometrical quantum formulation of special relativity for 1D trajectory of a free relativistic particle and Bohmian deterministic trajectory.[16]

In addition to the mathematical convenience, 4-vector calculus provides formal foundational connections to quantum mechanics (QM). In this introductory paper it is shown that essential foundational relations of QM exist in SR.

2. Classical Relativistic 4-Vectors

For convenience the well known basic 4-vectors of SR and EM, with SI units are shown in Table 1.

Table 1. Basic SR 4-vectors.

4-Vector Name	4-Vector Definition	4-Vector Dimensionality, SI Units	Equation Number
4-position	$\mathbf{R} = R^\mu = (ct, \mathbf{r})$	m	(2.1)
4-velocity	$\mathbf{U} = U^\mu = \gamma(c, \mathbf{u}) = (\mathbf{U}\cdot\partial)\mathbf{R} = (d/d\tau)\mathbf{R} = d\mathbf{R}/d\tau$	m/s	(2.2)
4-momentum	$\mathbf{P} = P^\mu = (E/c, \mathbf{p}) = (E_o/c^2)\mathbf{U} = (mc, \mathbf{p}) = m_o\mathbf{U}$	kg·m/s = N·s	(2.3)
4-wave vector	$\mathbf{K} = K^\mu = (\omega/c, \mathbf{k}) = (\omega_o/c^2)\mathbf{U}$	radian/m	(2.4)
4-gradient	$\partial = \partial^\mu = (\partial_t/c, -\nabla) = \partial/\partial R_\mu$	1/m	(2.5)
4-(dust) number flux	$\mathbf{N} = N^\mu = (nc, \mathbf{n}) = n_o\mathbf{U}$	number/(m²·s) = (number/m³)·(m/s)	(2.6)
4-current density, 4-charge flux	$\mathbf{J} = J^\mu = (\rho c, \mathbf{j}) = \rho_o\mathbf{U}$	C/(m²·s) = (C/m³)·(m/s) = A/m²	(2.7)
4-(EM) vector potential	$\mathbf{A} = A^\mu = (\varphi/c, \mathbf{a}) = (\varphi/c^2)\mathbf{U}$	kg·m/(C·s) = T·m	(2.8)

Useful invariant 4-scalars derived from the Lorentz scalar product rule are listed in Table 2.

Table 2. Basic invariant SR 4-scalars.

Lorentz scalar product of 4-Vectors	4-Scalar Name	4-Scalar Symbol	4-Scalar Dimensionality, SI Units	Equation Number
$\mathbf{R}\cdot\mathbf{R} = (ct)^2 - \mathbf{r}\cdot\mathbf{r} = (ct_o)^2 = (c\tau)^2 = (i\lvert\mathbf{r}_o\rvert)^2$	Proper time, Proper length	$t_o = \tau$, $\lvert\mathbf{r}_o\rvert$	s, m	(2.9)
$\mathbf{U}\cdot\mathbf{U} = \gamma^2[c^2 - \mathbf{u}\cdot\mathbf{u}] = c^2$	Invariant Light speed	c	m/s	(2.10)
$\mathbf{P}\cdot\mathbf{P} = (E/c)^2 - \mathbf{p}\cdot\mathbf{p} = (E_o/c)^2 = (m_oc)^2$	Invariant energy	$E_o = m_oc^2$	kg·m²/s² = J	(2.11)
$\mathbf{K}\cdot\mathbf{K} = (\omega/c)^2 - \mathbf{k}\cdot\mathbf{k} = (\omega_o/c)^2$	Invariant angular frequency	ω_o	radian/s	(2.12)
$\partial\cdot\partial = (\partial_t/c)^2 - \nabla\cdot\nabla = (\partial_\tau/c)^2$	Invariant d'Alembertian 4-wave equation	$\partial\cdot\partial$	1/m²	(2.13)
$\mathbf{N}\cdot\mathbf{N} = (nc)^2 - \mathbf{n}\cdot\mathbf{n} = (n_oc)^2$	Invariant number density	n_o	number/m³	(2.14)
$\mathbf{J}\cdot\mathbf{J} = (\rho c)^2 - \mathbf{j}\cdot\mathbf{j} = (\rho_oc)^2$	Invariant charge density	ρ_o	C/m³	(2.15)
$\mathbf{A}\cdot\mathbf{A} = (\varphi/c)^2 - \mathbf{a}\cdot\mathbf{a} = (\varphi_o/c)^2$	Invariant electric potential	φ_o	V = J/C	(2.16)

3. An Invariant Called *ħ*

Applying the general method of dividing the magnitudes of two arbitrary 4-vectors $\mathbf{A} = A^\mu = (a^0, \mathbf{a})$ and $\mathbf{B} = B^\mu = (b^0, \mathbf{b})$ by using a third arbitrary 4-vector $\mathbf{V} = V^\mu = (v^0, \mathbf{v})$ such that $\lvert\mathbf{A}\rvert/\lvert\mathbf{B}\rvert = (\mathbf{A}\cdot\mathbf{V})/(\mathbf{B}\cdot\mathbf{V}) = (a^0{}_o v^0{}_o)/(b^0{}_o v^0{}_o) = (a^0{}_o/b^0{}_o)$, which is a Lorentz scalar invariant, to the 4-momentum **P** and 4-wave vector **K** gives:

$$\lvert\mathbf{P}\rvert/\lvert\mathbf{K}\rvert = (\mathbf{P}\cdot\mathbf{U})/(\mathbf{K}\cdot\mathbf{U}) = \gamma(E - \mathbf{p}\cdot\mathbf{u})/[\gamma(\omega - \mathbf{k}\cdot\mathbf{u})] = E_o/\omega_o\ , \tag{3.1}$$

$$\lvert\mathbf{P}\rvert/\lvert\mathbf{K}\rvert = (\mathbf{P}\cdot\mathbf{K})/(\mathbf{K}\cdot\mathbf{K}) = (E\omega/c^2 - \mathbf{p}\cdot\mathbf{k})/[(\omega/c)^2 - \mathbf{k}\cdot\mathbf{k}] = m_o\omega_o/(\omega_o/c)^2 = E_o/\omega_o\ , \tag{3.2}$$

$$\lvert\mathbf{P}\rvert/\lvert\mathbf{K}\rvert = (\mathbf{P}\cdot\mathbf{P})/(\mathbf{K}\cdot\mathbf{P}) = (E^2/c^2 - \mathbf{p}\cdot\mathbf{p})/(E\omega/c^2 - \mathbf{p}\cdot\mathbf{k}) = (m_oc)^2/(m_o\omega_o) = E_o/\omega_o\ , \tag{3.3}$$

$$\lvert\mathbf{P}\rvert/\lvert\mathbf{K}\rvert = (\mathbf{P}\cdot\mathbf{R})/(\mathbf{K}\cdot\mathbf{R}) = (Et - \mathbf{p}\cdot\mathbf{r})/(\omega t - \mathbf{k}\cdot\mathbf{r}) = (-S_{\text{action,free particle}})/(-\Phi_{\text{phase,planewave}}) = E_o/\omega_o\ . \tag{3.4}$$

It is an empirical fact that the Lorentz invariant $E_o/\omega_o = \gamma E_o/(\gamma\omega_o) = E/\omega = \hbar$ is a theory-independent measurable quantity just like the values of electric charge q and free space light speed c and the other fundamental constants. $\hbar$ can be measured classically from the photoelectric effect, from the inverse photoelectric effect, from atomic line spectra (Rydberg spectra), from the Duane-Hunt Law in bremsstrahlung, from electron diffraction in crystals, from the Kibble balance, from incandescent blackbody intensity-temperature relations, from Compton scattering, from the Stern-Gerlach experiment and others independent of their theoretical explanations. Computer simulations of many experiments, suitable for pedagogical and other purposes, are available.[17]

This implies the following relation, derived from standard 4-tensor arguments of SR,

$$\mathbf{P} = \hbar\mathbf{K} = (E/c, \mathbf{p}) = \hbar(\omega/c, \mathbf{k}) \,. \tag{3.5}$$

The temporal part $E = \hbar\omega$ is Einstein's photoelectric quantum postulate and the spatial part $\mathbf{p} = \hbar\mathbf{k}$ is de Broglie's matter-wave postulate. The simplicity of this relation is very similar to Einstein's well known SR relations,

$$\mathbf{P} = m_o\mathbf{U} = (E/c, \mathbf{p}) = m_o\gamma(c, \mathbf{u}) = m(c, \mathbf{u}) = (mc, m\mathbf{u}) = E/c^2\,(c, \mathbf{u}) = \gamma E_o/c^2\,(c, \mathbf{u}) = E_o/c^2\,\mathbf{U} \,. \tag{3.6}$$

The temporal part $E = \gamma m_o c^2 = mc^2 = \gamma E_o$ is the SR energy-mass relation and the spatial part $\mathbf{p} = \gamma m_o\mathbf{u} = m\mathbf{u} = E\mathbf{u}/c^2$ is relativistic momentum. Just as a note, the 4-momentum $\mathbf{P}$ is used in purely relativistic particle collision calculations and the 4-wave vector $\mathbf{K}$ is used in purely relativistic Doppler effect calculations. Both are used in the relativistic Compton effect photon-electron scattering calculations.

4. Existence of a Complex Function ψ

It is an empirical and mathematical fact that all waves (classical, relativistic and quantum) can be modeled using complex plane-waves. In SR 4-vector terms this gives the 4-vector relation,

$$\partial = -i\mathbf{K} \,, \tag{4.1}$$

with a wavefunction ψ of the form

$$\psi = ae^{\pm i(\mathbf{K}\cdot\mathbf{X})} = ae^{\pm i\varphi} = ae^{\pm iS/\hbar} \,, \tag{4.2}$$

where $S = \hbar\,\varphi$ = action. The amplitude a is an arbitrary constant. It can be a quantum scalar 4D (0,0)-tensor A, an EM/photonic 4D (1,0)-tensor A^{μ} or a gravitational wave 4D (2,0)-tensor $A^{\mu\nu}$. Then

$$\partial[\psi] = \partial[ae^{\pm i(\mathbf{K}\cdot\mathbf{X})}] = \pm i\mathbf{K}[ae^{\pm i(\mathbf{K}\cdot\mathbf{X})}] = \pm i\mathbf{K}[\psi] \,. \tag{4.3}$$

The selection of the minus sign in Eq (4.1) is a historical convention, and leads to standard convention of later equations derived from this relation.

Using Eqs. (3.5) and (4.1), the Lorentz scalar product relations from Table 2. can be re-visited to give the chain of relations $\mathbf{R}\cdot\mathbf{R} = (c\tau)^2$, $\mathbf{U}\cdot\mathbf{U} = c^2$, $\mathbf{P}\cdot\mathbf{P} = (m_o c)^2$, $\mathbf{K}\cdot\mathbf{K} = (m_o c/\hbar)^2$ and $\partial\cdot\partial = (im_o c/\hbar)^2 = -(m_o c/\hbar)^2$. The latter is the fundamental Klein-Gordon wave relation of relativistic quantum mechanics (RQM).

$$\partial\cdot\partial = (im_o c/\hbar)^2 = -(m_o c/\hbar)^2 \tag{4.4}$$

The Klein-Gordon (K-G) relation [18] implies there can exist a "wavefunction" Ψ which solves it, in the same way that the relativistic 4D Euler-Lagrange relation, $(\mathbf{U}\cdot\partial_{\mathbf{R}})\,\partial_{\mathbf{U}} = (d/d\tau)\partial_{\mathbf{U}} = \partial_{\mathbf{R}}$, implies there can exist a Lagrangian function (L) that solves it. One does not need a presupposed quantum axiom.

The Klein-Gordon (K-G) relation gives a 2nd order, linear PDE. The fact that it is a linear PDE leads to principles of quantum superposition. As a note, the standard Schrödinger quantum wave equation is the non-relativistic ($|\mathbf{v}| \ll c$) limit-case of the K-G relativistic quantum wave equation, which continues to show superposition.

5. Existence of a Non-Zero Commutator

A non-zero commutation relation between 4-position $\mathbf{X}$ and 4-momentum $\mathbf{P}$ exists in SR. To prove this let f be an arbitrary function in SR. Using 4-position $\mathbf{X} = (ct, \mathbf{r})$, Eq (2.1), and 4-gradient $\partial = (\partial_t/c, -\nabla)$, Eq (2.5), gives the following two primitive relations:

$$\mathbf{X}[f] = \mathbf{X}f \,, \tag{5.1}$$

$$\partial[f] = \partial[f] \,. \tag{5.2}$$

$\mathbf{X}$, whether an operator or not, has no effect on f, but ∂ absolutely has an effect on f. Continuing the analysis, using only the standard rules of calculus gives:

$$\mathbf{X}[\partial[f]] = \mathbf{X}\partial[f] \,, \tag{5.3}$$

$$\partial[\mathbf{X}[f]] = \partial[\mathbf{X}f] = \partial[\mathbf{X}]f + \mathbf{X}\partial[f] \,, \tag{5.4}$$

$$\partial[\mathbf{X}f] - \mathbf{X}\partial[f] = \partial[\mathbf{X}]f, \tag{5.5}$$

$$\partial[\mathbf{X}[f]] - \mathbf{X}[\partial[f]] = \partial[\mathbf{X}]f. \tag{5.6}$$

Recognizing this last as a commutation relation, $\mathbf{A}[\mathbf{B}[f]] - \mathbf{B}[\mathbf{A}[f]] = [\mathbf{A}, \mathbf{B}]\, f$, and using commutator notation gives,

$$[\partial, \mathbf{X}]f = \partial[\mathbf{X}]f. \tag{5.7}$$

Since f is an arbitrary function it can be set to unity leaving

$$[\partial, \mathbf{X}] = \partial[\mathbf{X}] = (\partial_t/c, -\nabla)[(ct, \mathbf{r})] = (\partial_t/c, -\partial_x, -\partial_y, -\partial_z)[(ct, x, y, z)]_{\{\text{Cartesian}\}} = \text{Diag}[+1,-1,-1,-1] = \eta^{\mu\nu} \tag{5.8}$$

or, in explicit tensor notation

$$[\partial, \mathbf{X}] = [\partial^\mu, X^\nu] = \partial^\mu[X^\nu] = \eta^{\mu\nu} \,. \tag{5.9}$$

Thus, there is a SR non-zero commutation relation between the SR 4-gradient and 4-position which leads to the Minkowski Metric. Note also that $\mathbf{X}[f] = \mathbf{X}f$ doesn't necessarily imply that $\mathbf{X}$ is an operator (it could be an operator or just a number). However the 4-gradient, ∂, is an operator, because it is already an operator function in pure SR.

Now, from complex plane-waves used in classical EM, $\mathbf{K} = i\partial$, Eq (4.1), and (3.5) gives:

$$[\partial, \mathbf{X}] = [\partial^\mu, X^\nu] = \eta^{\mu\nu} \,, \tag{5.10}$$

$$[i\partial, \mathbf{X}] = [i\partial^\mu, X^\nu] = i\eta^{\mu\nu} \,, \tag{5.11}$$

$$[\mathbf{K}, \mathbf{X}] = [K^\mu, X^\nu] = i\eta^{\mu\nu} \,, \tag{5.12}$$

$$[\hbar\mathbf{K}, \mathbf{X}] = [\hbar K^\mu, X^\nu] = i\hbar\eta^{\mu\nu} \,, \tag{5.13}$$

$$[\mathbf{P}, \mathbf{X}] = [P^\mu, X^\nu] = i\hbar\eta^{\mu\nu} \,, \tag{5.14}$$

$$[\mathbf{X}, \mathbf{P}] = [X^\mu, P^\nu] = -i\hbar\eta^{\mu\nu} \,. \tag{5.15}$$

Eq (5.15) is the major result that SR 4-position $\mathbf{X}$ and 4-momentum $\mathbf{P}$ do not commute. The temporal part $[x^0, p^0] = [ct, E/c] = [t, E] = -i\hbar\eta^{00} = -i\hbar$ is the "oft misunderstood" time-energy commutation. The spatial part $[x^j, p^k] = -i\hbar\eta^{jk} = i\hbar\delta^{jk}$ is the standard canonical commutation relation of QM. For the mixed parts $[x^0, p^k] = [x^j, p^0] = \eta^{0k} = \eta^{j0} = 0$, meaning these parts commute normally. Similar 4-vector arguments lead to the standard angular-momentum quantum commutation relations via the anti-symmetric 4-angular momentum $M^{\mu\nu} = X^\mu \wedge P^\nu$. In fact, the entire Poincaré Algebra can be generated in this fashion. Likewise, the general mathematical uncertainty relations, $\sigma_A{}^2\sigma_B{}^2 \geq 1/2\, |\langle[\hat{\mathbf{A}},\hat{\mathbf{B}}]\rangle|$, based on commutation relations, lead to the standard physical quantum Heisenberg uncertainty relations. Also note that the commutator order of operations is in accord with SR causality conditions. While space-like separated events $|\text{here}\rangle$ and $|\text{there}\rangle$ may occur in any temporal order depending on observer reference frames, all observers will see the same temporal order of time-like separated events. Thus, if measurement $\hat{\mathbf{A}}$ occurs temporally before measurement $\hat{\mathbf{B}}$ then this would be written in operator notation as $|\Psi'\rangle = \hat{\mathbf{A}}|\Psi\rangle$ then $|\Psi''\rangle = \hat{\mathbf{B}}|\Psi'\rangle = \hat{\mathbf{B}}\hat{\mathbf{A}}|\Psi\rangle$. Due to non-zero commutation relations, $\hat{\mathbf{A}}\hat{\mathbf{B}}|\Psi\rangle$ would likely give a different result.

6. Existence of a Conserved Current $\mathbf{J}_{\text{prob}}$

Special relativity supports a conserved current. Consider the vector identity,

$$\partial\cdot(f\,\partial[g] - \partial[f]\,g) = f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g \,, \tag{6.1}$$

where f and g are Lorentz scalar functions. The 4-divergence relation follows,

$$\partial\cdot(f\,\partial[g] - \partial[f]\,g) = \partial\cdot(f\,\partial[g]) - \partial\cdot(\partial[f]\,g)$$
$$= (f\,\partial\cdot\partial[g] + \partial[f]\cdot\partial[g]) - (\partial[f]\cdot\partial[g] + \partial\cdot\partial[f]\,g)$$
$$= f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g\,. \quad (6.2)$$

Multiplying by a constant Lorentz invariant scalar constant, s, for dimensional units gives

$$s(f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g) = s\,\partial\cdot(f\,\partial[g] - \partial[f]\,g) = \partial\cdot[s(f\,\partial[g] - \partial[f]\,g)] = \partial\cdot\mathbf{J}\,. \quad (6.3)$$

Thus, from the d'Alembertian 4-wave equation, Eq (2.13), there exists a 4-current

$$\mathbf{J} = [s(f\,\partial[g] - \partial[f]\,g)] \quad (6.4)$$

for any arbitrary scalar invariant s.

Let the K-G relation, Eq (4.4), written in the form $\partial\cdot\partial + (m_oc/\hbar)^2 = 0$ act on Lorentz invariant functions f and g giving

$$\partial\cdot\partial[f] + (m_oc/\hbar)^2\,[f] = 0\,[f] \text{ and } \partial\cdot\partial[g] + (m_oc/\hbar)^2\,[g] = 0\,[g]\,. \quad (6.5)$$

Post-multiplying the first by g and pre-multiplying the second by f gives

$$\partial\cdot\partial[f]\,g + (m_oc/\hbar)^2[f]\,g = 0\,[f]g \text{ and } f\,\partial\cdot\partial[g] + f\,(m_oc/\hbar)^2[g] = f\,0\,[g]\,, \quad (6.6)$$

so

$$f\,\partial\cdot\partial[g] + (m_oc/\hbar)^2 fg = 0 \text{ and } \partial\cdot\partial[f]g + (m_oc/\hbar)^2 fg = 0\,. \quad (6.7)$$

Subtracting one from the other gives

$$f\,\partial\cdot\partial[g] - \partial\cdot\partial[f]\,g = 0\,. \quad (6.8)$$

As noted from the mathematical vector identity Eq (6.1), this can be written as a 4-divergence with the additional constraint that it now equates to 0, meaning that it is conserved 4-current $\mathbf{J}$,

$$\partial\cdot[s(f\,\partial[g] - \partial[f]\,g)] = \partial\cdot\mathbf{J} = 0\,. \quad (6.9)$$

Thus, there exists a conserved 4-vector current,

$$\mathbf{J_{prob}} = (\rho_{\text{prob}}c, \mathbf{j_{prob}}) = s(f\,\partial[g] - \partial[f]\,g)\,, \quad (6.10)$$

for which $\partial\cdot\mathbf{J_{prob}} = 0$ and which also solves the K-G relation.

Using Eq (4.1) and a complex plane-wave function $g = ae^{-i(\mathbf{K}\cdot\mathbf{X})} = \psi$ choose $f = g^* = ae^{+i(\mathbf{K}\cdot\mathbf{X})} = \psi^*$ as its complex conjugate. Let $s = i\hbar/(2m_o) = ic^2/(2\omega_o)$ which is Lorentz scalar invariant, in order to make the probability have dimensionless units and be normalized to unity in the invariant case. Then a probability current can be written as

$$\mathbf{J_{prob}} = (\rho_{prob}c, \mathbf{j_{prob}}) = i\hbar/(2m_o)\,(\psi^*\partial[\psi] - \partial[\psi^*]\,\psi) = s\,(\psi^*\partial[\psi] - \partial[\psi^*]\,\psi)\,. \quad (6.11)$$

Examining the temporal component, the relativistic probability density is

$$\rho_{\text{prob}}c = i\hbar/(2m_o)\,(\psi^*\,(\partial_t/c)[\psi] - (\partial_t/c)[\psi^*]) \quad (6.12)$$

so

$$\rho_{\text{prob}} = i\hbar/(2m_oc^2)\,(\psi^*\,\partial_t[\psi] - \partial_t[\psi^*]\,\psi) = i/(2\omega_o)\,(\psi^*\,\partial_t[\psi] - \partial_t[\psi^*]\,\psi)\,. \quad (6.13)$$

Assume a wave solution in following general form

$$\psi = A\,f\,[k]\,e^{-i\omega t} \text{ and } \psi^* = A^*\,f\,[k]^*\,e^{+i\omega t} \quad (6.14)$$

then

$$\partial_t[\psi] = -i\omega\, A\, f[k]\, e^{-i\omega t} = -i\omega\, \psi \text{ and } \partial_t[\psi^*] = i\omega\, A^*\, f[k]^*\, e^{+i\omega t} = i\omega\, \psi^* , \tag{6.15}$$

so

$$\begin{aligned}\rho_{\text{prob}} &= i/(2\omega_o)\,(\psi^*\, \partial_t[\psi] - \partial_t[\psi^*]\, \psi) = \mathrm{i}/(2\omega_o)\,(-i\omega\, \psi^*\psi - i\omega\, \psi^*\psi)\\ &= i/(2\omega_o)\,(-2i\omega\, \psi^*\psi) = \omega/\omega_o\, \psi^*\psi = \gamma\omega_o/\omega_o\, \psi^*\psi = \gamma\psi^*\psi\\ &= \gamma(\rho_{\text{prob o}}) .\end{aligned} \tag{6.16}$$

Ivancevic and Ivancevic show the EM 4-current is formally analogous to a 4-probability current.[19] Multiplying $\mathbf{J}_{\mathbf{prob}}$, Eqs (6.10) and (6.11), by charge q gives the standard SR EM

$$\text{4-current density} = \text{4-charge flux} = q\mathbf{J}_{\mathbf{prob}} = q(c\rho_{\text{prob}}, \mathbf{j}_{\mathbf{prob}}) = \mathbf{J} = (\rho c, \mathbf{j}) \tag{6.17}$$

and the generalized 4-current within a 4-vector potential **A**

$$\text{4-probability current density } \mathbf{J}_{\mathbf{prob}} = (c\rho_{\text{prob}}, \mathbf{j}_{\mathbf{prob}}) = i\hbar/(2m_o)\,(\psi^*\partial[\psi] - \partial[\psi^*]\psi) + q/m_o\,(\psi^*\psi)\mathbf{A} . \tag{6.18}$$

The temporal component is

$$\rho_{\text{prob}} = i\hbar/(2m_o c^2)\,(\psi^*\, \partial_t[\psi] - \partial_t[\psi^*]\, \psi) + q/m_o\,(\psi^*\psi)(\varphi/c^2) , \tag{6.19}$$

so

$$\rho_{\text{prob}} \rightarrow \gamma(\psi^*\psi) + \gamma(q\varphi_o/m_o\mathrm{c}^2)(\psi^*\psi) = \gamma[1 + q\varphi_o/E_o](\psi^*\psi) . \tag{6.20}$$

7. Born Probability Interpretation

Examine the low-potential-energy limit. Take $q\varphi_o << \mathrm{E}_o$ which gives the EM factor $q\varphi_o/E_o \sim 0$. Now, taking the low-velocity limit $\gamma \rightarrow 1$ of $\rho_{\text{prob}} = \gamma[1 + {\sim}0](\psi^*\psi)$, $\rho_{\text{prob}} \rightarrow \psi^*\psi = \rho_{\text{prob o}}$ for $|\mathbf{u}| << c$. The standard Born probability interpretation, $\psi^*\psi = \rho_{\text{prob}}$, emerges in the low-potential-energy and low-velocity limit. This is why the non-positive-definite probabilities and |probabilities| > 1in the RQM Klein-Gordon equation puzzled physicists and is the reason why one must regard the probabilities as charge density conservation $\boldsymbol{\partial}\cdot\mathbf{J}$ instead.

The original definition from SR is continuity of worldlines $\boldsymbol{\partial}\cdot\mathbf{J}_{\mathbf{prob}} = 0$ for which all is good and well in the RQM version. The definition says there are no external sources or sinks of probability implying conservation of probability. The Born idea that total probability $\rho_{\text{prob}} \rightarrow \Sigma\,(\psi^*\psi) = 1$ is just the low-velocity QM limit. It is not a fundamental axiom. Multiplying by charge q gives the

$$\text{4-charge current density } \mathbf{J} = (c\rho, \mathbf{j}) = q\mathbf{J}_{\mathbf{prob}} = q(c\rho_{\text{prob}}, \mathbf{j}_{\mathbf{prob}}) \tag{7.1}$$

which is the standard SR EM 4-current density.

Note from Eq 2.6, the 4-vector **N** has dimensional units of [number-flux] and the 4-scalar invariant number-density (n_o) has dimensional units of [number/volume]. This is the same as the dimensional units of an invariant probability-density (ρ_o), also [number/volume]. This leads to the idea that the QM 4-probability current $\mathbf{J}_{\mathbf{prob}}$ is equivalent to the SR 4-(dust) number flux **N**. The concepts are actually quite similar if one considers the fluid approximation of individual particles. The fluid allows densities that are less than unity, much as probabilities of expected positions of particles are less than unity and only sum to unity over the entire volume. This argument is further strengthened by noting that in QM one also has $\mathbf{J} = q\mathbf{J}_{\mathbf{prob}}$, Eq (7.1).

The Koopman-von Neuman (KvN) formalism shows that classical mechanics can be formulated as an operational theory on a Hilbert space of complex L^2 wavefunctions analogous to QM. To accomplish this the Liouville equation is extended by introducing a phase space function of debatable physical interpretation. Klein [20] developed an alternative function $S(t, q, p)$ that expresses the classical action while retaining a mathematical analogy to QM. The idea that Hilbert Space requires a quantum axiom is disproved by the KvN formulation of classical mechanics in which Hilbert Space mathematical formulation is successfully applied and results in the classical Liouville equation. The Hilbert Space framework is purely mathematical and can be applied to both classical and quantum systems which is one of the more powerful connections between classical mechanics and QM.

The main difference between which system emerges is the commutation relation between position and momentum. In the classical case, one assumes a zero-valued commutation relation. In the quantum case, there is a non-zero commutation relation. SR contains a non-zero commutation relation, thus leading to the QM case.

8. Schrödinger QM relations

The Schrödinger QM relations are now easily derived from SR as follows. Using the positive form of Eq (4.3) for complex plane-waves used in classical EM, $\mathbf{K} = i\partial$, and the empirical Eq (5.3), $\mathbf{P} = \hbar\mathbf{K}$,

$$\mathbf{P} = i\hbar\partial = (E/c, \mathbf{p}) = i\hbar(\partial_t/c, -\nabla). \tag{8.1}$$

The temporal part $E = i\hbar\partial_t = i\hbar\, \partial/\partial t$ gives the unitary QM time evolution operator and the spatial part $\mathbf{p} = -i\hbar\nabla$ gives the QM momentum operator.

9. CPT Symmetry

Additionally, SR contains the essence of CPT Symmetry. The Lorentz transformations, $\Lambda^{\mu}{}_{\nu}$, play a fundamental role in SR of describing the inherent symmetries of spacetime. The main ones usually mentioned are the continuous transforms which include the temporal-spatial velocity boost $\Lambda^{\mu}{}_{\nu} \rightarrow B^{\mu}{}_{\nu}$ with parameter $\boldsymbol{\beta} = \mathbf{u}/c$ and the spatial-spatial rotation $\Lambda^{\mu}{}_{\nu} \rightarrow R^{\mu}{}_{\nu}$ with parameter θ. However, there also exist discrete Lorentz transformations. One is the 4-identity $\Lambda^{\mu}{}_{\nu} \rightarrow \delta^{\mu}{}_{\nu}$, which leaves a system completely unchanged. It is a special case of both the boost and rotation transforms when their parameters are $|\boldsymbol{\beta}| = \theta = 0$ which then gives $B^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$.

Most well known of the discrete Lorentz transformations are the Parity $\Lambda^{\mu}{}_{\nu} \rightarrow P^{\mu}{}_{\nu}$ which reverses the three spatial coordinates $\mathbf{x} \rightarrow -\mathbf{x}$, and the Time Reversal $\Lambda^{\mu}{}_{\nu} \rightarrow T^{\mu}{}_{\nu}$ which reverses the single temporal coordinate $t \rightarrow -t$. Less well known are the other Lorentz transformations which include rotations of a fixed amount and spatial flips which reverse only two of the spatial coordinates. It turns out that one can individually reverse any combination of the coordinates and still have a valid Lorentz transformation, (i.e., having a determinant = ±1). Reversal of all the coordinates, $t \rightarrow -t$ and $\mathbf{x} \rightarrow -\mathbf{x}$, gives the Combo(PT) transformation

$$\Lambda^{\mu}{}_{\nu} \rightarrow (PT)^{\mu}{}_{\nu} = C^{\mu}{}_{\nu} . \tag{9.1}$$

Examination of all possible combinations of discrete Lorentz transformations leads to CPT symmetry. In other words, one can go from the identity transformation (all +1) to the negative identity transformation (all -1) by doing a charge reversal Lorentz transformation here called the Combo(PT). This negative identity has the interpretation of antimatter without any need of Dirac's formulation using RQM. The Feynman-Stueckelberg CPT interpretation, that antimatter moving temporally backward in spacetime is equivalent to normal matter moving temporally forward in spacetime, aligns with this.[21, 22] Let $A^{\mu}{}_{\nu}$ be an arbitrary (1,1)-tensor, of which the Lorentz transformation is an example. Two interesting properties are,

$$\mathrm{Tr}(A^{\mu}{}_{\nu}) = \Sigma_j\, a_j \tag{9.2}$$

and

$$\mathrm{Det}(A^{\mu}{}_{\nu}) = \Pi_j\, a_j , \tag{9.3}$$

where a_j are the eigenvalues of $A^{\mu}{}_{\nu}$, $j = 1,2,3,\ldots$. As 4D tensors, each Lorentz transformation has 4 eigenvalues. Now, create an anti-transformation which has all eigenvalue tensor invariants negated,

$$\Sigma_{j=1}{}^{4} -a_j = -\Sigma_{j=1}{}^{4}\, a_j . \tag{9.4}$$

The anti-transform has negative trace of the transformation. The transformation and anti-transformation have equal determinants, meaning that a proper transform has a proper anti-transform,

$$\Pi_{j=1}{}^{4} -a_j = \Pi_{j=1}{}^{4}\, a_j . \tag{9.5}$$

Thus, the trace invariant identifies a "dual" negative-side for all Lorentz transformations. This (NM = normal matter, AM = antimatter) interpretation can be analyzed using tensor determinant and trace operations:

$$\mathrm{Tr[NM\text{-}identity]} = +4 , \quad +4 \leq \mathrm{Tr[NM\text{-}boost]} < +\infty , \quad 0 \leq \mathrm{Tr[NM\text{-}rotation]} \leq +4 , \tag{9.6a, b, c}$$
$$\mathrm{Tr[AM\text{-}identity]} = -4 , \quad -\infty < \mathrm{Tr[AM\text{-}boost]} \leq -4 , \quad -4 \leq \mathrm{Tr[AM\text{-}rotation]} \leq 0. \tag{9.7a, b, c}$$

Table 3. shows the complete (+/-) symmetry between the two, which agrees with all known experiments with normal matter and antimatter to-date.[23]

Table 3. The complete symmetry of discrete Lorentz transformations.

t	x	y	z	Discrete Normal Matter (NM) Lorentz Transform Type	Trace	Determinant
+1	+1	+1	+1	NM-Minkowksi Identity: AM-Flip-txyz=AM-Combo(PT)	Tr = +4	Det = +1 Proper
+1	+1	+1	–1	NM-Flip-z	Tr = +2	Det = –1 Improper
+1	+1	–1	+1	NM-Flip-y	Tr = +2	Det = –1 Improper
+1	+1	–1	–1	NM-Flip-yz=NM-Rotate-yz(π)	Tr = 0	Det = +1 Proper
+1	–1	+1	+1	NM-Flip-x	Tr = +2	Det = –1 Improper
+1	–1	+1	–1	NM-Flip-xz=NM-Rotate-xz(π)	Tr = 0	Det = +1 Proper
+1	–1	–1	+1	NM-Flip-xy=NM-Rotate-xy(π)	Tr = 0	Det = +1 Proper
+1	–1	–1	–1	NM-Flip-xyz=NM-ParityInverse: AM-Flip-t=AM-TimeReversal	Tr = –2	Det = –1 Improper
–1	+1	+1	+1	AM-Flip-xyz = AM-ParityInverse: NM-Flip-t=NM-TimeReversal	Tr = +2	Det = –1 Improper
–1	+1	+1	–1	AM-Flip-xy = AM-Rotate-xy(π)	Tr = 0	Det = +1 Proper
–1	+1	–1	+1	AM-Flip-xz =AM-Rotate-xz(π)	Tr = 0	Det = +1 Proper
–1	+1	–1	–1	AM-Flip-x	Tr = –2	Det = –1 Improper
–1	–1	+1	+1	AM-Flip-yz = AM-Rotate-yz(π)	Tr = 0	Det = +1 Proper
–1	–1	+1	–1	AM-Flip-y	Tr = –2	Det = –1 Improper
–1	–1	–1	+1	AM-Flip-z	Tr = –2	Det = –1 Improper
–1	–1	–1	–1	AM-Minkowksi Identity: NM-Flip-txyz = NM-Combo(PT)	Tr = –4	Det = +1 Proper
$\bar{t}$	$\bar{x}$	$\bar{y}$	$\bar{z}$	**Discrete AntiMatter (AM) Lorentz Transform Type**	**Trace**	**Determinant**

Table 4. shows the transformation grouped by the trace values. It very clearly shows that Combo(PT) transformation is equivalent to a charge transformation, which flips matter ↔ antimatter. Also this charge transform is proper, with a determinant of +1, the same as the boost and rotation transformations.

Table 4. Symmetries sorted by Trace.

Discrete Normal Matter (NM) Lorentz Transform Type	Trace	Determinant
NM-Minkowksi Identity AM-Flip-txyz = AM-Combo(PT) = AM-NegateIdentity ~ AM-NegateCharge	Tr = +4	Det = +1 Proper
NM-Flip-t ,NM-Flip-x, NM-Flip-y, NM-Flip-z AM-Flip-xyz = AM-ParityInverse	Tr = +2	Det = –1 Improper
NM-Flip-xy=NM-Rotate-xy(π), NM-Flip-xz=NM-Rotate-xz(π), NM-Flip-yz = NM-Rotate-yz(π) AM-Flip-xy = AM-Rotate-xy(π), AM-Flip-xz = AM-Rotate-xz(π), AM-Flip-yz = AM-Rotate-yz(π)	Tr = 0	Det = +1 Proper
NM-Flip-xyz=NM-ParityInverse AM-Flip-t, AM-Flip-x, AM-Flip-y, AM-Flip-z	Tr = –2	Det = –1 Improper
NM-Flip-txyz = NM-Combo(PT) = NM-NegateIdentity ~ NM-NegateCharge AM-Minkowksi Identity	Tr = –4	Det = +1 Proper
Discrete AntiMatter (AM) Lorentz Transform Type	**Trace**	**Determinant**

10. Relativity - Quantum - Classical Correspondence Principle

In SR one finds the Newtonian classical limiting-case approximation by using $|\mathbf{v}| << c$. In QM, there have been a variety of approaches to the Newtonian classical limiting-case approximation, including the idea of number of particles >> 1 and the idea of the action $S >> \hbar$. In the standard view of the theories of relativity and quantum mechanics, it is interesting to speculate on how the two "different" theories "conspire" to end up at the same classical mechanics approximation. However, in the SRQM view, this difficulty disappears. SR leads to RQM via the approach that has been shown. RQM then goes to QM as a limiting-case approximation by using $|\mathbf{v}| << c$. QM then goes to CM as a limiting-case in its own manner. There is a single chain of relationships, rather than two different theories "amazingly" approaching the same classical limit-case.

11. Conclusion

Using the 4-vector calculus of Einstein-Minkowski spacetime it is shown that foundational features of spacetime common to both special relativity and quantum mechanics exist. $\hbar$ is shown to be an empirically measurable constant and a Lorentz scalar, just like c. The 4-vector relations $\mathbf{P} = \hbar\mathbf{K}$, the wave view, and $\mathbf{P} = m_o\mathbf{U}$, the particle view, are shown to be isomorphic in the sense that both are derivable from SR. The mathematical relation $\mathbf{K} = i\partial$ and existence of complex wavefunction ψ is shown applicable to all types of waves: classical, quantum, and relativistic. The waves are all described by tensor amplitudes and the Lorentz scalar product function $e^{\pm i(\mathbf{K}\cdot\mathbf{X})}$ propagator. The combination of these relations lead to a K-G relativistic quantum wave relation $(\partial\cdot\partial) = (im_o c/\hbar)^2 = -(m_o c/\hbar)^2$ and to the 4-vector form of the standard Schrödinger relations $\mathbf{P} = i\hbar\partial$. There exists a non-zero commutation relation in SR, $[X^\mu, P^\nu] = -i\hbar\eta^{\mu\nu}$. There exists a conserved current $\mathbf{J}_{prob}$ based on a simple vector identity. The standard Born probability interpretation, $\psi^*\psi = \rho_{prob\ o}$, emerges in the low-potential-energy and low-velocity limit. CPT Symmetry emerges from an analysis of the Lorentz Transformations. The correspondence principle of both SR and QM to Newtonian classical physics is discussed.

A few of the more useful 4-vectors and invariants showing connections from SR to QR are collected here in Table 5. for convenience.

Table 5. Useful SRQM 4-vectors and invariants.

Standard SR 4-Vectors	**Empirical Relation**	**Lorentz scalar product**	**Notes**
4-position $\mathbf{R} = (ct, \mathbf{r})$		$\mathbf{R}\cdot\mathbf{R} = (c\tau)^2$	(τ) is the invariant proper time
4-velocity $\mathbf{U} = \gamma(c, \mathbf{u})$	$\mathbf{U} = d\mathbf{R}/d\tau$	$\mathbf{U}\cdot\mathbf{U} = c^2$	(c) is the invariant light speed
4-momentum $\mathbf{P} = (E/c, \mathbf{p})$	$\mathbf{P} = m_o\mathbf{U}$	$\mathbf{P}\cdot\mathbf{P} = (m_o c)^2$	(m_o) is the invariant mass
4-wave vector $\mathbf{K} = (\omega/c, \mathbf{k})$	$\mathbf{K} = 1/\hbar\ \mathbf{P}$	$\mathbf{K}\cdot\mathbf{K} = (m_o c/\hbar)^2$	$(\hbar)$ is the reduced Planck constant invariant Relation derived in Section 3. $\hbar/(m_o c) = \lambda\!\!\!^{-}$ = reduced Compton wavelength
4-gradient $\partial = (\partial_t/c, -\nabla)$	$\partial = -i\mathbf{K}$	$\partial\cdot\partial = (-im_o c/\hbar)^2 = -(m_o c/\hbar)^2$	(i) is the imaginary constant invariant Relation derived in Section 4. A Klein-Gordon relation

The results shown here suggest that using only the principles of SR plus a few empirical facts, i.e., the measurable invariant relations between 4-vectors, one can derive what are normally considered the axioms of QM. Hence, SR→QM. The rigorous derivation of various QM axioms from SR will be discussed in future works.

Software and Data Availability

None used.

Disclosure

This work was funded by and is the responsibility of the author. No support or incentives were provided by any funding agency, grant or contract.

Acknowledgments

Comments, suggestions and corrections from the readers are appreciated.

References

1 Minkowski, H. (1908) .“Die Grundgleichungen fur die elektromagnetischen Vorgange in bewegten Korpern, Konigliche Gesellschaft der Wissenschaften zu Gottingen’, *Mathematisch-Physikalische Klasse. Nachrichten*, 53–111. Read before the Academy on 21 December 1907.

2 Hehl, F.W. (2008). “Maxwell's equations in Minkowski's world: their premetric generalization and the electromagnetic energy-momentum tensor”, *Annalen der Physik* 17(9-10): 691-704. doi: https://doi.org/10.1002/andp.200810320. Preprint: https://arxiv.org/pdf/0807.4249.pdf.

3 Itin,Y.; Friedman, Y. (2008). “Backwards on Minkowski's road. From 4D to 3D Maxwellian electromagnetism”, *Annalen der Physik* 17(9-10): 769-786. DOI: https://doi.org/10.1002/andp.200810321.

4 Minkowski, H. (1909) “Raum und Zeit [Space and Time]”, *Physik. Zeitschr* 10: 104–111. For an English translation see the book *The Principle of Relativity* by H.A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, transl. by W. Perrett and G.B. Jeffery (Dover: New York, 1952). ISBN: 978-0486600819.

5 Landau, L.D.; Lifshitz, E.M. (2002). [1939]. *The Classical Theory of Fields, Course of Theoretical Physics 2,* (4th ed.). Butterworth–Heinemann: 4. ISBN: 0-7506-2768-9.

6 Schutz, John W. (1997-10-08). *Independent Axioms for Minkowski Space-Time*, CRC Press. ISBN: 978-0582317604.

7 Born, M. (1926). “Zur Quantenmechanik der Stoßvorgänge [On the quantum mechanics of collisions]”, *Zeitschrift für Physik* 37: 863–867.
8 Born, M. (1926). “Quantenmechanik der Stoßvorgange [Quantum mechanics of impact processes]”, *Zeitschrift für Physik* 38: 803-827.
9 Koopman, B. O. (1931). "Hamiltonian Systems and Transformations in Hilbert Space", *Proceedings of the National Academy of Sciences* 17 (5): 315-318. DOI: https://doi.org/10.1073%2Fpnas.17.5.315.
10 von Neumann, J. (1932). "Zur Operatorenmethode In Der Klassischen Mechanik", *Annals of Mathematics* 33(3): 587-642. DOI: https://doi.org/10.2307%2F1968537.
11 von Neumann, J. (1932). "Zusatze Zur Arbeit "Zur Operatorenmethode…", *Annals of Mathematics* 33(4): 789-791. DOI: https://doi.org/10.2307%2F1968225.
12 Wigner, E. (1932). "On the Quantum Correction for Thermodynamic Equilibrium", *Physical Review* 40(5): 749-759. DOI: https://doi.org/10.1103/PhysRev.40.749.
13 Gozzi, E.; Reuter, M.; Thacker, W.D. (1989). “Hidden BRS invariance in classical mechanics. II”, Physical Review *D* 40: 3363-3377. DOI: https://doi.org/10.1103/PhysRevD.40.3363.
14 Gozzi, E.; Reuter, M. (1989). “Algebraic characterization of ergodicity”, *Phys. Lett. B* 233, 383. DOI: https://doi.org/10.1016/0370-2693(89)91327-0.
15 Lam, P.K. (2014). “Special Relativity at the Quantum Scale”, *PLoS ONE* 9(12): e115810. DOI: https://doi.org/10.1371/journal.pone.0115810.
16 Walleczek, J.: Grossing, G.; Pylkkanen, P., Hiley, B. (2019). *Emergent Quantum Mechanics: David Bohm Centennial Perspectives*, *Entropy,* MDPI. ISBN: 978-3-03897-616-5 (pbk); ISBN: 978-3-03897-617-2 (pdf).
17 Simulations suitable for pedagogical and other purposes are available at: http://scirealm.org/Physics-PlanckConstantViaAtomicLineSpectra.html, http://scirealm.org/Physics-PlanckConstantViaComptonScattering.html, http://scirealm.org/Physics-PlanckConstantViaElectronDiffraction.html, http://scirealm.org/Physics-PlanckConstantViaIncandescence.html, http://scirealm.org/Physics-PlanckConstantViaLEDs.html, and http://scirealm.org/Physics-PlanckConstantViaSternGerlach.html.
18 Gordon, W. (1926-09-29). “Der Comptoneffekt nach der Schrödingerschen Theorie [The Compton effect according to Schrödinger’s theory]”, *Zeit. f. Phys.* 40: 117-133. English translation by D. H. Delphenich: http://neo-classical-physics.info/uploads/3/0/6/5/3065888/gordon_-_compton_effect.pdf.
19 Ivancevic, Vladimir G.; Ivancevic, Tijana T. (2008). *Quantum leap: from Dirac and Feynman, across the universe, to human body and mind*. World Scientific Publishing Company: 419. ISBN: 978-981-281-927-7.
20 Klein, U. (2018). “From Koopman–von Neumann theory to quantum theory”, *Quantum Stud.: Math. Found.* 5: 219–227. DOI: https://doi.org/10.1007/s40509-017-0113-2.
21 Feynman, Richard P. (1948). "Space-time approach to non-relativistic quantum mechanics", *Reviews of Modern Physics*, 20(2): 367–387. DOI: https://doi.org/10.1103/RevModPhys.20.367 .
Available at: https://authors.library.caltech.edu/47756/1/FEYrmp48.pdf.
22 Stueckelberg, Ernst (1941), "La signification du temps propre en mécanique ondulatoire [The meaning of natural time in wave mechanics]", *Helv. Phys. Acta* 14: 322–323. DOI: https://doi.org/10.1007/978-3-7643-8878-2_18.
23 Ulmer, S.; Somorra, C.; Mooser, A.; et al. (2015). “High-precision comparison of the antiproton-to-proton charge-to-mass ratio”, *Nature* 524: 196–199. DOI: https://doi.org/10.1038/nature14861.